

Eulerian Cyclic Posets

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ABSTRACT: We introduce the notion of cyclic posets and Eulerian cyclic posets. The generalized Dehn–Sommerville relations and the **cd**-index extend to this setting allowing us to define the cyclic **cd**-index. We show that a level Eulerian poset where the associated matrix is primitive and has trace 2 yields Eulerian cyclic posets. For two classes of level Eulerian posets we compute the cyclic **cd**-indices.

Keywords: Cyclic **cd**-index; Cyclic compositions; Level Eulerian posets

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1. Introduction

In this paper, we introduce the notion of cyclic posets. They are directed graphs where the notion of maximal chains is played by the directed cycles of length n . We show that the theory of flag enumeration carries over to cyclic posets. We also define the notion of an Eulerian cyclic poset. In particular, we obtain that the Eulerian cyclic posets satisfy an Alexander duality result. Furthermore, there is a cyclic **cd**-index for Eulerian cyclic posets. Whereas the classical **cd**-index encodes the flag f -vector of an Eulerian poset using the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$, for the cyclic **cd**-index we need to introduce the notation $\mathbf{dwd} = \mathbf{b} \cdot \mathbf{w} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{w} \cdot \mathbf{b}$. This should be thought of as the variable $\mathbf{d}$ has been cut in half and wraps around the polynomial w .

One source of examples of cyclic posets arises from considering periodic chains in level posets. Level posets were introduced in [10], where the cover relations are the same under shifting the ranks. When the cover relations are given by the directed graph in Figure 1; see [10, Example 4.7] (or Example 4.2 in this paper), the associated level poset is Eulerian. Level Eulerian posets where the associated 0, 1-matrix is primitive and has

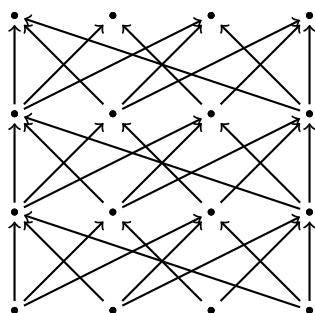


Figure 1: The level Eulerian poset of order 4.

the trace equal to 2 yield Eulerian cyclic posets when considering the periodic chains. For the two classes of Eulerian level posets from [9] we compute the cyclic **cd**-index of the associated cyclic posets. In both cases the cyclic **cd**-index is the sum over all cyclic **cd**-monomials where the coefficient of a given monomial is r to the power of the number of **ds** in the monomial.

*Tool and computational resource disclosure: The author did not use AI or LLM tools for any aspect of this research or the writing of this manuscript.

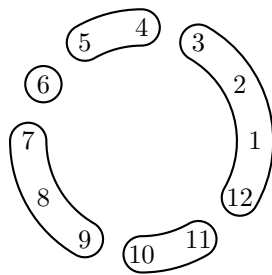


Figure 2: The cyclic composition encoded by $((4, 2, 1, 3, 2), 1)$. This cyclic composition correspond to the set $\{4, 6, 7, 10, 12\}$.

The paper is organized as follows. In Section 2 we review cyclic compositions, the **ab**-index of posets and the **cd**-index of Eulerian posets. In Section 3 we introduce the notion of cyclic posets and intervals in these posets. We discuss level posets in Section 4. Here we obtain some results about level Eulerian posets and we compute the Euler characteristic of the cyclic complex. We use these results to study the quasi-symmetric functions of Eulerian level posets in Section 5. We introduce Eulerian cyclic posets and show the Alexander duality results in Sections 6 and 7. Next we continue with the flag f -vector and introduce the **ab**-index in Section 8 and cyclic **cd**-polynomials in Section 9. In order to prove the existence of the cyclic **cd**-index we use the cyclic generalized Dehn–Sommerville relations; see Section 10. In the classical **cd**-index theory, sparse subsets were used to describe a basis. Instead, we use subsets corresponding to cyclic compositions with odd parts. Finally, we compute two classes of examples in Section 11. We end with open questions in the concluding remarks. Here the most important question is the topological question: What is the topology of the cyclic complex of a level Eulerian poset?

2. Preliminaries

2.1 Cyclic compositions

Recall that a composition of n into k parts corresponds to decomposing a path on n vertices into k paths. A cyclic composition into k parts is a decomposition of a cycle on n vertices into k paths; see for instance [11, Section 3]. To give an explicit description, recall that a composition $\vec{c} = (c_1, c_2, \dots, c_k)$ of n is a list of positive integers such that $c_1 + c_2 + \dots + c_k = n$. This composition corresponds to the intervals

$$I_1 = [1, c_1], I_2 = [c_1 + 1, c_1 + c_2], \dots, I_k = [c_1 + \dots + c_{k-1} + 1, c_1 + \dots + c_k].$$

A cyclic composition is a pair $(\vec{c}, p)$ where $\vec{c}$ is a composition of n and $0 \leq p \leq c_1 - 1$. This pair encodes the following collection of intervals:

$$I_1 - p, I_2 - p, \dots, I_k - p,$$

that is, c_1 is the number of elements of the interval that contains the element 1 and p is how much this interval is shifted from $[1, c_1]$. As an example, the cyclic composition in Figure 2 is encoded by the composition $\vec{c} = (4, 2, 1, 3, 2)$ and the shift $p = 1$.

Note that there is a natural bijection between k element subsets of $\mathbb{Z}_n$ and cyclic compositions into k parts. Make $\mathbb{Z}_n$ into a directed cycle by letting $i \rightarrow i + 1$ be the directed edges for all i in $\mathbb{Z}_n$, where the addition is modulo n . Then each interval induces a directed path. Given a cyclic composition $(\vec{c}, p)$ let S be the set of the sources of each interval. Explicitly, these elements are

$$S = \{1 - p, c_1 + 1 - p, \dots, c_1 + \dots + c_{k-1} + 1 - p\}.$$

For a sequence of complex numbers $\{\alpha_n\}_{n \geq 1}$ let $\varphi(t)$ be the associated generating function $\sum_{n \geq 1} \alpha_n \cdot t^n$. Let $\gamma_{n,k}$ be the sum

$$\gamma_{n,k} = \sum_{(\vec{c}, p)} \alpha_{c_1} \cdot \alpha_{c_2} \cdots \alpha_{c_k},$$

where the sum is over all cyclic compositions of n into k parts. For instance $\gamma_{3,2} = 3 \cdot \alpha_1 \alpha_2$ since there are 3 cyclic compositions of 3 into 2 parts. Then [11, Proposition 3.1] states that

$$\sum_{n \geq k} \gamma_{n,k} \cdot t^n = t \cdot \varphi'(t) \cdot \varphi(t)^{k-1}, \quad (1)$$

$$\sum_{n \geq 1} \sum_{k=1}^n \gamma_{n,k} \cdot t^n = \frac{t \cdot \varphi'(t)}{1 - \varphi(t)}. \quad (2)$$

The Fibonacci numbers are defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Recall that F_n is the number of compositions of n into parts 1 and 2. Similarly, F_{n-1} is the number of compositions of n into odd parts.

Lemma 2.1. (a) The number of cyclic compositions of n into parts 1 and 2 is given by $F_n + F_{n-2}$. (b) The number of cyclic compositions of n into odd parts is given by $F_n + F_{n-2}$ if n is odd and $F_n + F_{n-2} - 2$ if n is even.

Proof. To prove part (a), either we have the pair $\{n, 1\}$ or not. Compositions without this pair are counted by F_n . Compositions with this pair are counted by F_{n-2} . Part (b) follows part (a) by considering the map sending a cyclic composition with odd parts into a cyclic composition with parts 1 and 2 by replacing each odd part as follows

$$2r + 1 \mapsto \underbrace{2, 2, \dots, 2}_r, 1.$$

This yields a bijection except that the two cyclic compositions with all parts equal to 2 which are not in the image. $\square$

2.2 The ab-index and cd-index of graded posets

Let P be a graded poset of rank $n + 1$, that is, the poset P has a minimal element $\widehat{0}$, a maximal element $\widehat{1}$ and there is a rank function $\rho : P \rightarrow \{0, 1, \dots, n + 1\}$ such that $\rho(\widehat{0}) = 0$ and the cover relation $x < y$ implies $\rho(x) + 1 = \rho(y)$. The flag f -vector enumerates the number of chains in the poset given a set of ranks. More specifically, let $S = \{s_1 < s_2 < \dots < s_k\}$ be a subset of $\{1, 2, \dots, n\}$. The flag f -vector entry f_S is defined by

$$f_S = |\{\widehat{0} < x_1 < \dots < x_k < \widehat{1} \subseteq P : \rho(x_i) = s_i\}|.$$

The flag h -vector is obtained by the inclusion-exclusion relation and its inverse

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T, \quad f_S = \sum_{T \subseteq S} h_T. \quad (3)$$

For each subset S of $\{1, 2, \dots, n\}$ define the **ab**-monomial $u_S = u_1 u_2 \dots u_n$, where $u_i = \mathbf{b}$ if $i \in S$ and $u_i = \mathbf{a}$ otherwise. The **ab**-index is defined as the sum

$$\Psi(P) = \sum_{S \subseteq \{1, 2, \dots, n\}} h_S \cdot u_S.$$

An alternative way to view the **ab**-index is by defining the weight $\text{wt}(c) = v_1 v_2 \dots v_n$ of a chain $c = \{\widehat{0} < x_1 < \dots < x_k < \widehat{1}\}$ in the poset P by

$$v_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \rho(x_2), \dots, \rho(x_k)\}, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

The **ab**-index is then given by the sum

$$\Psi(P) = \sum_c \text{wt}(c),$$

where c ranges over all chains in the poset P .

A poset P is *Eulerian* if every non-singleton interval contains the same number of elements of odd as of even rank. For an Eulerian poset P there are linear relations among the flag f -entries, known as the generalized Dehn–Sommerville relations; see [1].

Theorem 2.1 (Bayer–Billera). Let $S \subseteq \{1, 2, \dots, n\}$ and let i and k be two consecutive elements of the set $S \cup \{0, n + 1\}$. Then the following relation holds for the flag f -vector of an Eulerian poset of rank n :

$$(1 + (-1)^{k-i}) \cdot f_S = \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \cdot f_{S \cup \{j\}}.$$

One such relation is the symmetry of the flag h -vector, stating that $h_S = h_{\overline{S}}$, where $\overline{S}$ is the complement $\{1, 2, \dots, n\} - S$. The **cd**-index offers an explicit basis for the subspace cut out by the generalized Dehn–Sommerville relations. Its existence was first proved by Bayer and Klapper [4]. See also [15, Section 3.17] for a poset oriented proof.

Theorem 2.2 (Bayer–Klapper). The **ab**-index of an Eulerian poset P belongs to the subalgebra $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ where $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$.

3. Cyclic posets

We define a cyclic poset P as follows.

Definition 3.1. A graded cyclic poset of rank n is a directed graph P with a rank function $\rho : P \rightarrow \mathbb{Z}_n$ such that if $x \rightarrow y$ is a directed edge then the equality $\rho(x) + 1 = \rho(y)$ holds.

To every cyclic poset P we associate a simplicial complex, defined as follows.

Definition 3.2. For a cyclic poset P of rank n define the cyclic complex $\Delta(P)$ to be the simplicial complex where the facets are the directed cycles of length n .

In the case when every element x in P belongs to a directed cycle of length n , we note that the cyclic complex is a balanced simplicial complex and its dimension is $n - 1$.

Consider the example in Figure 3. Note that there is a directed path $a \rightarrow g \rightarrow i$. However, since there is no directed edge from i to a , the pair $\{a, i\}$ is not a face in the cyclic complex.

Example 3.1. Let Q be a graded poset of rank $n + 1$. We obtain a cyclic poset by letting $\mathcal{C}(Q) = Q - \{\widehat{0}, \widehat{1}\}$. The directed edges are given by the following two rules: (i) if $x \prec_Q y$ then $x \rightarrow y$ and; (ii) if $\rho_Q(x) = n$ and $\rho_Q(y) = 1$ then $x \rightarrow y$.

Assume that $\{x, y\}$ is a face in the cyclic complex and there is a directed path from the element x to y of length $k < n$. Then the interval is given by

$$[x, y] = \{z \in P : \text{there is a directed path from } x \text{ to } z \text{ of length } i \geq 0 \text{ and} \\ \text{there is a directed path from } z \text{ to } y \text{ of length } k - i \geq 0\}.$$

Note that the order relation of this interval is given by $z \leq w$ if there are three directed paths x to z , z to w and w to y of lengths i , j and respectively $k - i - j$.

4. Level posets and cyclic level posets

Let J denote the matrix where all the entries are 1. For a matrix $A = (a_{i,j})$ where all the entries are nonnegative integers let $\text{Bin}(A)$ be the matrix where the (i, j) entry is 1 if $a_{i,j} \geq 1$ and 0 otherwise. Let M be a square 0, 1-matrix of order q . We define the *level poset* of the matrix M on the set $\{1, 2, \dots, q\} \times \mathbb{Z}$ by the cover relation $(i, k) \prec (j, k + 1)$ if $m_{i,j} = 1$. Note that this poset is ranked and all the cover relations between elements of two consecutive ranks are identical independent of the rank.

We call the matrix M *level Eulerian* if the associated level poset is Eulerian. The Eulerian condition is equivalent to the following matrix identity:

$$\sum_{k=0}^n (-1)^{n-k} \cdot \text{Bin}(M^k) \cdot \text{Bin}(M^{n-k}) = \delta_{n,0} \cdot I,$$

where δ denotes the Kronecker delta. We can express this identity in terms of the generating function $\Phi(t) = \sum_{n \geq 0} \text{Bin}(M^n) \cdot t^n$. This generating function should be thought of as a matrix where the entries are generating functions in the variable t ; see Example 4.2. A matrix M is level Eulerian if and only if

$$\Phi(t) \cdot \Phi(-t) = I. \quad (4)$$

Definition 4.1. Let M be a square 0, 1-matrix of order q and let n be a positive integer. Define the cyclic poset of period n associated with the matrix M on the elements $[q] \times \mathbb{Z}_n$ where the rank function is given by $\rho((i, k)) = k$ and $(i, k) \rightarrow (j, k + 1)$ is a directed edge if and only if $m_{i,j} = 1$. Let $\Delta_n(M)$ denote the associated cyclic complex.

Example 4.1. For $n = 1$ the cyclic complex consists of all vertices $(i, 1)$ such that $m_{i,i} = 1$. The number of such vertices is given by the trace of the matrix M .

We next compute the Euler characteristic of the cyclic chain complex arising from a level poset. We will use a matrix analogue of equation (1) to obtain the generating function of the Euler characteristic.

Theorem 4.1. Let M be a square 0, 1-matrix. Then the generating function for the Euler characteristic for the cyclic complex $\Delta_n(M)$ is given by

$$\sum_{n \geq 1} \chi(\Delta_n(M)) \cdot t^n = t \cdot \frac{d}{dt} \log(\det(\Phi(t))).$$

Proof. Let k be a positive integer. The number of faces of dimension $k - 1$ is given by the trace

$$f_{k-1} = \text{tr} \left(\sum_{(\vec{c}, p)} \text{Bin}(M^{c_1}) \cdot \text{Bin}(M^{c_2}) \cdots \text{Bin}(M^{c_k}) \right),$$

where the sum is over all cyclic compositions of n into k parts. Let $F(t) = \sum_{n \geq 1} \text{Bin}(M^n) \cdot t^n$, that is, $F(t) = \Phi(t) - I$. Similar to the proof of Proposition 3.1 in [11], we have the identity

$$\sum_{n \geq k} f_{k-1} \cdot t^n = \text{tr} (t \cdot F'(t) \cdot F(t)^{k-1}).$$

Hence the sought-after generating function is

$$\begin{aligned} \sum_{n \geq 1} \chi(\Delta_n(M)) \cdot t^n &= \sum_{n \geq 1} \sum_{k=1}^n (-1)^{k-1} \cdot f_{k-1} \cdot t^n \\ &= \sum_{k \geq 1} (-1)^{k-1} \cdot \text{tr} (t \cdot F'(t) \cdot F(t)^{k-1}) \\ &= t \cdot \text{tr} (F'(t) \cdot (I + F(t))^{-1}) \\ &= t \cdot \frac{d}{dt} \text{tr} (\log(\Phi(t))) \\ &= t \cdot \frac{d}{dt} \log(\det(\Phi(t))). \end{aligned} \quad \square$$

Corollary 4.1. *Let M be a 0,1-matrix such that the associated poset is level Eulerian. Then the generating function for the Euler characteristic for the cyclic complex $\Delta_n(M)$ is an odd function. In other words, for n even the Euler characteristic satisfies $\chi(\Delta_n(M)) = 0$.*

Proof. Applying the determinant and the logarithm to equation (4) yields

$$\log(\det(\Phi(t))) + \log(\det(\Phi(-t))) = 0.$$

Hence the function $\log(\det(\Phi(t)))$ is odd and so is $t \cdot \frac{d}{dt} \log(\det(\Phi(t)))$. $\square$

This corollary should be compared with the statement that the Euler characteristic of an odd-dimensional manifold is zero.

We call a 0,1-matrix M *primitive* if there is a positive integer γ such that $\text{Bin}(M^\gamma) = J$. The smallest such power is called the *exponent*.

Theorem 4.2. *Let M be a primitive 0,1-matrix such that the associated poset is level Eulerian. Then the determinant of the matrix $\Phi(t)$ is given by*

$$\det(\Phi(t)) = \left(\frac{1+t}{1-t} \right)^{\text{tr}(M)/2},$$

where the power $\text{tr}(M)/2$ is an integer. Furthermore, the Euler characteristic of the cyclic complex $\Delta_n(M)$ is given by

$$\chi(\Delta_n(M)) = \begin{cases} \text{tr}(M) & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

Proof. The identity $\text{Bin}(M^\gamma) = J$ implies that every entry of $\Phi(t)$ is a rational function of the form a polynomial in t plus the term $t^\gamma/(1-t)$. Hence each entry is a rational function with the denominator $1-t$. By factoring $1/(1-t)$ from every row of the matrix, we know that the determinant is of the form $p(t)/(1-t)^s$ for some polynomial $p(t)$ and a nonnegative integer s . Assume that this fraction is in its lowest terms, that is, $1-t$ is not a factor of $p(t)$. Note also that $p(0) = 1$. Since the matrix M is level Eulerian, equation (4) implies the equality

$$p(t) \cdot p(-t) = (1-t)^s \cdot (1+t)^s.$$

Since $p(t)$ does not have the factor $1-t$ and $p(-t)$ does not have the factor $1+t$, the only solution is $p(t) = (1+t)^s$.

Consider the coefficient of t on both sides of the equality $\det(\Phi(t)) = (1+t)^s/(1-t)^s$. Since every off-diagonal entry of $\Phi(t)$ has no constant, the only way to obtain a linear term in the variable t is by multiplying the diagonal entries. Hence the coefficient of the linear term is given by the trace of M . The result follows from observing that the coefficient of the linear term of $(1+t)^s/(1-t)^s$ is $2s$.

The last conclusion follows by Theorem 4.1 and

$$\begin{aligned} \sum_{n \geq 1} \chi(\Delta_n(M)) \cdot t^n &= t \cdot \frac{d}{dt} \log(\det(\Phi(t))) \\ &= s \cdot t \cdot \frac{d}{dt} (\log(1+t) - \log(1-t)) \\ &= \frac{2s \cdot t}{1-t^2}. \end{aligned} \quad \square$$

Corollary 4.2. *Let M be a primitive $0,1$ -matrix such that the associated poset is level Eulerian and that $\text{tr}(M) = 2$. Then the reduced Euler characteristic of the cyclic complex $\Delta_n(M)$ is the given by $\tilde{\chi}(\Delta_n(M)) = (-1)^{n-1}$. In other words, $\tilde{\chi}(\Delta_n(M)) = \tilde{\chi}(S^{n-1})$ where S^{n-1} is the $(n-1)$ -dimensional sphere.*

Example 4.2. Let M be the following 4×4 matrix:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}, \quad \text{Bin}(M^2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

and $\text{Bin}(M^n) = J$ for $n \geq 3$. It is straightforward to verify that M yields a level Eulerian poset; see Example 4.7 in [10] and Figure 1. Directly we have

$$\Phi(t) = \begin{pmatrix} \frac{1}{1-t} & \frac{t}{1-t} & \frac{t}{1-t} & \frac{t^2}{1-t} \\ \frac{t}{1-t} & 1 + \frac{t^2}{1-t} & \frac{t}{1-t} & \frac{t}{1-t} \\ \frac{t^2}{1-t} & t + \frac{t^3}{1-t} & 1 + \frac{t^2}{1-t} & \frac{t}{1-t} \\ \frac{t}{1-t} & \frac{t^2}{1-t} & \frac{t}{1-t} & \frac{1}{1-t} \end{pmatrix}.$$

By Theorem 4.2 the determinant of this matrix is $(1+t)/(1-t)$ and the reduced Euler characteristic is given by $\tilde{\chi}(\Delta_n(M)) = (-1)^{n-1} = \tilde{\chi}(S^{n-1})$.

5. Quasi-symmetric functions

Another way to encode the flag f -vector of a graded poset is via quasi-symmetric functions; see [7]. A quick review is as follows. Let P be a graded poset of rank n and let $\rho(x, y)$ denote the rank difference $\rho(y) - \rho(x)$. Define the polynomial $F_m(P)$ by the sum over multichains taking m steps:

$$F_m(P) = \sum_{\hat{0}=x_0 \leq x_1 \leq \dots \leq x_m = \hat{1}} t_1^{\rho(x_0, x_1)} \cdot t_2^{\rho(x_1, x_2)} \dots t_m^{\rho(x_{m-1}, x_m)}.$$

Observe that $F_m(P)$ is a homogeneous polynomial of degree n in the variables t_1 through t_m . The quasi-symmetric function of the poset P is defined as the limit

$$F(P) = \lim_{m \rightarrow \infty} F_m(P).$$

For a $0,1$ -matrix M we define the matrix $F_m(M)$ where the (i, j) entry is given by

$$(F_m(M))_{i,j} = \sum_{n \geq 0} F_m([(i, 0), (j, n)]).$$

Each entry is a generating function in the variables t_1 through t_m . Note that we set $F_m([(i, 0), (j, n)])$ to be zero if $(i, 0) \not\leq (j, n)$. Observe that this matrix factors as

$$F_m(M) = \Phi(t_1) \cdot \Phi(t_2) \cdots \Phi(t_m).$$

Next let $F(M)$ be the limit

$$F(M) = \lim_{m \rightarrow \infty} F_m(M).$$

Hence the (i, j) entry of the matrix $F(M)$ is the sum $\sum_{n \geq 0} F([(i, 0), (j, n)])$. Furthermore, the matrix $F(M)$ factors as the infinite product $\Phi(t_1) \cdot \Phi(t_2) \cdots$.

Corollary 5.1. *Let M be a primitive $0,1$ -matrix such that the associated poset is level Eulerian and that $\text{tr}(M) = 2$. Then the n th homogeneous part of the determinant of $F(M)$ is given by the sum*

$$\sum_{i+j=n} e_i \cdot h_j,$$

where e_i and h_j are the elementary, respectively, complete symmetric functions.

Proof. By Theorem 4.2 we have

$$\begin{aligned} \det(F(M)) &= \det(\Phi(t_1)) \cdot \det(\Phi(t_2)) \cdots \\ &= \frac{1+t_1}{1-t_1} \cdot \frac{1+t_2}{1-t_2} \cdots \\ &= \left(\sum_{n \geq 0} e_n \right) \cdot \left(\sum_{n \geq 0} h_n \right). \end{aligned} \quad \square$$

6. Eulerian cyclic posets

In this section, we introduce an important class of cyclic posets. Before doing so, we define the notion of a long interval of a cyclic poset. Let x be an element that belongs to a directed cycle of length n . Let $x_{\widehat{0}}$ and $x_{\widehat{1}}$ be two copies of the element x . Define the *open long interval* be the set

$$(x_{\widehat{0}}, x_{\widehat{1}}) = \{z \in P : \text{there is a directed path from } x \text{ to } z \text{ of length } i > 0 \text{ and} \\ \text{there is a directed path from } z \text{ to } x \text{ of length } n - i > 0\}.$$

The order relation of this open interval is given by $z \leq w$ if there are three directed paths x to z , z to w and w to x of lengths $i \geq 1$, $n - i - j$ and respectively $j \geq 1$. Finally, define the (closed) *long interval* $[x_{\widehat{0}}, x_{\widehat{1}}]$ to be set $[x_{\widehat{0}}, x_{\widehat{1}}] = \{x_{\widehat{0}}, x_{\widehat{1}}\} \cup (x_{\widehat{0}}, x_{\widehat{1}})$ where we attach $x_{\widehat{0}}$ and $x_{\widehat{1}}$ as the minimal, respectively maximal element of the interval. This interval should be viewed as having $x_{\widehat{0}}$ as a minimal element and then looping around the directed graph, finishing with $x_{\widehat{1}}$ as the maximal element.

Definition 6.1. *A cyclic poset P of rank n is Eulerian if*

- (i) *every long interval $[x_{\widehat{0}}, x_{\widehat{1}}]$ is an Eulerian poset, and*
- (ii) *the reduced Euler characteristic of the cyclic complex $\Delta(P)$ is given by $\widetilde{\chi}(\Delta(P)) = (-1)^{n-1}$.*

Note that condition (i) implies that every short interval $[x, y]$ is also an Eulerian poset.

For a face F of a simplicial complex Δ , recall that the link is defined to be $\text{link}_{\Delta}(F) = \{G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta\}$. The link describes how the simplicial complex looks around the face F .

Lemma 6.1. *Let P be a cyclic poset of rank n . For a face F of the cyclic complex $\Delta(P)$ the reduced Euler characteristic of the link of F is given by*

$$\widetilde{\chi}(\text{link}_{\Delta(P)}(F)) = (-1)^{n-\dim(F)}.$$

Observe that the Euler characteristic of a link of a face in the cyclic complex $\Delta(P)$ is the same as the Euler characteristic of the link of a face of a $(n-1)$ -dimensional sphere.

Example 6.1. Let Q be an Eulerian poset of rank $n+1$. Construct the cyclic poset $\mathcal{C}(Q)$ as in Example 3.1. Then the cyclic poset $\mathcal{C}(Q)$ is Eulerian.

Example 6.2. Let M be primitive $0,1$ -matrix such that the associated poset is level Eulerian and that $\text{tr}(M) = 2$. Corollary 4.2 implies that the cyclic poset of the matrix M is Eulerian.

Let G be a planar graph where each vertex degree is 3 and every region has an even number of edges. The regions of such a graph can be colored with three colors 1, 2 and 3 such that adjacent regions obtain different colors. Assume furthermore that the following condition holds: given three regions of three different colors such that they are pairwise adjacent then the three regions meet in a vertex. Define a cyclic poset $P(G)$ of rank 3 on the set of regions of the graph G . Let the rank function ρ be the color of the region. Finally, let $x \rightarrow y$ be a directed edge in P if the regions x and y are adjacent and $\rho(x) + 1 \equiv \rho(y) \pmod{3}$.

Proposition 6.1. *Let G be a planar graph described in the previous paragraph. Then the cyclic poset $P(G)$ is Eulerian.*

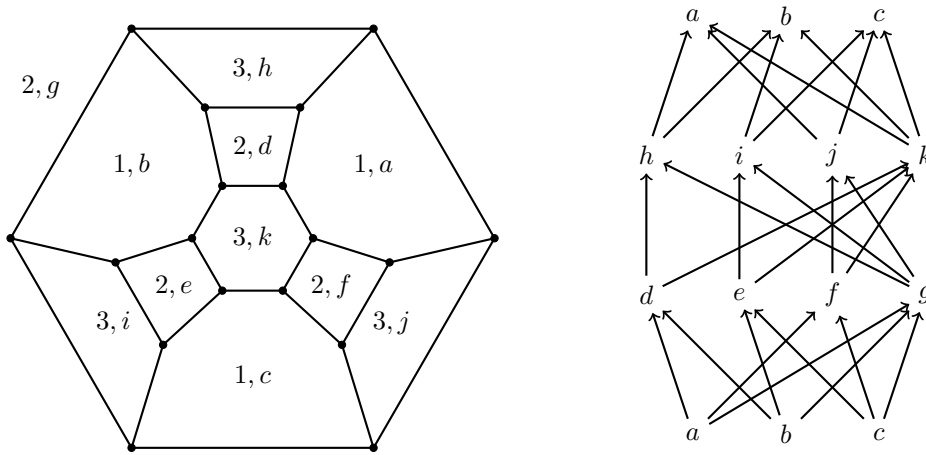


Figure 3: On the left a planar graph with a 3 coloring and labeling of its regions. To the right its associated cyclic poset of rank 3. Note that the rank 1 elements are repeated on top for clarity.

Proof. Since the graph is planar, this implies $\tilde{\chi}(\Delta(P)) = 1$. Next, assume that x and z are two regions of colors i and $i + 2$. Assume there is a region y of color $i + 1$ such that $x \rightarrow y \rightarrow z$. If there is no edge from z to x , then the set $\{x, z\}$ is not an edge. If there is an edge $z \rightarrow x$, the three regions meet in a vertex v . On the other side of the edge between regions x and z there is another region y' of color $i + 1$. Hence the interval $[x, z]$ is a diamond. Hence every 2-interval is Eulerian. By [13, Lemma 2.6] we know that every long interval $[u_{\hat{0}}, u_{\hat{1}}]$ is an Eulerian poset. $\square$

Note that the cyclic complex of the cyclic poset in Proposition 6.1 is the dual of the graph G . For an example of this proposition; see Figure 3.

7. Alexander duality

We now obtain Alexander duality for Eulerian cyclic posets. For comparison see [15, Proposition 3.16.5].

Theorem 7.1. *Let P be an Eulerian cyclic poset of rank n . Let P be the disjoint union of two sets Q and R , that is, $P = Q \sqcup R$. Then the reduced Euler characteristic of the two induced complexes $\Delta_Q = \Delta(P)|_Q$ and $\Delta_R = \Delta(P)|_R$ are related by*

$$\tilde{\chi}(\Delta_Q) = (-1)^n \cdot \tilde{\chi}(\Delta_R).$$

Proof. We now proceed by induction on the cardinality of the set R . Observe that the statement is true when $R = \emptyset$ and $Q = P$ since the reduced Euler characteristic of the simplicial complex consisting of just the empty set is -1 . Assume now that the result is true for the pair Q, R . Furthermore, we can assume that the set Q is nonempty. Let x be an element in Q . Then we have

$$\tilde{\chi}(\Delta_{Q-\{x\}}) = \tilde{\chi}(\Delta_Q) - \mu([x_{\hat{0}}, x_{\hat{1}}] \cap Q),$$

by partitioning the faces of Δ_Q according to whether they contain the element x or not. The Möbius function enters via Philip Hall's theorem. The induction hypothesis implies $\tilde{\chi}(\Delta_Q) = (-1)^n \cdot \tilde{\chi}(\Delta_R)$. Furthermore the interval $[x_{\hat{0}}, x_{\hat{1}}]$ is an Eulerian poset of rank n , so it satisfies Alexander duality; see [15, Proposition 3.16.5]. Hence we have $\mu([x_{\hat{0}}, x_{\hat{1}}] \cap Q) = (-1)^{n-1} \cdot \mu([x_{\hat{0}}, x_{\hat{1}}] \cap R)$. Now we conclude

$$\begin{aligned} \tilde{\chi}(\Delta_{Q-\{x\}}) &= (-1)^n \cdot \tilde{\chi}(\Delta_R) - (-1)^{n-1} \cdot \mu([x_{\hat{0}}, x_{\hat{1}}] \cap R) \\ &= (-1)^n \cdot (\tilde{\chi}(\Delta_R) + \mu([x_{\hat{0}}, x_{\hat{1}}] \cap R)) \\ &= (-1)^n \cdot \tilde{\chi}(\Delta_{R \cup \{x\}}), \end{aligned}$$

proving the induction step. $\square$

8. Flag vectors and the ab-index

Let P be a cyclic poset of rank n . The flag f -vector enumerates the number of faces of the cyclic complex that takes steps in certain levels. More specifically, let S be a subset of $\{1, 2, \dots, n\} = \mathbb{Z}_n$. For the empty set, let

$f_\emptyset = 1$. For a non-empty subset $S = \{s_1 < s_2 < \dots < s_k\}$ we let

$$f_S = \{\{x_1, x_2, \dots, x_k\} \in \Delta(P) : \rho(x_i) = s_i\}.$$

The flag h -vector is obtained by the inclusion-exclusion relation displayed in equation (3).

Corollary 8.1. *The flag h -vector of an Eulerian cyclic poset satisfies the following symmetry*

$$h_S = h_{\overline{S}},$$

where $\overline{S}$ is the complement set $\mathbb{Z}_n - S$.

Proof. Let $Q = \{x \in P : \rho(x) \in S\}$ and $R = \{x \in P : \rho(x) \notin S\}$. Then we have by Alexander duality

$$h_S = (-1)^{|S|-1} \cdot \tilde{\chi}(\Delta_Q) = (-1)^{n-|S|-1} \cdot \tilde{\chi}(\Delta_R) = h_{\overline{S}}. \quad \square$$

For each subset S of $\{1, 2, \dots, n\}$ define the **ab**-monomial u_S and **ab**-polynomial v_S as in Subsection 2.2. Then the **ab**-index is similarly defined as the two sums

$$\Psi(P) = \sum_{S \subseteq \mathbb{Z}_n} h_S \cdot u_S = \sum_F \text{wt}(F),$$

where the second sum is over all faces F of the cyclic complex $\Delta(P)$.

Example 8.1. Let Q be a graded poset of rank $n + 1$. Construct a cyclic poset $\mathcal{C}(Q)$ as in Example 3.1. Then we have the equality $\Psi(\mathcal{C}(Q)) = \Psi(Q)$.

Definition 8.1. Define the shift operator $E : \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle \rightarrow \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ by $E(1) = 1$ and $E(u \cdot \mathbf{a}) = \mathbf{a} \cdot u$ and $E(u \cdot \mathbf{b}) = \mathbf{b} \cdot u$ and extend by linearity.

Note that $E(u \cdot \mathbf{c}) = \mathbf{c} \cdot u$ and $E(u \cdot (\mathbf{a} - \mathbf{b})) = (\mathbf{a} - \mathbf{b}) \cdot u$.

Example 8.2. Let P be a cyclic poset of rank n . Let $E(P)$ be the cyclic poset where we increase each rank by 1, that is, $\rho_{E(P)}(x) = \rho_P(x) + 1$. Then the **ab**-index of the cyclic poset $E(P)$ is given by $E(\Psi(P))$, that is, $\Psi(E(P)) = E(\Psi(P))$.

Let $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle_n$ be the Abelian group of homogenous degree n **ab**-polynomials. For $0 \leq i \leq n - 1$ define $\Gamma_i : \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle_n \rightarrow \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle_{n-1}$ by

$$\Gamma_i(u \cdot \mathbf{a} \cdot v) = \Gamma_i(u \cdot \mathbf{b} \cdot v) = v \cdot u,$$

for homogenous **ab**-polynomial u and v such that $\deg(u) = i$ and $\deg(v) = n - 1 - i$. Similar to Lemma 2.2 in [12] we have

Lemma 8.1. *The kernel of the map $\Gamma_i : \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle_n \rightarrow \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle_{n-1}$ is given by*

$$\ker(\Gamma_i) = \{u \cdot (\mathbf{a} - \mathbf{b}) \cdot v : \deg(u) = i, \deg(v) = n - 1 - i\}.$$

Proposition 8.1. *For a cyclic poset P of rank n and $i \in \mathbb{Z}_n$ we have*

$$\Gamma_i(\Psi(P)) = \sum_{\substack{x \in P \\ \rho(x)=i}} \Psi([x_{\widehat{0}}, x_{\widehat{1}}]).$$

Proof. If a face of the cyclic complex does not contain any element of rank i then it does not contribute anything to the term $\Gamma_i(\Psi(P))$. If the face contains an element of rank i , say the element x , then the remaining terms of the weight contribute to the classical **ab**-index of the long interval $[x_{\widehat{0}}, x_{\widehat{1}}]$. $\square$

9. Cyclic cd-polynomials

In this section we introduce cyclic **cd**-polynomials. Recall that $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Note that the shift operator E from the previous section satisfies $E(v \cdot \mathbf{c}) = \mathbf{c} \cdot v$. We introduce the notation

$$\underline{\mathbf{d}}v\underline{\mathbf{d}} = E(v \cdot \mathbf{d}) = \mathbf{b} \cdot v \cdot \mathbf{a} + \mathbf{a} \cdot v \cdot \mathbf{b}.$$

The two underlined $\underline{\mathbf{d}}$'s symbolizes that the variable $\mathbf{d}$ was cut in half and the two pieces were placed on both sides of the polynomial v .

S	f_S	h_S	$\mathbf{c}^3$	$1/2 \cdot \mathbf{dc}$	$3/2 \cdot \mathbf{cd}$	$1/2 \cdot \mathbf{dcd}$
$\emptyset$	1	1	1	0	0	0
$\{0\}$	3	2	1	1/2	0	1/2
$\{1\}$	4	3	1	1/2	3/2	0
$\{2\}$	4	3	1	0	3/2	1/2
$\{0, 1\}$	9	3	1	0	3/2	1/2
$\{0, 2\}$	9	3	1	1/2	3/2	0
$\{1, 2\}$	9	2	1	1/2	0	1/2
$\{0, 1, 2\}$	18	1	1	0	0	0

Table 1: The computation of the cyclic $\mathbf{cd}$ -index of the Eulerian cyclic poset in Figure 3. Note that the sum of the last four columns is equal to the column containing the flag h -vector. Hence the cyclic $\mathbf{cd}$ -index is given by $\mathbf{c}^3 + 1/2 \cdot \mathbf{dc} + 3/2 \cdot \mathbf{cd} + 1/2 \cdot \mathbf{dcd}$.

Definition 9.1. A cyclic $\mathbf{cd}$ -monomial of degree n is an $\mathbf{ab}$ -polynomial of the form $E^i(v)$ where v is a $\mathbf{cd}$ -monomial of degree n and $0 \leq i \leq n - 1$.

Either the shift operator cuts a variable $\mathbf{d}$ or not. If it does not cut a $\mathbf{d}$, then it is a $\mathbf{cd}$ -monomial. If it cuts a $\mathbf{d}$ it is of the form $\mathbf{dvd}$ where v is a $\mathbf{cd}$ -monomial of degree $n - 2$. By Lemma 2.1(a) there are $F_n + F_{n-2}$ cyclic $\mathbf{cd}$ -monomials of degree n . A cyclic $\mathbf{cd}$ -polynomial is a linear combination of cyclic $\mathbf{cd}$ -monomials. In conclusion, a cyclic $\mathbf{cd}$ -polynomial of degree n is of the form

$$u + \mathbf{dvd} = u + \mathbf{b} \cdot v \cdot \mathbf{a} + \mathbf{a} \cdot v \cdot \mathbf{b},$$

where u and v are $\mathbf{cd}$ -polynomials such that $\deg(u) = n$ and $\deg(v) = n - 2$.

Our goal is to prove that the $\mathbf{ab}$ -index of an Eulerian cyclic poset is a cyclic $\mathbf{cd}$ -polynomial. The resulting $\mathbf{cd}$ -polynomial will be called the *cyclic $\mathbf{cd}$ -index* of the cyclic poset. Let us first note that this result is directly true for the following example of Eulerian cyclic posets.

Example 9.1. Let Q be an Eulerian poset of rank $n + 1$. By combining Examples 3.1, 6.1 and 8.2 we have that $E^i(\mathcal{C}(Q))$ is an Eulerian cyclic poset and its $\mathbf{ab}$ -index can be expressed as an cyclic $\mathbf{cd}$ -index.

Lemma 9.1. Each cyclic $\mathbf{cd}$ -monomial of degree n is in the linear span of the cyclic $\mathbf{cd}$ -indices of Eulerian cyclic posets of the form $E^i(\mathcal{C}(Q))$, where $0 \leq i \leq n - 1$ and Q ranges over Eulerian posets of rank $n + 1$.

Proof. Each $\mathbf{cd}$ -monomial is in the linear span of Eulerian posets; see for instance [15, Theorem 3.17.2]. Hence by Example 6.1 each $\mathbf{cd}$ -monomial is in the linear span of Eulerian cyclic posets of the form $\mathcal{C}(Q)$. By Examples 8.2 and 9.1 it follows that every cyclic $\mathbf{cd}$ -monomial of the form $\mathbf{dvd}$ is also in the linear span of Eulerian cyclic posets of the form $E^i(\mathcal{C}(Q))$. $\square$

When the degree n is even, there is a linear relation between the cyclic $\mathbf{cd}$ -monomials of degree n . Observe that

$$E((\mathbf{c}^2 - 2\mathbf{d})^{n/2}) = E((\mathbf{a} - \mathbf{b})^n) = (\mathbf{a} - \mathbf{b})^n = (\mathbf{c}^2 - 2\mathbf{d})^{n/2}. \quad (5)$$

Especially note that $\mathbf{d}\mathbf{d} = \mathbf{d}$.

Lemma 9.2. Assume that $u \in \mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$ is a $\mathbf{cd}$ -polynomial that factors as $u = (\mathbf{a} - \mathbf{b}) \cdot v$ in $\mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$. Then the polynomial u factors as $u = (\mathbf{c}^2 - 2\mathbf{d}) \cdot y$ in $\mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$.

Proof. The $\mathbf{cd}$ -polynomial u can be written as $u = \mathbf{c} \cdot x - \mathbf{d} \cdot 2y$. Hence we have $(\mathbf{a} - \mathbf{b}) \cdot v = \mathbf{c} \cdot x - \mathbf{d} \cdot 2y$. Now apply the map Γ_0 to obtain $0 = 2 \cdot x - \mathbf{c} \cdot 2y$. Hence $x = \mathbf{c} \cdot y$ and we have $u = (\mathbf{c}^2 - 2\mathbf{d}) \cdot y$. $\square$

Proposition 9.1. Let $w \in \mathbb{Q}\langle \mathbf{a}, \mathbf{b} \rangle$ be an $\mathbf{ab}$ -polynomial of degree n such that $(\mathbf{a} - \mathbf{b}) \cdot w$ and $w \cdot (\mathbf{a} - \mathbf{b})$ both belong to $\mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$. If n is even then w is zero. If n is odd then w is a scalar multiple of $(\mathbf{a} - \mathbf{b})^n$.

Proof. Proof by induction on n . The two cases $n = 0$ and $n = 1$ are direct. Assume now that the statement is true for n and consider such a polynomial w of degree $n + 2$. By Lemma 9.2 we know that $(\mathbf{a} - \mathbf{b}) \cdot w$ factors as $(\mathbf{c}^2 - 2\mathbf{d}) \cdot w' = (\mathbf{a} - \mathbf{b})^2 \cdot w'$, that is, $w = (\mathbf{a} - \mathbf{b}) \cdot w'$. By the reverse of Lemma 9.2 we know that $w \cdot (\mathbf{a} - \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot w' \cdot (\mathbf{a} - \mathbf{b})$ factors as $(\mathbf{a} - \mathbf{b}) \cdot w'' \cdot (\mathbf{c}^2 - 2\mathbf{d}) = (\mathbf{a} - \mathbf{b}) \cdot w'' \cdot (\mathbf{a} - \mathbf{b})^2$. Hence we have $w = (\mathbf{a} - \mathbf{b}) \cdot w'' \cdot (\mathbf{a} - \mathbf{b})$ and w'' has degree n . Furthermore, we know that $(\mathbf{a} - \mathbf{b}) \cdot w''$ and $w'' \cdot (\mathbf{a} - \mathbf{b})$ both belong to $\mathbb{Q}\langle \mathbf{c}, \mathbf{d} \rangle$. By the induction hypothesis, we now reach the conclusion: If n is even w'' is zero and hence $w = 0$. If n is odd $w'' = c \cdot (\mathbf{a} - \mathbf{b})^n$ where c is a scalar and hence we have $w = c \cdot (\mathbf{a} - \mathbf{b})^{n+2}$. $\square$

Proposition 9.2. *When n is odd the cyclic $\mathbf{cd}$ -monomials of degree n are linearly independent. When n is even there is only one linear relation among the cyclic $\mathbf{cd}$ -monomials of degree n which is given by equation (5).*

Proof. Assume there is a non-trivial relation among the degree n cyclic $\mathbf{cd}$ -monomials of degree n . Then it can be written as $\mathbf{c} \cdot x + \mathbf{d} \cdot y = \mathbf{dzd}$, where x, y and z are $\mathbf{cd}$ -polynomials of degrees $n-1, n-2, n-2$, respectively. Applying Γ_0 yields $2 \cdot x + \mathbf{c} \cdot y = z \cdot \mathbf{c}$. Multiplying this relation with $\mathbf{c}$ and subtracting the first relation times 2 yields $(\mathbf{c}^2 - 2\mathbf{d}) \cdot y = \mathbf{c} \cdot z \cdot \mathbf{c} - 2 \cdot \mathbf{dzd}$. In terms of $\mathbf{a}$'s and $\mathbf{b}$'s, this states that $(\mathbf{a} - \mathbf{b})^2 \cdot y = (\mathbf{a} - \mathbf{b}) \cdot z \cdot (\mathbf{a} - \mathbf{b})$. Cancelling the factor of $\mathbf{a} - \mathbf{b}$ yields $(\mathbf{a} - \mathbf{b}) \cdot y = z \cdot (\mathbf{a} - \mathbf{b})$. Hence there is an $\mathbf{ab}$ -polynomial w of degree $n-3$ such that $y = w \cdot (\mathbf{a} - \mathbf{b})$ and $z = (\mathbf{a} - \mathbf{b}) \cdot w$. Proposition 9.1 implies that if n is odd then $w = 0$ and there is no relation. If n is even, we let $w = (\mathbf{a} - \mathbf{b})^{n-3}$. We get $y = z = (\mathbf{c}^2 - 2\mathbf{d})^{n/2-1}$ and the relation is (5). $\square$

By combining Proposition 9.2 and Lemma 9.1, we have the following lower bound.

Corollary 9.1. *The dimension of the linear span of $\mathbf{ab}$ -indices of Eulerian cyclic posets is bounded below by*

$$\begin{cases} F_n + F_{n-2} & \text{if } n \text{ is odd,} \\ F_n + F_{n-2} - 1 & \text{if } n \text{ is even.} \end{cases}$$

10. The cyclic generalized Dehn–Sommerville relations and the $\mathbf{cd}$ -index

The flag f -vector of an Eulerian cyclic poset satisfies a cyclic analogue of the generalized Dehn–Sommerville relations. For a subset S of $\mathbb{Z}_n$, we say that two elements i and k in S are consecutive if none of the elements $i+1, i+2, \dots, k-1$ belong to S . Note that the element i could be equal to k if the set S is the singleton $\{i\}$. Furthermore, we let the difference $k-i$ be the smallest positive integer congruent to $k-i$.

Theorem 10.1. *Let $S \subseteq \mathbb{Z}_n$ and let i and k be two consecutive elements from the set S . Then the following relation holds for the flag f -vector of an Eulerian cyclic poset*

$$(1 + (-1)^{k-i}) \cdot f_S = \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \cdot f_{S \cup \{j\}}.$$

Proof. Fix a face F of the cyclic complex $\Delta(P)$ whose ranks are given by the set S . Let x and y be the two elements in the face F such that their ranks are given by $\rho(x) = i$ and $\rho(y) = k$. Apply the Euler–Poincaré–Schläfli relation to the interval $[x, y]$:

$$1 + (-1)^{k-i} = \sum_{\substack{z \in [x, y] \\ z \neq x, y}} (-1)^{\rho(z)-i-1}.$$

Summing this identity over all faces F yields the relation. In the case when S is a singleton element and thus $i = k$, apply the Euler–Poincaré–Schläfli relation to the long interval $[x_{\hat{0}}, x_{\hat{1}}]$. $\square$

Theorem 10.2. *The flag f -vector of an Eulerian cyclic poset of rank n is completely determined by the following entries of the flag f -vector:*

- (a) *When n is odd, it suffices to know f_S where the subsets S correspond to cyclic compositions where all the parts are odd.*
- (b) *When n is even it suffices to know f_S where the subsets S correspond to cyclic compositions where all the parts are odd and the empty set $\emptyset$.*

Proof. We begin to express a flag vector entry f_S for S a non-empty set in terms of entries which correspond to cyclic compositions into odd parts. The proof is by induction on the set S . The base case $S = \mathbb{Z}_n$ is direct since this subset corresponds to the cyclic composition $(1, 1, \dots, 1)$. Assume that S is a non-empty subset which has two consecutive elements i and k such that the difference $k-i$ is even. The cyclic generalized Dehn–Sommerville relation expresses f_S in terms of larger subsets:

$$2 \cdot f_S = \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \cdot f_{S \cup \{j\}}.$$

By the induction hypothesis, the entries $f_{S \cup \{j\}}$ can be expressed in the desired form, completing the induction. The case when the set S is empty remains. The condition that the cyclic poset is Eulerian implies

$$(1 - (-1)^n) \cdot f_\emptyset = \sum_{\emptyset \subsetneq S \subseteq \mathbb{Z}_n} (-1)^{|S|-1} \cdot f_S.$$

In other words, when n is odd we have that $f_\emptyset$ is a linear combination of the other flag f -vector entries. $\square$

Theorem 10.3. *Let P be an Eulerian cyclic poset. Then the **ab**-index $\Psi(P)$ can be written as a cyclic **cd**-polynomial.*

Proof. Theorem 10.2 yields an upper bound for the dimension of the linear span of the **ab**-indices of Eulerian level posets. By Lemma 2.1 this is the same as the lower bound in Corollary 9.1. $\square$

11. Two classes of level Eulerian posets

In this section, we consider the two classes of level Eulerian posets from [9, Section 4]. Our goal is to compute the cyclic **cd**-index of their cyclic complexes. To do so, we first introduce the anti-diagonal flip: for a $m \times n$ matrix $M = (M_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ define the anti-flip to be the $n \times m$ matrix

$$(M^{\triangleright})_{i,j} = M_{m+1-j, n+1-i}.$$

Let r be a nonnegative integer. Define the following three $(r+1) \times (r+1)$ matrices:

- (A) $A = (a_{i,j})_{0 \leq i,j \leq r}$ where $a_{i,j} = 1$ if $i = 0$ and $a_{i,j} = 0$ otherwise.
- (A') $A' = (a'_{i,j})_{0 \leq i,j \leq r}$ where $a'_{i,j} = 1$ if $i = 0$ and $0 \leq j \leq r-1$, or $i = 1$ and $j = r-1$. Otherwise we let $a'_{i,j} = 0$.
- (B) $B = (b_{i,j})_{0 \leq i,j \leq r}$ where $b_{i,j} = 1$ if $0 \leq i-j \leq 1$ and $b_{i,j} = 0$ otherwise.

Let M and N be the two block matrices

$$M = \left(\begin{array}{c|c} A & B \\ \hline J & A^{\triangleright} \end{array} \right), \quad N = \left(\begin{array}{c|c} A' & B \\ \hline J & A'^{\triangleright} \end{array} \right).$$

For $r = 3$ these two matrices are

$$M = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right), \quad N = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right).$$

Propositions 4.2 and 5.2 in [9] shows that the matrix M is level Eulerian for $r \geq 0$, respectively, and N is level Eulerian for $r \geq 2$. Note that the matrix M in the case $r = 0$ yields the butterfly poset and that Example 4.2 corresponds to the case $r = 1$ by switching the 2nd and 3rd rows and switching the 2nd and 3rd columns.

Let ϕ denote the non-commutative power series

$$\phi = \sum_{n \geq 0} \phi_n = \frac{1}{1 - \mathbf{c} - r \cdot \mathbf{d}},$$

where ϕ_n denotes the degree n component. Then [9, Propositions 4.3 and 5.4] implies that the **cd**-index of the interval $[(i, 0), (i, n)]$ in these two examples is given by

$$\Psi([(i, 0), (i, n)]_M) = \begin{cases} \phi_{n-1} & \text{if } i = 0, 2r+1, \\ \mathbf{c} \cdot \phi_{n-2} & \text{if } 1 \leq i \leq r, \\ \phi_{n-2} \cdot \mathbf{c} & \text{if } r+1 \leq i \leq 2r, \end{cases}$$

and $\Psi([(i, 0), (i, n)]_N)$ is given by

$$\begin{cases} \mathbf{c} \cdot \phi_{n-2} + (r-1) \cdot \mathbf{d} \cdot \phi_{n-3} + \delta_{n,1} & \text{if } i = 0, \\ \mathbf{c} \cdot \phi_{n-2} + \mathbf{d} \cdot \phi_{n-3} & \text{if } i = 1, \\ \mathbf{c} \cdot \phi_{n-2} & \text{if } 2 \leq i \leq r, \\ \phi_{n-2} \cdot \mathbf{c} & \text{if } r+1 \leq i \leq 2r-1, \\ \phi_{n-2} \cdot \mathbf{c} + \phi_{n-3} \cdot \mathbf{d} & \text{if } i = 2r, \\ \phi_{n-2} \cdot \mathbf{c} + (r-1) \cdot \phi_{n-3} \cdot \mathbf{d} + \delta_{n,1} & \text{if } i = 2r+1. \end{cases}$$

Lemma 11.1. *The following identity holds:*

$$\sum_{i=0}^{2r+1} \Psi([(i, 0), (i, n)]_M) = \sum_{i=0}^{2r+1} \Psi([(i, 0), (i, n)]_N) = 2 \cdot \phi_{n-1} + r \cdot \mathbf{c} \cdot \phi_{n-2} + r \cdot \phi_{n-2} \cdot \mathbf{c}.$$

Proof. For the case of the matrix M , the sum is direct. For the matrix N case, we have that the first half of the sum is given by

$$\sum_{i=0}^r \Psi([(i, 0), (i, n)]_N) = (r+1) \cdot \mathbf{c} \cdot \phi_{n-2} + r \cdot \mathbf{d} \cdot \phi_{n-3} + \delta_{n,1} = \phi_{n-1} + r \cdot \mathbf{c} \cdot \phi_{n-2},$$

using the recursion $\phi_{n-1} = \delta_{n,1} + \mathbf{c} \cdot \phi_{n-2} + r \cdot \mathbf{d} \cdot \phi_{n-3}$. The second half follows by symmetry. $\square$

Proposition 11.1. *The cyclic $\mathbf{cd}$ -index of the complexes $\Delta_n(M)$ and $\Delta_n(N)$ is the sum over all cyclic $\mathbf{cd}$ -monomials of degree n with the coefficient r to the power of the number of $\mathbf{ds}$ in the monomial, that is,*

$$\Psi(\Delta_n(M)) = \Psi(\Delta_n(N)) = \phi_n + r \cdot \underline{\mathbf{d}}\phi_{n-2}\underline{\mathbf{d}}.$$

Proof. Let w denote the cyclic $\mathbf{cd}$ -polynomial described in the statement of the proposition. Note that w is shift invariant, that is, $E(w) = w$. Hence we know $\Gamma_0(w) = \Gamma_i(w)$ for all i . By Proposition 8.1 and Lemma 11.1 we know that

$$\begin{aligned} \Gamma_i(\Delta_n(M)) &= \Gamma_i(\Delta_n(N)) = 2 \cdot \phi_{n-1} + r \cdot \mathbf{c} \cdot \phi_{n-2} + r \cdot \phi_{n-2} \cdot \mathbf{c} \\ &= \Gamma_0(\mathbf{c} \cdot \phi_{n-1}) + \Gamma_0(r \cdot \mathbf{d} \cdot \phi_{n-2}) + \Gamma_0(r \cdot \underline{\mathbf{d}}\phi_{n-2}\underline{\mathbf{d}}) \\ &= \Gamma_0(w) = \Gamma_i(w). \end{aligned}$$

Since this identity holds for all i , we have that $\Psi(\Delta_n(M)) + \alpha \cdot (\mathbf{a} - \mathbf{b})^n = \Psi(\Delta_n(N)) + \beta \cdot (\mathbf{a} - \mathbf{b})^n = w$. Note that the coefficient of $\mathbf{a}^n$ of this identity is $1 + \alpha = 1 + \beta = 1$, that is $\alpha = \beta = 0$ and the result follows. $\square$

Proposition 11.2. *The two non-zero eigenvalues of the matrix M (and N) are $1 \pm \sqrt{2r+1}$.*

Proof. In the cyclic complex $\Delta_n(M)$ the flag f -vector entry $f_{\{1,2,\dots,n\}}$ is the sum of all the entries of the flag h -vector. This sum can be computed by setting $\mathbf{a} = \mathbf{b} = 1$ in the cyclic $\mathbf{cd}$ -index $\Psi(\Delta_n(M))$. This is equivalent to setting $\mathbf{c} = \mathbf{d} = 2$ in the expression $\phi_n + r \cdot \underline{\mathbf{d}}\phi_{n-2}\underline{\mathbf{d}}$. Hence this is the sum over all cyclic compositions into parts 1 and 2 where the part 1 is weighted by 2 and the part 2 is weighted by $2r$. Thus using $\varphi(t) = 2 \cdot t + 2r \cdot t^2$ and equation (2) we obtain

$$\begin{aligned} \sum_{n \geq 1} f_{\{1,2,\dots,n\}} t^n &= \frac{t \cdot \varphi'(t)}{1 - \varphi(t)} = \frac{2 \cdot t + 4r \cdot t^2}{(1-t)^2 - (2r+1) \cdot t^2} \\ &= \frac{1}{1 - (1 + \sqrt{2r+1})t} + \frac{1}{1 - (1 - \sqrt{2r+1})t} - 2, \end{aligned}$$

that is, for $n \geq 1$ we have

$$f_{\{1,2,\dots,n\}} = (1 + \sqrt{2r+1})^n + (1 - \sqrt{2r+1})^n.$$

Observe the flag f -vector entry $f_{\{1,2,\dots,n\}}$ is given by the trace of M^n . If the non-zero eigenvalues of M with multiplicities are λ_1 through λ_k then $f_{\{1,2,\dots,n\}} = \lambda_1^n + \dots + \lambda_k^n$. Hence we conclude that the matrix M only has two non-zero eigenvalues: $1 + \sqrt{2r+1}$ and $1 - \sqrt{2r+1}$. The same reasoning holds for the matrix N . $\square$

12. Concluding remarks

Our argument in Theorem 10.3 only shows that the cyclic **cd**-index has rational coefficients. Hence it is natural to ask the following question.

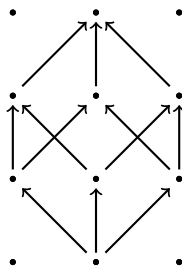
Question 12.1. *What is the smallest ring $R \subseteq \mathbb{Q}$ such that the cyclic **cd**-index of every Eulerian cyclic poset belongs to $R\langle \mathbf{c}, \mathbf{d} \rangle$?*

There has been a lot of work on inequalities among flag f -vectors of graded posets [6], Eulerian posets [2, 3] and polytopes [5, 8, 14]. Hence we state the following questions:

Question 12.2. *What is the set of inequalities that hold among the flag f -vector entries of the cyclic posets? What is the set of inequalities that hold between the coefficients of the cyclic **cd**-index of Eulerian cyclic posets?*

A first, but still hard, step would be to consider linear inequalities. Furthermore, the set of cyclic **cd**-indices of Eulerian cyclic sets has holes. By the first example in Section 11 we know that $\mathbf{c}^3$ and $\mathbf{c}^3 + \mathbf{dc} + \mathbf{cd} + \mathbf{dcd}$ arise from Eulerian cyclic posets. But their midpoint does not, as the next example illustrates.

Example 12.1. There is no rank 3 Eulerian cyclic poset with the cyclic **cd**-index $w = \mathbf{c}^3 + 1/2 \cdot (\mathbf{dc} + \mathbf{cd} + \mathbf{dcd})$. Begin by observing that $\Gamma_i(w) = 3\mathbf{c}^2 + \mathbf{d}$. Hence every rank has three elements. Next $\Gamma_0(w)$ is the sum of three **cd**-indices by Proposition 8.1. Hence they must be two $\mathbf{c}^2$ terms and one $\mathbf{c}^2 + \mathbf{d}$. Thus there is one element x of rank 0 such that the long interval $[x_{\hat{0}}, x_{\hat{1}}]$ is isomorphic to the Boolean algebra B_3 . The cyclic poset must have the following directed edges:



Note that there cannot be any more edges between ranks 1 and 2. However, for a different element y of rank 0 we have $\Psi([y_{\hat{0}}, y_{\hat{1}}]) = \mathbf{c}^2$, that is, the interval $[y_{\hat{0}}, y_{\hat{1}}]$ is isomorphic to the butterfly poset. In order to have that we need the bowtie configuration $\bowtie$ between ranks 1 and 2. Since this configuration does not appear in the Boolean algebra B_3 , we reach a contradiction.

We also are interested in the topology of the cyclic chain complex.

Question 12.3. *For which primitive 0,1-matrices M such that the associated poset is level Eulerian, is the cyclic chain complex $\Delta_n(M)$ is homeomorphic to an $(n-1)$ -sphere?*

For instance, for the first class of level Eulerian posets in Section 11 we know that the chain complex of each interval is homeomorphic to a sphere; see [9, Corollary 6.4]. This result is proved by presenting a vertex shelling order. Can such a shelling order be extended to prove special cases of Question 12.3? See also [10, Example 6.11] for a level Eulerian poset which has non-shellable intervals of rank 3 and hence the associated cyclic chain complex is not a sphere.

Finally, it is clearly necessary to find more examples of level Eulerian posets and Eulerian cyclic posets.

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