

Closed Formula for Kronecker Structure Constants of a Molecular Basis of Homogeneous Symmetric Functions

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ABSTRACT: In this paper, we give an explicit formula for the coefficients $b_{\alpha,\beta}^\lambda$ appearing in the species decomposition

$$\mathbf{C}_\alpha \times \mathbf{C}_\beta = \sum_{\lambda \vdash n} b_{\alpha,\beta}^\lambda \mathbf{C}_\lambda.$$

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1. Introduction

Let Λ^n denote the degree- n homogeneous component of the ring of symmetric functions. Given a partition $\alpha = (a_1, \dots, a_k) \vdash n$, let $\sigma_\alpha \in S_n$ be a permutation with cycles of lengths $a_1, \dots, a_k$. Informally, a $\mathbf{C}_\alpha$ -structure on an n -element set U is an arrangement of the elements of U into k words of lengths $a_1, \dots, a_k$, where two arrangements are identified if one is obtained from the other by cyclically rotating all the words simultaneously by the same amount. Formally, $\mathbf{C}_\alpha = X^n / \langle \sigma_\alpha \rangle$ is the *molecular species* (a species with a single isomorphism type) attached to the cyclic group $\langle \sigma_\alpha \rangle$; see Section 2. Its cycle-index series $Z_{\mathbf{C}_\alpha} = \frac{1}{o(\alpha)} \sum_{\tau \in \langle \sigma_\alpha \rangle} p_{\lambda(\tau)}$ is a symmetric function, and the series $\{Z_{\mathbf{C}_\lambda}\}_{\lambda \vdash n}$ studied in [3] form a basis of Λ^n . The family $\{\mathbf{C}_\lambda\}$ is closed under the Hadamard (Kronecker) product $\times$ of species:

$$\mathbf{C}_\alpha \times \mathbf{C}_\beta = \sum_{\lambda \vdash n} b_{\alpha,\beta}^\lambda \mathbf{C}_\lambda, \quad b_{\alpha,\beta}^\lambda \in \mathbb{N}.$$

By Mackey's theorem, $b_{\alpha,\beta}^\lambda$ is the number of isomorphism types of $\mathbf{C}_\alpha \times \mathbf{C}_\beta$ whose automorphism group is cyclic of type λ [3]. This extends to the basis $\{Z_{\mathbf{C}_\lambda}\}$ the analogous closure for $\{h_\lambda\}$, where

$$h_\lambda \star h_\mu = \sum_{\nu} c_{\lambda,\mu}^\nu h_\nu,$$

and $c_{\lambda,\mu}^\nu$ is the number of non-negative integer matrices with row sums λ , column sums μ , and entries forming the partition ν . This interpretation goes back to Redfield [10, p. 448]; see also Garsia–Remmel [7, 241] and its species-theoretic derivation in [3].

The corresponding question for the Schur basis, finding a positive combinatorial rule for the Kronecker coefficients $g_{\lambda\mu}^\nu$ in $s_\lambda \star s_\mu = \sum_{\nu} g_{\lambda\mu}^\nu s_\nu$, has been open at least since Murnaghan [13], and according to Richard Stanley (see [17]) it remains one of the most important open problems in algebraic combinatorics. For $b_{\alpha,\beta}^\lambda$, no explicit formula was previously known; finding one was left open as [3, Question 5.1].

This paper answers that question with an explicit formula. In Section 3, we exploit the fact that for every divisor d of $o(\alpha)$ the cyclic group $\langle \sigma_\alpha \rangle$ has a unique subgroup of order d , and that subgroup inclusion corresponds to divisibility. This reduces the computation of $b_{\alpha,\beta}^\lambda$ to a Möbius inversion over the divisors of $\gcd(o(\alpha), o(\beta))$. Note that the formula involves signs, so it is an explicit expression rather than a positive combinatorial interpretation; the latter remains open.

*Anthropic's Claude was used for mathematical exploration and language proofreading. All mathematical results were independently verified by the authors, who take full responsibility for the content of this paper.

2. Background and notation

A *combinatorial species* is a functor

$$F: \mathcal{B} \longrightarrow \text{Set},$$

where $\mathcal{B}$ denotes the category of finite sets and bijections. Thus, to every finite set U , a species F associates a set $F[U]$ of F -structures on U , and to every bijection $f: U \rightarrow V$ it associates a transport map

$$F[f]: F[U] \longrightarrow F[V].$$

These maps satisfy the usual functoriality conditions.

Given a subgroup $H \leq S_n$, the associated *molecular species* X^n/H is defined by

$$(X^n/H)[U] = \{\lambda H \mid \lambda: [n] \xrightarrow{\sim} U\},$$

for $|U| = n$, with transport of structure

$$(X^n/H)[f](\lambda H) = (f \circ \lambda)H.$$

The species X^n/H has a single isomorphism type, hence the terminology “molecular species.”

Its cycle-index series is

$$Z_{X^n/H} = \frac{1}{|H|} \sum_{\tau \in H} p_{\lambda(\tau)},$$

where $\lambda(\tau)$ denotes the cycle type of τ , and $p_{\lambda(\tau)}$ is the corresponding power-sum symmetric function. We refer the reader to Bergeron–Labelle–Leroux [4] and Joyal [9] for the general theory of combinatorial species.

Two molecular species X^n/H and X^n/K are isomorphic if and only if the subgroups H and K are conjugate in S_n . In particular, the isomorphism class of a molecular species is completely determined by the conjugacy class of its stabilizer.

For a partition

$$\alpha = (a_1, \dots, a_k) \vdash n,$$

we define the *standard permutation* of shape α by

$$\sigma_\alpha = (1 \cdots a_1)(a_1 + 1 \cdots a_1 + a_2) \cdots \in S_n.$$

Its order is denoted by

$$o(\alpha) := o(\sigma_\alpha) = \text{lcm}(a_1, \dots, a_k).$$

We then define the cyclic molecular species

$$\mathbf{C}_\alpha := X^n / \langle \sigma_\alpha \rangle.$$

By [3, Lemma 2.1], any permutation of cycle type α gives an isomorphic species. Hence $\mathbf{C}_\alpha$ depends only on the partition α and not on the particular choice of the permutation σ_α .

Recall also the classical construction associated with Young subgroups. If

$$\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$$

and

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k},$$

then

$$X^n/S_\lambda = \mathbf{E}_{\lambda_1} \mathbf{E}_{\lambda_2} \cdots \mathbf{E}_{\lambda_k} = \mathbf{E}_\lambda.$$

Its cycle-index series is the complete homogeneous symmetric function

$$Z_{\mathbf{E}_\lambda} = h_\lambda.$$

Equivalently, this cycle index is the Frobenius characteristic of the permutation representation

$$\text{Ind}_{S_\lambda}^{S_n} \mathbf{1}.$$

The family $\{\mathbf{C}_\lambda\}_{\lambda \vdash n}$ is the cyclic-subgroup analogue of the family $\{\mathbf{E}_\lambda\}_{\lambda \vdash n}$: the Young subgroup S_λ is replaced by the cyclic subgroup $\langle \sigma_\lambda \rangle \leq S_\lambda$. We write

$$\mathbf{C}_\alpha(\mathbf{x}) := Z_{\mathbf{C}_\alpha}.$$

Theorem 2.1 ([3]). *The set*

$$\{\mathbf{C}_\lambda(\mathbf{x})\}_{\lambda \vdash n}$$

forms a basis of Λ^n .

We now recall the species-theoretic description of the Hadamard product. This description will be the starting point for the enumeration developed in Section 3.

The *Hadamard product* of two species F and G is the species defined by

$$(F \times G)[U] = F[U] \times G[U],$$

with transport maps defined componentwise:

$$(F \times G)[f](s, t) = (F[f](s), G[f](t)).$$

Its cycle-index series satisfies

$$Z_{F \times G} = Z_F \star Z_G,$$

where $\star$ denotes the Kronecker product of symmetric functions [4].

For molecular species, the Hadamard product admits a standard decomposition in terms of double cosets; see Bergeron–Labelle–Leroux [4].

Proposition 2.1 (Molecular double-coset decomposition [4]). *Let $H, K \leq S_n$. Then*

$$(X^n/H) \times (X^n/K) \cong \sum_{\pi \in H \backslash S_n / K} X^n / (H \cap \pi K \pi^{-1}).$$

Passing to cycle-index series gives

$$Z_{X^n/H} \star Z_{X^n/K} = \sum_{\pi \in H \backslash S_n / K} Z_{X^n / (H \cap \pi K \pi^{-1})}.$$

Thus the decomposition of the Kronecker product of two molecular species is governed by the intersections

$$H \cap \pi K \pi^{-1}, \quad \pi \in H \backslash S_n / K.$$

We now specialize this standard decomposition to the cyclic molecular species $\mathbf{C}_\alpha$ and $\mathbf{C}_\beta$.

Lemma 2.1. *Let H_1 and H_2 be cyclic subgroups of S_n . Then $H_1 \cap H_2$ is cyclic.*

Proof. The intersection $H_1 \cap H_2$ is a subgroup of the cyclic group H_1 . Since every subgroup of a cyclic group is cyclic, the result follows. $\square$

Consequently, for every $\pi \in S_n$,

$$H_\pi := \langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1}$$

is cyclic. Let τ_π be a generator of H_π , and let $\lambda(\tau_\pi)$ denote its cycle type. All generators of the same cyclic subgroup have the same cycle type, so this partition is independent of the choice of the generator.

If

$$\lambda(\tau_\pi) = \lambda,$$

then H_π is conjugate to $\langle \sigma_\lambda \rangle$, and therefore

$$X^n / H_\pi \cong X^n / \langle \sigma_\lambda \rangle = \mathbf{C}_\lambda.$$

Applying Proposition 2.1 with

$$H = \langle \sigma_\alpha \rangle, \quad K = \langle \sigma_\beta \rangle,$$

gives

$$\mathbf{C}_\alpha \times \mathbf{C}_\beta \cong \sum_{\pi \in \langle \sigma_\alpha \rangle \backslash S_n / \langle \sigma_\beta \rangle} X^n / (\langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1}).$$

Grouping together the double cosets for which the intersection subgroup has the same generator cycle type gives the following decomposition.

Proposition 2.2. *Let $\alpha, \beta \vdash n$. Then*

$$\mathbf{C}_\alpha \times \mathbf{C}_\beta = \sum_{\lambda \vdash n} b_{\alpha, \beta}^\lambda \mathbf{C}_\lambda,$$

where

$$b_{\alpha, \beta}^\lambda = \# \left\{ \langle \sigma_\alpha \rangle \pi \langle \sigma_\beta \rangle \in \langle \sigma_\alpha \rangle \backslash S_n / \langle \sigma_\beta \rangle : \langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1} \text{ is conjugate to } \langle \sigma_\lambda \rangle \right\}.$$

Proof. By Proposition 2.1,

$$\mathbf{C}_\alpha \times \mathbf{C}_\beta = \sum_{\pi \in \langle \sigma_\alpha \rangle \backslash S_n / \langle \sigma_\beta \rangle} X^n / H_\pi,$$

where

$$H_\pi = \langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1}.$$

By Lemma 2.1, every H_π is cyclic. Hence H_π is conjugate to $\langle \sigma_\lambda \rangle$ for a unique generator cycle type λ . The corresponding molecular species is $\mathbf{C}_\lambda$.

Grouping all double cosets producing the same molecular species $\mathbf{C}_\lambda$ gives the stated coefficient $b_{\alpha,\beta}^\lambda$. $\square$

Proposition 2.2 gives an algebraic double-coset description of the coefficients $b_{\alpha,\beta}^\lambda$. We next recall Mackey's theorem in order to give these coefficients their species-theoretic interpretation as numbers of isomorphism types.

Lemma 2.2 (Mackey's theorem for group actions). *Let G be a group acting transitively on two sets X and Y . Fix $x \in X$ and $y \in Y$. Then there is a natural bijection*

$$G \backslash (X \times Y) \longrightarrow \text{Aut}(x) \backslash G / \text{Aut}(y),$$

given by

$$G \cdot (x, g \cdot y) \longmapsto \text{Aut}(x) g \text{Aut}(y).$$

Here G acts diagonally on $X \times Y$.

This theorem is especially natural for molecular species, since the automorphism group of a molecular structure is conjugate to the subgroup defining the molecule.

Lemma 2.3. *Let $H \leq S_n$, and let*

$$\lambda H \in (X^n / H)[U].$$

Then

$$\text{Aut}(\lambda H) = \lambda H \lambda^{-1}.$$

In particular,

$$\text{Aut}(\lambda H) \simeq H.$$

Proof. A bijection $f: U \rightarrow U$ belongs to $\text{Aut}(\lambda H)$ if and only if

$$(f \circ \lambda)H = \lambda H.$$

This is equivalent to

$$\lambda^{-1} f \lambda \in H,$$

or

$$f \in \lambda H \lambda^{-1}.$$

Hence

$$\text{Aut}(\lambda H) = \lambda H \lambda^{-1}.$$

$\square$

In particular, for

$$x \in \mathbf{C}_\alpha[n], \quad y \in \mathbf{C}_\beta[n],$$

we have

$$\text{Aut}(x) \simeq \langle \sigma_\alpha \rangle, \quad \text{Aut}(y) \simeq \langle \sigma_\beta \rangle.$$

Applying Lemma 2.2 with

$$G = S_n, \quad X = \mathbf{C}_\alpha[n], \quad Y = \mathbf{C}_\beta[n],$$

therefore gives a bijection

$$S_n \backslash (\mathbf{C}_\alpha[n] \times \mathbf{C}_\beta[n]) \longrightarrow \langle \sigma_\alpha \rangle \backslash S_n / \langle \sigma_\beta \rangle.$$

The left-hand side is precisely the set of isomorphism types, or unlabeled structures, of the species $\mathbf{C}_\alpha \times \mathbf{C}_\beta$.

More precisely, if the double coset

$$\langle \sigma_\alpha \rangle \pi \langle \sigma_\beta \rangle$$

corresponds to an isomorphism type of $\mathbf{C}_\alpha \times \mathbf{C}_\beta$, then the automorphism group of a representative of that isomorphism type is conjugate to

$$\langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1}.$$

This allows us to refine the Hadamard product according to the automorphism type.
For $\lambda \vdash n$, define

$$\mathbf{C}_{\alpha,\beta}^\lambda[U] := \left\{ (s, t) \in (\mathbf{C}_\alpha \times \mathbf{C}_\beta)[U] : \text{Aut}(s) \cap \text{Aut}(t) \simeq \langle \sigma_\lambda \rangle \right\}.$$

The transport of structure preserves automorphism groups up to conjugacy, so $\mathbf{C}_{\alpha,\beta}^\lambda$ is a subspecies of $\mathbf{C}_\alpha \times \mathbf{C}_\beta$.

We can now state the combinatorial meaning of the coefficient $b_{\alpha,\beta}^\lambda$.

Proposition 2.3. *The coefficient $b_{\alpha,\beta}^\lambda$ is equal to the number of isomorphism types, or equivalently the number of unlabeled structures, of the subspecies $\mathbf{C}_{\alpha,\beta}^\lambda$. Thus*

$$b_{\alpha,\beta}^\lambda = |S_n \backslash \mathbf{C}_{\alpha,\beta}^\lambda[n]|.$$

Proof. By Mackey's theorem,

$$S_n \backslash (\mathbf{C}_\alpha[n] \times \mathbf{C}_\beta[n]) \cong \langle \sigma_\alpha \rangle \backslash S_n / \langle \sigma_\beta \rangle.$$

Thus the isomorphism types of $\mathbf{C}_\alpha \times \mathbf{C}_\beta$ are naturally indexed by the double cosets

$$\langle \sigma_\alpha \rangle \pi \langle \sigma_\beta \rangle.$$

For the isomorphism type represented by such a double coset, the corresponding automorphism intersection is conjugate to

$$\langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1}.$$

Therefore, the isomorphism types belonging to the subspecies $\mathbf{C}_{\alpha,\beta}^\lambda$ correspond precisely to the double cosets satisfying

$$\langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1} \sim \langle \sigma_\lambda \rangle,$$

where $\sim$ denotes conjugacy in S_n .

By Proposition 2.2, the number of these double cosets is exactly $b_{\alpha,\beta}^\lambda$. Hence $b_{\alpha,\beta}^\lambda$ is the number of isomorphism types of $\mathbf{C}_{\alpha,\beta}^\lambda$. $\square$

Thus the coefficient $b_{\alpha,\beta}^\lambda$ has two equivalent interpretations:

$$b_{\alpha,\beta}^\lambda = \#\{\text{double cosets with cyclic intersection type } \lambda\},$$

and, by Mackey's theorem,

$$b_{\alpha,\beta}^\lambda = \#\{\text{isomorphism types of } \mathbf{C}_\alpha \times \mathbf{C}_\beta \text{ of automorphism type } \lambda\}.$$

Passing to cycle-index series gives

$$\mathbf{C}_\alpha(\mathbf{x}) \star \mathbf{C}_\beta(\mathbf{x}) = \sum_{\lambda \vdash n} b_{\alpha,\beta}^\lambda \mathbf{C}_\lambda(\mathbf{x}).$$

Hence the integers $b_{\alpha,\beta}^\lambda$ are the structure constants of the basis $\{\mathbf{C}_\lambda(\mathbf{x})\}_{\lambda \vdash n}$ under the Kronecker product, and simultaneously count the isomorphism types of $\mathbf{C}_\alpha \times \mathbf{C}_\beta$ having cyclic automorphism type λ .

The double-coset description above determines the coefficients $b_{\alpha,\beta}^\lambda$ conceptually, but does not by itself give a closed enumeration. The purpose of Section 3 is to exploit the special structure of the cyclic subgroups $\langle \sigma_\alpha \rangle$ and $\langle \sigma_\beta \rangle$ in order to classify the possible stabilizer intersections and derive an explicit formula for $b_{\alpha,\beta}^\lambda$ by Möbius inversion.

3. A Möbius-inversion formula for $b_{\alpha,\beta}^\lambda$

Double coset enumeration has long been studied, especially in the context of Coxeter groups and Hecke algebras, where one counts parabolic double cosets $W_I \backslash W / W_J$ (see, e.g., [2]), and has rich applications in algebraic combinatorics (see, e.g., [1, 6, 11]). Very recently, Schwob [16] gave general formulas for the double cosets $H \backslash G / H$ and their self-inverse members. Our setting differs from both: we count double cosets $H \backslash S_n / K$ for two *different* cyclic subgroups $H = \langle \sigma_\alpha \rangle$ and $K = \langle \sigma_\beta \rangle$. Since the associated double-translation group is abelian (Lemma 3.1) and the subgroup lattice of a cyclic group is isomorphic to the divisor lattice, the enumeration reduces to a single Möbius inversion over divisors.

Throughout, fix $\alpha, \beta \vdash n$ and write

$$D := \gcd(o(\alpha), o(\beta)).$$

3.1 The double translation action and its stabilizers

The group $G := \langle \sigma_\alpha \rangle \times \langle \sigma_\beta \rangle$ acts on S_n by

$$(\sigma_\alpha^x, \sigma_\beta^y) \cdot \pi := \sigma_\alpha^x \pi \sigma_\beta^{-y}, \quad x \in \mathbb{Z}_{o(\alpha)}, \quad y \in \mathbb{Z}_{o(\beta)},$$

and the orbits of this action are exactly the double cosets $\langle \sigma_\alpha \rangle \pi \langle \sigma_\beta \rangle$. The stabilizer of π is

$$\text{Stab}(\pi) \cong H_\pi := \langle \sigma_\alpha \rangle \cap \pi \langle \sigma_\beta \rangle \pi^{-1},$$

via the projection $(\sigma_\alpha^x, \sigma_\beta^y) \mapsto \sigma_\alpha^x$, and $b_{\alpha,\beta}^\lambda$ counts the orbits for which a generator of H_π (a cyclic group) has cycle type λ .

Lemma 3.1 (Stabilizers are orbit-invariant). *Since G is abelian, H_π depends only on the double coset of π : if $\pi' = (x, y) \cdot \pi$, then $H_{\pi'} = H_\pi$. Consequently every double coset C has a well-defined stabilizer order $d_C := |H_\pi|$ for any $\pi \in C$, and*

$$|C| = \frac{o(\alpha)o(\beta)}{d_C}.$$

Proof. For $g \in G$, $\text{Stab}(g \cdot \pi) = g \text{Stab}(\pi) g^{-1}$. Since G is abelian, conjugation by g is trivial, so $\text{Stab}(g \cdot \pi) = \text{Stab}(\pi)$. The orbit-size formula is then the orbit-stabilizer theorem applied to the (constant) stabilizer order d_C . $\square$

3.2 Cyclic subgroups force the stabilizer type

Lemma 3.2 (Unique subgroup per order, with forced cycle type). *For each divisor $d \mid o(\alpha)$, the cyclic group $\langle \sigma_\alpha \rangle$ has a unique subgroup of order d , namely $\langle \sigma_\alpha^{o(\alpha)/d} \rangle$. Writing $\alpha = (a_1, \dots, a_k)$, every generator of this subgroup has the same cycle type*

$$\lambda_d(\alpha) := \bigsqcup_{i=1}^k \left(\underbrace{\frac{a_i}{\gcd(a_i, o(\alpha)/d)}, \dots, \frac{a_i}{\gcd(a_i, o(\alpha)/d)}}_{\gcd(a_i, o(\alpha)/d) \text{ parts}} \right) \vdash n.$$

Proof. Uniqueness of the order- d subgroup of a cyclic group is standard. Its generators are $\sigma_\alpha^{o(\alpha)/d \cdot e}$ for e coprime to d ; raising a single cycle of length $\ell \mid o(\alpha)$ to a power coprime to d (hence coprime to any divisor of ℓ dividing d , and in particular to $\gcd(\ell, d)$, acting on the relevant orbit) preserves the cycle-length partition of that cycle, since raising an ℓ -cycle to a power coprime to ℓ yields a single ℓ -cycle. Applying $\sigma_\alpha^{o(\alpha)/d}$ to the i -th cycle (length a_i) splits it into $\gcd(a_i, o(\alpha)/d)$ cycles of length $a_i / \gcd(a_i, o(\alpha)/d)$, and this partition is unchanged by any further coprime power. $\square$

Example 3.1 (Illustrating $\lambda_d(\alpha)$). Let $\alpha = (4, 2) \vdash 6$, so $o(\alpha) = \text{lcm}(4, 2) = 4$. For each divisor $d \mid 4$, split each part a_i into $\gcd(a_i, o(\alpha)/d)$ equal pieces of size $a_i / \gcd(a_i, o(\alpha)/d)$:

d	$o(\alpha)/d$	split of 4	split of 2
4	1	4	2
2	2	2, 2	1, 1
1	4	1, 1, 1, 1	1, 1

so $\lambda_4(\alpha) = (4, 2)$, $\lambda_2(\alpha) = (2, 2, 1, 1)$, and $\lambda_1(\alpha) = (1^6)$. Concretely, $\lambda_2(\alpha)$ is the cycle type of σ_α^2 : squaring a 4-cycle gives two 2-cycles, and squaring a 2-cycle gives two fixed points.

Consequently, if H_π has order d , it is forced to equal both $\langle \sigma_\alpha^{o(\alpha)/d} \rangle$ (cycle type $\lambda_d(\alpha)$) and $\pi \langle \sigma_\beta^{o(\beta)/d} \rangle \pi^{-1}$ (cycle type $\lambda_d(\beta)$, since conjugation preserves cycle type). Hence a double coset with stabilizer order d can exist only if

$$d \mid D = \gcd(o(\alpha), o(\beta)) \quad \text{and} \quad \lambda_d(\alpha) = \lambda_d(\beta),$$

in which case its stabilizer type is exactly $\lambda = \lambda_d(\alpha) = \lambda_d(\beta)$. Since the order of a permutation of cycle type λ is the lcm of its parts, distinct divisors d give distinct partitions λ ; thus d is recovered from λ as $d = \text{lcm}(\lambda)$, and the correspondence $d \leftrightarrow \lambda$ is a bijection onto its image.

3.3 The counting formula

For $d \mid D$, define

$$F(d) := \#\{\pi \in S_n : \sigma_\alpha^{o(\alpha)/d} \in \pi\langle\sigma_\beta\rangle\pi^{-1}\} = \#\{\pi \in S_n : d \mid d_{\langle\pi\rangle}\},$$

where $d_{\langle\pi\rangle}$ denotes the stabilizer order of the double coset containing π . The equality holds because $\langle\sigma_\alpha\rangle$ has a unique subgroup of each order dividing $o(\alpha)$, and the order- d subgroup is contained in the order- e subgroup exactly when $d \mid e$.

Proposition 3.1.

$$F(d) = \varphi(d) \cdot z_{\lambda_d(\alpha)} \cdot \delta_{\lambda_d(\alpha), \lambda_d(\beta)},$$

where φ is Euler's totient function and, for a partition λ with m_i parts equal to i , $z_\lambda := \prod_i i^{m_i} m_i!$ is the order of the centralizer in S_n of a permutation of cycle type λ [8, §1.2] and $\delta_{a,b}$ is the Kronecker delta symbol.

Proof. By Lemma 3.2, $\sigma_\alpha^{o(\alpha)/d}$ lies in $\pi\langle\sigma_\beta\rangle\pi^{-1}$ if and only if $\pi^{-1}\sigma_\alpha^{o(\alpha)/d}\pi$ is one of the $\varphi(d)$ generators of the unique order- d subgroup $\langle\sigma_\beta^{o(\beta)/d}\rangle$, all of which share cycle type $\lambda_d(\beta)$. If $\lambda_d(\alpha) \neq \lambda_d(\beta)$, no such conjugation is possible and $F(d) = 0$. Otherwise, for each fixed generator y , the number of π with $\pi^{-1}\sigma_\alpha^{o(\alpha)/d}\pi = y$ equals the centralizer order $z_{\lambda_d(\alpha)}$, and there are $\varphi(d)$ choices of y . $\square$

Theorem 3.1 (Closed-form formula for $b_{\alpha,\beta}^\lambda$). *Let $\lambda \vdash n$ with $d := \text{lcm}(\lambda) \mid D$ and $\lambda_d(\alpha) = \lambda_d(\beta) = \lambda$. Then*

$$b_{\alpha,\beta}^\lambda = \frac{d}{o(\alpha)o(\beta)} \sum_{\substack{d \mid d' \\ d' \mid D}} \mu\left(\frac{d'}{d}\right) F(d')$$

where μ is the classical (number-theoretic) Möbius function and F is as in Proposition 3.1. If no such d exists, $b_{\alpha,\beta}^\lambda = 0$.

Proof. By Lemma 3.1, each double coset C has a well-defined stabilizer order $d_C \mid D$, and $|C| = o(\alpha)o(\beta)/d_C$. Set $N_d := b_{\alpha,\beta}^\lambda$ for $\lambda = \lambda_d(\alpha) = \lambda_d(\beta)$, i.e. the number of double cosets with $d_C = d$ (and $N_d := 0$ if $\lambda_d(\alpha) \neq \lambda_d(\beta)$).

We first check that $F(d)$ literally counts the elements π whose double coset has stabilizer order divisible by d . By definition $H_\pi = \langle\sigma_\alpha\rangle \cap \pi\langle\sigma_\beta\rangle\pi^{-1}$, so the condition $\sigma_\alpha^{o(\alpha)/d} \in \pi\langle\sigma_\beta\rangle\pi^{-1}$ defining $F(d)$ says exactly that $\sigma_\alpha^{o(\alpha)/d} \in H_\pi$. Since $H_\pi \leq \langle\sigma_\alpha\rangle$ is cyclic and $\sigma_\alpha^{o(\alpha)/d}$ generates the unique order- d subgroup of $\langle\sigma_\alpha\rangle$ (Lemma 3.2), and a cyclic group has a unique subgroup of each order dividing its order, with the order- d subgroup contained in the order- e subgroup exactly when $d \mid e$. Therefore, $\sigma_\alpha^{o(\alpha)/d}$ lies in H_π if and only if H_π contains that order- d subgroup, i.e. if and only if $d \mid |H_\pi| = d_{\langle\pi\rangle}$. Hence

$$F(d) = \#\{\pi \in S_n : d \mid d_{\langle\pi\rangle}\} = \sum_{\pi : d \mid d_{\langle\pi\rangle}} 1.$$

Now partition this index set according to the double coset $C(\pi) := \langle\sigma_\alpha\rangle\pi\langle\sigma_\beta\rangle$ containing π . By Lemma 3.1, the value $d_{\langle\pi\rangle}$ depends only on $C(\pi)$, not on the chosen representative π ; write d_C for this common value. So the condition " $d \mid d_{\langle\pi\rangle}$ " is constant on each double coset — it either holds for every element of C or for none — and therefore selects some double cosets *in their entirety* and excludes the rest completely:

$$\{\pi \in S_n : d \mid d_{\langle\pi\rangle}\} = \bigsqcup_{\substack{C \subseteq S_n \text{ a double coset} \\ d \mid d_C}} C,$$

a disjoint union because distinct double cosets are disjoint. Summing $|C| = o(\alpha)o(\beta)/d_C$ over this union gives

$$F(d) = \sum_{C : d \mid d_C} |C| = \sum_{C : d \mid d_C} \frac{o(\alpha)o(\beta)}{d_C}.$$

Finally, regroup the double cosets on the right by the *exact* value $d' := d_C$ of their stabilizer order. Since d_C always divides D (Lemma 3.1), d' ranges over divisors of D with $d \mid d'$, and by definition of $N_{d'}$ there are exactly $N_{d'}$ double cosets with $d_C = d'$, each contributing $o(\alpha)o(\beta)/d'$ to the sum. This yields

$$F(d) = \sum_{\substack{d \mid d' \\ d' \mid D}} N_{d'} \cdot \frac{o(\alpha)o(\beta)}{d'}.$$

This is a divisor-sum relation on the poset of divisors of D between d and D ; Möbius inversion on this poset gives

$$N_d \cdot \frac{o(\alpha)o(\beta)}{d} = \sum_{d|d'|D} \mu\left(\frac{d'}{d}\right) F(d'),$$

which rearranges to the stated formula. $\square$

Corollary 3.1 (Total count and the classical case). *Summing over all valid $d \mid D$,*

$$\sum_{\lambda \vdash n} b_{\alpha,\beta}^\lambda = |\langle \sigma_\alpha \rangle \backslash S_n / \langle \sigma_\beta \rangle|.$$

For $\alpha = \beta = (n)$ (so $o(\alpha) = o(\beta) = D = n$), $\lambda_d(\alpha) = \lambda_d(\beta)$ automatically for every $d \mid n$, and Theorem 3.1 reduces to a Möbius-inversion count of classical Steggall patterns [5, 18], i.e. of permutation matrices up to simultaneous cyclic shift of rows and columns. In this case

$$\sum_{\lambda \vdash n} b_{(n),(n)}^\lambda = |\langle \sigma_{(n)} \rangle \backslash S_n / \langle \sigma_{(n)} \rangle| = \frac{1}{n^2} \sum_{d|n} \varphi(d)^2 \left(\frac{n}{d}\right)! d^{n/d},$$

which is exactly sequence A002619 in the OEIS [14], 1, 1, 2, 3, 8, 24, 108, 640, 4492, 36336, ... for $n = 1, 2, 3, \dots$

3.4 Worked examples

We first illustrate Theorem 3.1 on the smallest case with a non-trivial divisor lattice, namely $\alpha = \beta = (4, 2) \vdash 6$. A second example below treats $\alpha \neq \beta$. All numerical computations in this section – including the values of $F(d)$, the resulting coefficients $b_{\alpha,\beta}^\lambda$, the consistency checks against $|S_n|$, and the OEIS sequence identified in Corollary 3.1 – were carried out and independently verified using SageMath [15], as were the analogous computations in the predecessor paper [3].

Example 3.2 ($\alpha = \beta = (4, 2)$). Here $o(\alpha) = o(\beta) = \text{lcm}(4, 2) = 4$ and $D = \text{gcd}(o(\alpha), o(\beta)) = 4$, so the divisors to consider are $d \in \{1, 2, 4\}$.

Step 1: forced cycle types. By Lemma 3.2, for each $d \mid 4$ we compute $\lambda_d(\alpha)$ by splitting the i -th part a_i into $\text{gcd}(a_i, o(\alpha)/d)$ parts of size $a_i / \text{gcd}(a_i, o(\alpha)/d)$:

d	$o(\alpha)/d$	split of 4	split of 2
4	1	4	2
2	2	2, 2	1, 1
1	4	1, 1, 1, 1	1, 1

so $\lambda_4(\alpha) = (4, 2)$, $\lambda_2(\alpha) = (2, 2, 1, 1)$, and $\lambda_1(\alpha) = (1^6)$. Since $\beta = \alpha$, we have $\lambda_d(\beta) = \lambda_d(\alpha)$ for every d , so all three divisors contribute (no vanishing from a cycle-type mismatch).

Step 2: the values $F(d)$. By Proposition 3.1, $F(d) = \varphi(d) z_{\lambda_d(\alpha)}$, using $z_\lambda = \prod_i i^{m_i} m_i!$:

d	$\lambda_d(\alpha)$	$\varphi(d)$	$z_{\lambda_d(\alpha)}$	$F(d)$
4	(4, 2)	2	$4 \cdot 2 = 8$	16
2	(2, 2, 1, 1)	1	$(2^2 \cdot 2!)(1^2 \cdot 2!) = 16$	16
1	(1 ⁶)	1	$6! = 720$	720

Step 3: Möbius inversion. With $o(\alpha)o(\beta) = 16$, Theorem 3.1 gives $b_{\alpha,\beta}^{\lambda_d(\alpha)} = \frac{d}{16} \sum_{d|d'|4} \mu(d'/d) F(d')$:

$$\begin{aligned} b_{\alpha,\beta}^{(4,2)} &= \frac{4}{16} [\mu(1)F(4)] = \frac{4}{16}(16) = 4, \\ b_{\alpha,\beta}^{(2,2,1,1)} &= \frac{2}{16} [\mu(1)F(2) + \mu(2)F(4)] = \frac{2}{16}(16 - 16) = 0, \\ b_{\alpha,\beta}^{(1^6)} &= \frac{1}{16} [\mu(1)F(1) + \mu(2)F(2) + \mu(4)F(4)] = \frac{1}{16}(720 - 16 + 0) = 44. \end{aligned}$$

All other $b_{\alpha,\beta}^\lambda$ vanish, since only $d = 1, 2, 4$ can occur as a stabilizer order. Thus

$$\mathbf{C}_{(4,2)} \times \mathbf{C}_{(4,2)} = 4 \mathbf{C}_{(4,2)} + 44 \mathbf{C}_{(1,1,1,1,1,1)},$$

the class $\mathbf{C}_{(2,2,1,1)}$ occurring with coefficient 0.

Consistency check. By Lemma 3.1, a double coset with stabilizer order d has size $o(\alpha)o(\beta)/d = 16/d$, so the double cosets must partition S_6 :

$$b_{\alpha,\beta}^{(4,2)} \cdot \frac{16}{4} + b_{\alpha,\beta}^{(2,2,1,1)} \cdot \frac{16}{2} + b_{\alpha,\beta}^{(1^6)} \cdot \frac{16}{1} = 4 \cdot 4 + 0 \cdot 8 + 44 \cdot 16 = 16 + 0 + 704 = 720 = 6!,$$

confirming the computation and, via Corollary 3.1, that $|\langle \sigma_{(4,2)} \rangle \backslash S_n / \langle \sigma_{(4,2)} \rangle| = 4 + 0 + 44 = 48$.

Our second example takes $\alpha \neq \beta$, with $o(\alpha) \neq o(\beta)$ and D strictly smaller than both, so that the divisor lattice genuinely reflects an interaction between two different shapes.

Example 3.3 ($\alpha = (4, 2)$, $\beta = (2, 2, 1, 1)$). Here $o(\alpha) = \text{lcm}(4, 2) = 4$ while $o(\beta) = \text{lcm}(2, 2, 1, 1) = 2$, so $D = \text{gcd}(o(\alpha), o(\beta)) = 2$: only the divisors $d \in \{1, 2\}$ of D can possibly occur, even though $o(\alpha)$ itself has the further divisor 4. This already illustrates the vanishing clause of Theorem 3.1: no λ with $\text{lcm}(\lambda) = 4$ can appear in $\mathbf{C}_{(4,2)} \times \mathbf{C}_{(2,2,1,1)}$, because $4 \nmid D$.

Step 1: forced cycle types. For $\alpha = (4, 2)$ (so $o(\alpha) = 4$) the same computation as in Example 3.2 gives $\lambda_2(\alpha) = (2, 2, 1, 1)$ and $\lambda_1(\alpha) = (1^6)$. For $\beta = (2, 2, 1, 1)$ (so $o(\beta) = 2$) we split each part b_i into $\text{gcd}(b_i, o(\beta)/d)$ pieces of size $b_i / \text{gcd}(b_i, o(\beta)/d)$:

d	$o(\beta)/d$	split of 2	split of 2	split of 1	split of 1
2	1	2	2	1	1
1	2	1, 1	1, 1	1	1

so $\lambda_2(\beta) = (2, 2, 1, 1)$ and $\lambda_1(\beta) = (1^6)$. In both cases $\lambda_d(\alpha) = \lambda_d(\beta)$, so — unlike a generic pair of shapes — neither admissible divisor is killed by a cycle-type mismatch here; we address the mismatched case in Remark 3.1 below.

Step 2: the values $F(d)$. As before $F(d) = \varphi(d) z_{\lambda_d(\alpha)}$:

d	$\lambda_d(\alpha) = \lambda_d(\beta)$	$\varphi(d)$	$z_{\lambda_d(\alpha)}$	$F(d)$
2	$(2, 2, 1, 1)$	1	$(2^2 \cdot 2!)(1^2 \cdot 2!) = 16$	16
1	(1^6)	1	$6! = 720$	720

Step 3: Möbius inversion. With $o(\alpha)o(\beta) = 4 \cdot 2 = 8$, Theorem 3.1 gives

$$b_{\alpha,\beta}^{(2,2,1,1)} = \frac{2}{8} [\mu(1)F(2)] = \frac{2}{8}(16) = 4,$$

$$b_{\alpha,\beta}^{(1^6)} = \frac{1}{8} [\mu(1)F(1) + \mu(2)F(2)] = \frac{1}{8}(720 - 16) = 88.$$

Every other $b_{\alpha,\beta}^\lambda$ vanishes, so

$$\mathbf{C}_{(4,2)} \times \mathbf{C}_{(2,2,1,1)} = 4 \mathbf{C}_{(2,2,1,1)} + 88 \mathbf{C}_{(1,1,1,1,1,1)}.$$

Consistency check. As in Example 3.2, $b_{\alpha,\beta}^{(2,2,1,1)} \cdot \frac{8}{2} + b_{\alpha,\beta}^{(1^6)} \cdot \frac{8}{1} = 4 \cdot 4 + 88 \cdot 8 = 16 + 704 = 720 = 6!$, and Corollary 3.1 gives $|\langle \sigma_{(4,2)} \rangle \backslash S_n / \langle \sigma_{(2,2,1,1)} \rangle| = 4 + 88 = 92$.

Remark 3.1 (When cycle types genuinely clash). The condition $\lambda_d(\alpha) = \lambda_d(\beta)$ is genuinely restrictive. For example, take

$$\alpha = (4), \quad \beta = (2, 1, 1) \vdash 4.$$

Then $o(\alpha) = 4$, $o(\beta) = 2$, and $D = 2$. For the non-trivial common divisor $d = 2$, we have

$$\lambda_2(\alpha) = (2, 2), \quad \lambda_2(\beta) = (2, 1, 1).$$

Hence

$$\lambda_2(\alpha) \neq \lambda_2(\beta),$$

and Proposition 3.1 gives

$$F(2) = 0.$$

Therefore no double coset can have stabilizer of order 2.

For $d = 1$, both forced cycle types are (1^4) , so

$$F(1) = 4! = 24.$$

Since $o(\alpha)o(\beta) = 8$, Theorem 3.1 gives

$$b_{(4),(2,1,1)}^{(1^4)} = \frac{1}{8}(F(1) - F(2)) = 3.$$

Thus

$$\mathbf{C}_{(4)} \times \mathbf{C}_{(2,1,1)} = 3 \mathbf{C}_{(1,1,1,1)}.$$

This example shows explicitly how a mismatch of the forced cycle types eliminates a possible non-trivial stabilizer order.

4. Open problems

Open Problem 4.1 (Extension to the species $\mathbf{K}_\alpha$). For $\alpha \vdash n$ let $G_\alpha = \langle \tau_1 \rangle \times \cdots \times \langle \tau_k \rangle$, where $\tau_1, \dots, \tau_k$ are the individual cycles of σ_α , and let $\mathbf{K}_\alpha := X^n/G_\alpha$ (one independent rotation per cycle, instead of a single common rotation as in $\mathbf{C}_\alpha$). Determine a closed-form formula for the coefficients $j_{\alpha,\beta}^\lambda$ in $\mathbf{K}_\alpha \times \mathbf{K}_\beta = \sum_\lambda j_{\alpha,\beta}^\lambda \mathbf{K}_\lambda$.

Open Problem 4.2 (Quasisymmetric refinement). The cycle index $\mathbf{C}_\lambda(\mathbf{x})$ is Schur-positive, so its expansion in fundamental quasisymmetric functions $\mathbf{C}_\lambda(\mathbf{x}) = \sum_{\nu \models n} c_{\lambda,\nu} F_\nu$ has non-negative integer coefficients $c_{\lambda,\nu} = \langle \mathbf{C}_\lambda, r_\nu \rangle$, where r_ν is the commutative ribbon Schur function. Find a combinatorial interpretation of $c_{\lambda,\nu}$, for instance through a descent statistic on orbits of permutations under $\langle \sigma_\lambda \rangle$, in the spirit of Loehr and Warrington [12].

5. Conclusion

We obtained a closed-form formula for the coefficient $b_{\alpha,\beta}^\lambda$ via Möbius inversion over the divisors of $\gcd(o(\alpha), o(\beta))$ (Theorem 3.1). The formula is explicit but signed. In contrast, for $h_\lambda \star h_\mu$ one has a positive combinatorial interpretation in terms of matrices with prescribed row and column sums, but no closed formula. A positive combinatorial interpretation of $b_{\alpha,\beta}^\lambda$ remains an interesting open question. Section 4 collects natural next steps, including extending this technique to the species $\mathbf{K}_\alpha$ (Open Problem 4.1).

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