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Injectivity of Symmetric Polynomial Maps on Partitions

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Abstract: Introduced by Ballantine, Beck, and Merca, the elementary symmetric partition function pre_k, defined on the set of partitions with at least k parts, has been a topic of recent interest. We prove that pre_k is injective on the set of m-ary partitions for positive integers $m \geq k$, generalizing a $k = 2$ result of Ballantine, Beck, Merca, and Sagan, and complementing a result of Hadelyn, Niergarth, Weiyou Li, and Wenhui Li showing that, for each $k \geq 3$, pre_k is not injective on partitions of n with length $2k$ for infinitely many n. We introduce the skew Schur partition function $\mathrm{prs}_{\lambda'/\mu'}$, prove injectivity results for particular choices of λ', μ', and describe an application to representation theory.

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1. Introduction

In this paper, we study certain functions constructed from integer partitions and symmetric polynomials, two classes of well-studied objects in combinatorics [1, 2, 4–6, 8, 10–12, 14, 15].

First defined by Ballantine, Beck, and Merca [2], the elementary symmetric partition function pre_k is a map sending an integer partition λ with at least k parts to the summands of the elementary symmetric polynomial e_k evaluated at λ. Though the elementary symmetric partition function was not formally defined until very recently, it has been previously studied in the literature, though not in this exact language. For example, in 1958, Selfridge and Straus proved that if $X = \{\{x_1, \ldots, x_n\}\}$ is a multiset of complex numbers, and $X' = \{\{x_i + x_j\}\}_{1 \leq i < j \leq n}$ is the multiset of pairwise sums of X, we can uniquely retrieve X from X' if n is not a power of 2 [9, 13]. This result implies that pre_2 is injective on the set of partitions whose length is not a power of 2.

More generally, one may ask whether pre_k is injective. A back-of-the-envelope check reveals that $\mathrm{pre}_2(2, 2) = \mathrm{pre}_2(4, 1)$, so it is natural to filter by the size of the partitions before asking about injectivity.

Question 1.1 ([2])**.** *questionmainQuest For what positive integers n and k is the map pre_k injective on the set of partitions of n of length greater than or equal to k?*

Devnani and Eyyunni [6] gave an infinite family of examples demonstrating that pre_k is not injective on the set of partitions of length k. They also posed the following refinement of Question 1.1.

Conjecture 1.1 ([6])**.** *For all positive integers n and k, the map pre_k is injective on the set of partitions of n with length strictly greater than k.*

Hadelyn et al. [10] disproved this conjecture. We give a slightly different disproof of the conjecture in Section 3.1.

Theorem 1.1. *There exist infinitely many n for which pre_3 is not injective on the set of partitions of n with length greater than 3.*

However, there exist restricted classes of partitions with at least k parts on which pre_k is injective. For example, Ballantine, Beck, and Merca showed the injectivity of pre_2 on the binary partitions of n, and Ballantine, Beck, Merca, and Sagan showed the injectivity of pre_2 on m-ary partitions of n, where m-ary partitions are

partitions whose parts are all powers of m [2,3]. This result was generalized by Li, who showed injectivity of pre_2 on all partitions of n with at least 2 parts [11]. As another example, Ballantine et al. showed that pre_k is injective on a family of partitions $\mathcal{S}_k$ consisting of partitions with at least k ones or $(k-1)$ ones and at least one prime part [4].

Extending this line of research, we prove injectivity on the m-ary partitions of n for $m \geq k$:

Theorem 1.2. *For integers $m \geq k \geq 2$ and $n \geq 0$, the map pre_k is injective on the set of m-ary partitions of n with at least k parts.*

In the literature, there are a variety of techniques used to establish injectivity. For instance, to show injectivity of pre_2, Li [11] utilized the fundamental structure of e_2 as well as the extremal principle. By contrast, Ballantine et al. [4] and Ballantine, Beck, and Merca [2] adopted a more algorithmic approach when proving their results, attempting to construct λ from $\mathrm{pre}_k(\lambda)$. We use a variety of approaches throughout the paper to prove our injectivity results.

In addition to considering injectivity, Devnani and Eyyunni [6] considered how many partitions of n are in the image of pre_2, finding at least one for each positive integer n. They defined $\mathrm{pre}_2(n)$ to be this number. They then asked the following question:

Question 1.2 ([6])**.** *Does there exist an $n \geq 1$ such that $\mathrm{pre}_2(n) = 1$?*

We fully address this question, as well as a generalization of it, in Section 3.2. In his recent note, Garg [8] also addressed this question, but we offer our own independently derived proof to showcase a different technique. We utilize a recursive-style argument rather than direct casework, as used in Garg [8].

Besides pre_k, we study other functions constructed from symmetric polynomials and partitions. Hadelyn et al. [10] defined prh_k and $\mathrm{prs}_{\lambda'}$, which are complete homogeneous symmetric polynomial and Schur polynomial analogs of pre, and they proved that prh_k is injective on the set of all partitions. They also asked about injectivity of $\mathrm{prs}_{\lambda'}$ [10, Question 5.3]. We offer an independently derived proof of prh_k injectivity, an explicit algorithm to determine λ from $\mathrm{prh}_k(\lambda)$ whenever λ contains a positive number of ones, and a theorem making progress towards [10, Question 5.3].

Theorem 1.3 ([10], independent proof here)**.** *For all positive integers $k \geq 2$, the map prh_k is injective on the set of all partitions.*

Theorem 1.4. *For all positive integers $n \geq 2$, the map $\mathrm{prs}_{(2,1)}$ is injective on the set of partitions of n with at least two parts.*

1.1 Skew Schur polynomials

We can generalize $\mathrm{prs}_{\lambda'}$ from ordinary Schur polynomials over an integer partition λ' to skew Schur polynomials over a skew partition λ'/μ'. To that end, we define the analogous map $\mathrm{prs}_{\lambda'/\mu'}$ and prove the following theorem:

Theorem 1.5. *For any nonempty skew partition λ'/μ' in which there is at most one square in each column and at most one square in each row, $\mathrm{prs}_{\lambda'/\mu'}$ is injective on $\mathcal{P}(n)$ where n is at least the number of squares in λ'/μ'.*

1.2 Organization of the paper

The remainder of this paper is organized as follows:

- Section 2 establishes preliminary definitions, notational conventions, and summarizes prior results on Question 1.1.
- Section 3 concerns pre_k, giving a proof of Theorem 1.2 and detailing some instances where injectivity holds and fails to hold.
- Section 4 concerns prh_k, giving an alternate proof of injectivity and a remark on the existence of an algorithm.
- Section 5 details progress on Schur polynomials—notably a proof of Theorem 1.4—and expands upon applications.
- Section 6 contains open problems and future directions.

2. Background

2.1 Key definitions

A *partition* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_\ell)$ of n is a nonincreasing sequence of positive integers that sum to n. The value n is also called the *size* of λ, and ℓ is called the *length* of λ. We sometimes also write the size of λ as $|\lambda|$ and the length of λ as $\ell(\lambda)$. Let $\mathcal{P}$ be the set of all partitions, let $\mathcal{P}(n)$ be the set of all partitions of n, and let $\mathcal{P}_\ell(n)$ be the set of all partitions of n of length ℓ.

Convention 2.1. *Define $\mathcal{P}_{\geq k}(n) := \mathcal{P}_k(n) \cup \mathcal{P}_{k+1}(n) \cup \cdots \cup \mathcal{P}_n(n)$, and similarly define $\mathcal{P}_{>k}(n) := \mathcal{P}_{k+1}(n) \cup \cdots \cup \mathcal{P}_n(n)$. Later in this section, we define some functions indexed by k with domain $\mathcal{P}_{\geq k}(n)$ for various values of n. For brevity, we sometimes abuse notation and write the domains as $\mathcal{P}(n)$ instead of $\mathcal{P}_{\geq k}(n)$.*

Convention 2.2. *In the literature, the notation λ' is sometimes reserved for the conjugate partition of λ. We do not adopt this convention and instead use λ' to denote some other partition in statements where λ is already used. We use λ^T for the conjugate of λ.*

Let the *multiplicity* $m_\lambda(a)$ of an integer a in λ be the number of times a appears in λ.

We use a variant of exponential notation for partitions, in which

$$\lambda = (d_1, \ldots, d_1, d_2, \ldots d_2, \ldots, d_q, \ldots, d_q)$$

with no two d_i equal can be abbreviated as $d_1^{m_\lambda(d_1)} d_2^{m_\lambda(d_2)} \cdots d_q^{m_\lambda(d_q)}$. For instance, $(3,3,2,2,2,1)$ is abbreviated as $3^2 2^3 1^1$, with exponents of 1 sometimes omitted.

Definition 2.1. *For positive integers j, ℓ such that $j \leq \ell$, the* j-th elementary symmetric polynomial *is the polynomial $e_j \in \mathbb{Z}[x_1, x_2, \ldots, x_\ell]$ defined by:*

$$e_j(x_1, x_2, \ldots, x_\ell) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq \ell} x_{i_1} x_{i_2} \ldots x_{i_j}.$$

When we say e_k is evaluated at a partition λ, we mean $e_k(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where ℓ is the length of λ. Using these definitions, we can now define the map pre_k:

Definition 2.2 ([2])**.** *For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n and a positive integer $k \leq \ell$, we define* pre_k *as the map such that* $\mathrm{pre}_k(\lambda)$ *is the partition composed of the summands of the k-th elementary symmetric polynomial evaluated at λ.*

Example 2.1. *When $\lambda = (4,2,1,1)$ and $k = 2$,*

$$\mathrm{pre}_2(\lambda) = (\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_1\lambda_4, \lambda_2\lambda_3, \lambda_2\lambda_4, \lambda_3\lambda_4) = (8,4,4,2,2,1).$$

Remark 2.1. *In general, the relative sizes of the parts of λ affect the ordering of the parts of* $\mathrm{pre}_k(\lambda)$*. For example, $\lambda = (4,3,2,1)$ satisfies $\lambda_2\lambda_3 \geq \lambda_1\lambda_4$, whereas $\lambda' = (7,1,1,1)$ satisfies $\lambda'_2\lambda'_3 \leq \lambda'_1\lambda'_4$.*

We now discuss the complete homogeneous symmetric polynomial and the map prh_k, the latter of which was first defined in [10].

Definition 2.3. *For positive integers j, n, the* j-th complete homogeneous symmetric polynomial *is the polynomial $h_j \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ defined by:*

$$h_j(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq n} x_{i_1} x_{i_2} \ldots x_{i_j}.$$

Definition 2.4 ([10])**.** *For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n and a positive integer k, we define* prh_k *as the map such that* $\mathrm{prh}_k(\lambda)$ *is the partition composed of the summands of the k-th complete homogeneous symmetric polynomial evaluated at λ.*

Example 2.2. *Let $\lambda = (2,2)$ and $\mu = (4,1)$. By our definition of h_k, we have $h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$. Note that* $\mathrm{prh}_2(\lambda) = (4,4,4)$ *and* $\mathrm{prh}_2(\mu) = (16,4,1)$.

Finally, we define the skew Schur polynomial $s_{\lambda/\mu}$ and the map $\mathrm{prs}_{\lambda'/\mu'}$.

A *Young diagram* for a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a collection of boxes arranged in left-justified rows with ℓ total rows, and with row i having λ_i boxes for each i. The *skew partition* λ/μ is a pair of partitions λ, μ with $\ell(\lambda) \geq \ell(\mu)$ and $\lambda_i \geq \mu_i$ for all $i \in \{1, \ldots, \ell(\mu)\}$. The *skew diagram* of λ/μ is the shape defined by the set of squares that belong to the Young diagram of λ but not of μ. A *skew semi-standard Young tableau* of shape λ/μ is a filling of the squares of λ/μ with positive integers such that the entries are weakly increasing along every row and the entries are strictly increasing down every column. When $\mu = \emptyset$, we use the terminology "semi-standard Young tableau (SSYT) of shape λ."

Example 2.3. *Here is a skew semi-standard Young tableau of shape λ/μ for $\lambda = (5,4,2,2)$, $\mu = (2,1)$.*

		2	5	5
	3	4	6	
1	7			
8	9			

We say that a skew semi-standard Young tableau T of shape λ/μ has *type* $\alpha = (\alpha_1, \alpha_2, \dots)$ if for each $i \geq 1$, α_i is the number of i in T. Write $x^T = x_1^{\alpha_1} x_2^{\alpha_2} \dots$. We now define the skew Schur polynomial.

Definition 2.5. *The skew Schur polynomial $s_{\lambda/\mu}(x)$ for λ/μ and $x = (x_1, x_2, \dots, x_\ell)$ is defined as*

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

where the sum runs over all SSYTs T of shape λ/μ with entries in $\{1, 2, \dots, \ell\}$. Note that ℓ does not have to equal $|\lambda| - |\mu|$, but ℓ must be at least the maximum column height of λ/μ. We define the usual Schur polynomial s_λ as $s_{\lambda/\emptyset}$.

Example 2.4. *Let $\lambda = (2,1)$, let $\mu = \emptyset$, and consider the set of $n \geq 3$ variables $x = (x_1, x_2, \dots, x_n)$. Now, consider three integers $1 \leq i, j, k \leq n$ such that $i < j < k$. The SSYTs of shape λ can be generated as follows:*

i	j
k	

i	k
j	

i	i
j	

i	j
j	

Thus, it follows that

$$s_{(2,1)}(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} (x_i^2 x_j + x_i x_j^2) + 2 \sum_{1 \leq i < j < k \leq n} x_i x_j x_k.$$

With this framework, we define the maps $\mathrm{prs}_{\lambda'}$ and $\mathrm{prs}_{\lambda'/\mu'}$.

Definition 2.6 ([10])**.** *For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of n and an arbitrary partition λ' for which $s_{\lambda'}(\lambda)$ is defined and nonzero, we define $\mathrm{prs}_{\lambda'}(\lambda)$ to be the partition composed of the summands of $s_{\lambda'}(\lambda)$. Note that if $s_{\lambda'}$ has a term with coefficient $m > 1$, then that term appears m times in $\mathrm{prs}_{\lambda'}$.*

Example 2.5. *Let $\lambda' = (2,1)$. Then $s_{\lambda'}(x_1, x_2, x_3) = (x_1^2 x_2 + x_1 x_2^2) + (x_1^2 x_3 + x_1 x_3^2) + (x_2^2 x_3 + x_2 x_3^2) + x_1 x_2 x_3 + x_1 x_2 x_3$. Let $\lambda = (10, 5, 3)$. Then $\mathrm{prs}_{\lambda'}(\lambda) = (500, 300, 250, 150, 150, 90, 75, 45)$.*

Definition 2.7. *For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of n and an arbitrary skew partition λ'/μ' for which $s_{\lambda'/\mu'}(\lambda)$ is defined and nonzero, we define $\mathrm{prs}_{\lambda'/\mu'}(\lambda)$ to be the partition composed of the summands of $s_{\lambda'/\mu'}(\lambda)$. Note that if $s_{\lambda'/\mu'}$ has a term of coefficient $m > 1$, then that term appears m times in $\mathrm{prs}_{\lambda'/\mu'}$.*

Example 2.6. *Let $\lambda' = (3,2)$ and $\mu' = (1)$. Then $s_{\lambda'/\mu'}(x_1, x_2) = x_1^3 x_2 + x_1 x_2^3 + x_1^2 x_2^2 + x_1^2 x_2^2$. Let $\lambda = (10, 7)$. Then $\mathrm{prs}_{\lambda'/\mu'}(\lambda) = (7000, 4900, 4900, 3430)$.*

Remark 2.2. *Since $e_k = s_{(1^k)}$ and $h_k = s_{(k)}$, $\mathrm{prs}_{(1^k)} = \mathrm{pre}_k$ and $\mathrm{prs}_{(k)} = \mathrm{prh}_k$.*

Proposition 2.1 (Jacobi-Trudi identities and Jacobi's bialternant formula [7])**.** *Consider a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$. Let the conjugate partition of λ be $\lambda^T = (\lambda_1^T, \lambda_2^T, \dots, \lambda_{\ell'}^T)$, and let $\ell' = \ell(\lambda^T)$. We then have the Jacobi-Trudi identities*

$$s_\lambda = \det(h_{\lambda_i + j - i})_{i,j=1}^{\ell},$$

$$s_\lambda = \det(e_{\lambda_i^T + j - i})_{i,j=1}^{\ell'},$$

where if $t = 0$, we set $h_t = e_t = 1$, and if $t < 0$, we set $h_t = e_t = 0$. Additionally, if we consider s_λ over m variables $(x_1, x_2, \dots, x_m)$ with $m \geq \ell(\lambda)$, then we have Jacobi's bialternant formula

$$s_\lambda = \frac{\det(x_j^{\lambda_i + m - i})_{i,j=1}^{m}}{\det(x_j^{m-i})_{i,j=1}^{m}}.$$

Note that if $i > \ell$, then λ_i is set to be 0.

Convention 2.3. *Let f be pre_k, prh_k, or $\mathrm{prs}_{\lambda'/\mu'}$. Given some input partition λ, we usually denote $f(\lambda)$ by ν.*

Convention 2.4. *We set* $\binom{n}{k} = 0$ *if* $n < k$.

Proposition 2.2 ([4])**.** *Let* λ *be a partition with* $m_\lambda(1) > 0$*, and let* $\nu = \mathrm{pre}_k(\lambda)$*. Then* $m_\nu(1) = \binom{m_\lambda(1)}{k}$.

We will also use the following definitions.

Definition 2.8 ([4])**.** *Let* k *be a positive integer. Define* $\mathcal{S}_k$ *as the set of partitions such that, for each* $\lambda \in \mathcal{S}_k$*, either the number of ones in* λ *is at least* k*, or the number of ones in* λ *is* $k-1$ *and there exists some prime* p *for which the number of* p*'s in* λ *is positive.*

Definition 2.9. *Let* $k \geq 2$*. Define the* k*-ary partitions of* n*, denoted* $\mathcal{B}_k(n)$*, as the set of all partitions of* n *whose parts are all powers of* k*.*

The latter definition is a slight modification of a notational convention in [2], where $\mathcal{B}_2(n)$ is simply denoted as $\mathcal{B}(n)$.

Definition 2.10 ([6])**.** *Let* $Pre_k(n)$ *be the set of partitions of* n *that lie in the image of* $\mathrm{pre}_k(\lambda)$ *for* $\lambda \in \mathcal{P}$*. Define* $\mathrm{pre}_k(n)$ *as the cardinality of* $Pre_k(n)$*.*

Remark 2.3. *This notation may appear at first to conflict with* pre_1 *applied to a partition with one part, but we only consider* pre_1 *in a very limited section of the paper, namely Theorem 3.2 and Corollary 3.1.*

Remark 2.4. *As noted by Ballantine, Beck, and Merca [2], if* λ *and* μ *have different lengths, it is impossible for* $\mathrm{pre}_k(\lambda)$ *to equal* $\mathrm{pre}_k(\mu)$*. This property also applies to* prh_k *and* $\mathrm{prs}_{\lambda'}$ *in the cases where we need it. Thus, in our proofs of injectivity, we often start by assuming that if* $f(\lambda) = f(\mu)$ *for* f *some function on partitions, then* $\ell(\lambda) = \ell(\mu)$*.*

3. Elementary symmetric polynomials

3.1 Non-injective families of partitions

A more general version of Theorem 1.1 was proven by Hadelyn et al. [10], and it states that for each $j \geq 3$, there are infinitely many pairs of partitions λ, μ of length $2j$ for which $\mathrm{pre}_j(\lambda) = \mathrm{pre}_j(\mu)$. However, we give our progress as it was derived independently of them. We first disprove Conjecture 1.1 by generating an infinite family of counterexamples. To motivate this, we provide an explicit counterexample:

Example 3.1. *Consider* $\lambda = (40, 16, 16, 16, 5, 5)$ *and* $\mu = (32, 32, 10, 10, 10, 4)$*. Note that both* λ *and* μ *are partitions of* 98*. Furthermore, it follows that*

$$\mathrm{pre}_3(\lambda) = 10240^3 4096^1 3200^6 1280^6 1000^1 400^3 = \mathrm{pre}_3(\mu).$$

Thus, pre_3 *is not injective on* $\mathcal{P}(98)$*.*

Theorem 1.1. *There exist infinitely many* n *for which* pre_3 *is not injective on the set of partitions of* n *with length greater than* 3*.*

Proof. Consider $\lambda := (a, b, b, b, c, c)$ and $\mu := (d, d, e, e, e, f)$ for $a, b, c, d, e, f \in \mathbb{Z}^+$, where λ, μ are partitions of n. It follows that $pre_3(\lambda)$ is the partition with 3 copies of part ab^2, 6 copies of part abc, 1 copy of part ac^2, 1 copy of part b^3, 6 copies of part b^2c, and 3 copies of part bc^2; and $pre_3(\mu)$ is the partition with 3 copies of part d^2e, 6 copies of part def, 1 copy of part d^2f, 1 copy of part e^3, 6 copies of part de^2, and 3 copies of part e^2f."

In order to directly compare these two partitions in an efficient manner, we can set terms of the same multiplicity equal. Mimicking the structure of Example 3.1, we obtain the following system of equations:

$$ab^2 = d^2e,$$
$$b^3 = d^2f,$$
$$bc^2 = e^2f,$$
$$ac^2 = e^3,$$
$$abc = de^2,$$
$$b^2c = def.$$

Using these six equations, we can restrict the problem to three degrees of freedom:

$$a = \frac{e^3}{c^2}, b = \frac{e^2f}{c^2}, d = \frac{e^3f}{c^3}.$$

Now, as λ and μ are partitions of the same number, it follows that $a+2c+3b = 2d+3e+f$. Substituting our expressions for a, b, d, we can simplify this equation as follows:

$$f = \frac{3ec^3 - 2c^4 - ce^3}{3ce^2 - c^3 - 2e^3}.$$

Let $c = k$ and $e = 2k$ for $k \in \mathbb{Z}^+$. Then, f simplifies to $\frac{4}{5}k$. Thus, let $k = 5q$ for positive integers q, so $f = 4q$. Thus, $a = 40q, b = 16q, d = 32q$, all of which are integers. Thus, by varying q over the positive integers, we generate infinitely many partitions $\lambda, \mu \in \mathcal{P}(n)$ for which $\mathrm{pre}_3(\lambda) = \mathrm{pre}_3(\mu)$. □

Remark 3.1. *There are many more counterexamples of this type. For instance, if* $\lambda = (189, 45, 45, 45, 7, 7)$ *and* $\mu = (135, 135, 21, 21, 21, 5)$*, we may calculate that* $\lambda, \mu \in \mathcal{P}(338)$ *and* $\mathrm{pre}_3(\lambda) = \mathrm{pre}_3(\mu)$*.*

Remark 3.2. *The existence of such counterexamples motivates a new question: how often is* pre_k *injective? Devnani and Eyyunni [6] asked this as an open question for the case of* $\mathcal{P}_k(n)$*. It turns out that in this scenario,* pre_k *is "almost" never injective. Particularly, for sufficiently large* n *relative to* k*,* pre_k *fails to be injective on* $\mathcal{P}_k(n)$*. We prove this now, starting with the case of* $k = 3$*.*

Lemma 3.1. *When* $n > 18$*, the map* pre_3 *is not injective on the set of partitions of* n *with* 3 *parts.*

Proof. Consider any positive integer $p \geq 13$ that can be written as $6q+1$ or $6q-1$ for some integer q.

Suppose $p = 6q+1$. Thus, consider $\lambda = (3q+3, 2q-2, q)$ and $\mu = (3q, 2q+2, q-1)$, both of which are partitions of p. Note the following equality:

$$\mathrm{pre}_3(\lambda) = q(2q-2)(3q+3) = (q-1)(2q+2)(3q) = \mathrm{pre}_3(\mu).$$

Thus, pre_3 is not injective on $\mathcal{P}_3(p)$.

Now, suppose $p = 6q-1$. Using an argument similar to above, letting λ be the partition with parts $3q-3, 2q+2, q$ and letting $\mu = (3q, 2q-2, q+1)$, it follows that $\mathrm{pre}_3(\lambda) = \mathrm{pre}_3(\mu)$. Therefore, pre_3 is not injective on $\mathcal{P}_3(p)$ for any positive integer $p \geq 13$ that is 1 or 5 (mod 6). We may also check manually that pre_3 is noninjective on $\mathcal{P}_3(n)$ when $n = 16$ ($\lambda = (10, 3, 3), \mu = (9, 5, 2)$) and when $n = 27$ ($\lambda = (16, 6, 5), \mu = (15, 8, 4)$).

Consider two nonequal partitions $\lambda = (a_1, b_1, c_1), \mu = (a_2, b_2, c_2) \in \mathcal{P}(n)$ such that $\mathrm{pre}_3(\lambda) = \mathrm{pre}_3(\mu)$. Note, for any positive integer k, it follows that $k\lambda = (ka_1, kb_1, kc_1)$ and $k\mu = (ka_2, kb_2, kc_2)$ are partitions of kn satisfying $\mathrm{pre}_3(k\lambda) = \mathrm{pre}_3(k\mu)$. Therefore, pre_3 is not injective on $\mathcal{P}_3(kn)$.

Write n as $2^r 3^s X$ for positive integer X not divisible by 2 or 3 and for nonnegative integers r, s. To finish the proof, it suffices to show that pre_3 is not injective on $\mathcal{P}_3(n)$ for $r < 4$, $s < 3$, $X < 13$, and $n > 18$, which we verify using a computer program*. □

We can now prove the following theorem:

Theorem 3.1. *When* $k \geq 3$ *and* $n > k + 15$*, the map* pre_k *is not injective on partitions of* n *with* k *parts.*

Proof. For integers $k \geq 3$ and $n > k + 15$, consider distinct partitions $\lambda, \mu \in \mathcal{P}_3(n-k+3)$ such that $\mathrm{pre}_3(\lambda) = \mathrm{pre}_3(\mu)$, which exist by Lemma 3.1. Now, define $\lambda', \mu' \in \mathcal{P}(n)$ as the partitions obtained by appending $k-3$ 1's to λ, μ, respectively. We have that $\mathrm{pre}_k(\lambda') = \mathrm{pre}_k(\mu')$. □

Remark 3.3. *For each* $k \geq 5$*, there are exactly* 8 *values of* n *for which* pre_k *is injective on* $\mathcal{P}_k(n)$*, so in some sense,* pre_k *is almost never injective on* $\mathcal{P}_k(n)$*.*

Furthermore, Devnani and Eyyunni [6] offered the question of whether a lattice-type technique, first introduced in [2], could be feasible for proving injectivity of pre_2. They offered such a technique for partitions of length 4, 5, and 6, and asked the open question of whether this is possible for other lengths. We offer an intuitive reason for why it is likely not possible:

Remark 3.4. *There are two possibilities that could arise from employing this lattice-type method: one-by-one casework or a nice set of cases that eliminates all others.*

The former approach, used by [6], is computationally challenging to extend to general ℓ*. For the latter approach, this would mean injectivity would follow under a significantly smaller set than all of* pre_2*. One way to think about this is by looking at each "level" of the lattice structure, and seeing if that is sufficient to show injectivity. However, as suggested by empirical results, this is not sufficient to show injectivity on all of* pre_2*.*

*See `https://github.com/katherineatung/primes-2026/blob/main/lemma34.py`

3.2 Injective families of partitions

While pre_k is not injective on $\mathcal{P}(n)$ for most values of n and k, there are still many positive results associated with injectivity on pre_k. We begin by extending Lemma 2.1 of Li [11].

Proposition 3.1. *Let $\ell \geq k \geq 2$ and $\lambda, \mu \in \mathcal{P}_\ell(n)$. If $\lambda_i = \mu_i$ for $1 \leq i \leq k-1$ and $\mathrm{pre}_k(\lambda) = \mathrm{pre}_k(\mu)$, then $\lambda = \mu$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be partitions of n of length ℓ and assume $\mathrm{pre}_k(\mu) = \mathrm{pre}_k(\lambda)$. Moreover, assume $\lambda_i = \mu_i$ for $1 \leq i \leq k-1$. We want to show $\lambda_i = \mu_i$ for all $1 \leq i \leq \ell$.

We use strong induction on i. For our base case, consider $i = k$. Note a maximum element in $\mathrm{pre}_k(\lambda)$ is $\lambda_1\lambda_2\ldots\lambda_{k-1}\lambda_k$. Also, a maximum element in $\mathrm{pre}_k(\mu)$ is

$$\mu_1\mu_2\ldots\mu_{k-1}\mu_k = \lambda_1\lambda_2\ldots\lambda_{k-1}\mu_k.$$

As $\mathrm{pre}_k(\mu) = \mathrm{pre}_k(\lambda)$, the maximum elements of each set will be equal. Thus, $\lambda_k = \mu_k$.

For our inductive step, suppose $\lambda_i = \mu_i$ for all $1 \leq i \leq k'-1$ for some $k' \geq k+1$. Consider the following multiset:

$$M_{k,k'}(\lambda) := \{\{\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq k'-1\}\}.$$

Thus, by the inductive hypothesis, $M_{k,k'}(\lambda) = M_{k,k'}(\mu)$. Therefore, it follows that

$$\mathrm{pre}_k(\lambda) \setminus M_{k,k'}(\lambda) = \mathrm{pre}_k(\mu) \setminus M_{k,k'}(\mu),$$

where $\setminus$ denotes multiset difference. This means that the maximum elements of both of the above sets are equal.

Now, consider the element $a = \lambda_1\lambda_2\ldots\lambda_{k-1}\lambda_{k'} \in \mathrm{pre}_k(\lambda)$, which we claim to be the maximum element of $\mathrm{pre}_k(\lambda) \setminus M_{k,k'}(\lambda)$. Consider an arbitrary element $s = \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k} \in \mathrm{pre}_k(\lambda)$. Suppose for the sake of contradiction that s is the maximum element. By definition, $\lambda_j \geq \lambda_{i_j}$. Note $\lambda_{k'} \geq \lambda_{i_k}$ if $i_k \geq k'$. Therefore, $i_k < k'$ is a necessary condition for $s > a$. However, if $i_k < k'$, then $s \in M_{k,k'}(\lambda)$, which is a contradiction. Thus, a is a maximum element. Likewise, we have that $\mu_1\mu_2\ldots\mu_{k-1}\mu_{k'}$ is a maximum element of $\mathrm{pre}_k(\mu) \setminus M_{k,k'}(\mu)$.

Thus, it follows that

$$\lambda_1\lambda_2\ldots\lambda_{k-1}\lambda_{k'} = \mu_1\mu_2\ldots\mu_{k-1}\mu_{k'} = \lambda_1\lambda_2\ldots\lambda_{k-1}\mu_{k'}.$$

Hence, $\lambda_{k'} = \mu_{k'}$. Therefore, by strong induction, we conclude $\lambda = \mu$, as desired. □

We now offer a proof of Theorem 3.2, which was obtained independently of Devnani and Eyyunni [6], in order to offer an analog to a theorem we prove about Schur polynomials in Section 5.

Theorem 3.2 ([6], independent proof here)**.** *Let ℓ be an integer greater than or equal to 2. For each k such that $1 \leq k \leq \ell-1$, the map pre_k is injective on $\mathcal{P}_\ell(n)$ if and only if the map $\mathrm{pre}_{\ell-k}$ is injective on $\mathcal{P}_\ell(n)$.*

Proof. Suppose there exist two partitions $\lambda, \mu \in \mathcal{P}_\ell(n)$ such that $\mathrm{pre}_k(\lambda) = \mathrm{pre}_k(\mu)$. Note the product of all elements in $\mathrm{pre}_k(\lambda)$ is $(\lambda_1\lambda_2\ldots\lambda_\ell)^{\binom{\ell-1}{k-1}}$. Likewise, the product of all elements in $\mathrm{pre}_k(\mu)$ is $(\mu_1\mu_2\ldots\mu_\ell)^{\binom{\ell-1}{k-1}}$. Clearly, both of these are equal, so $\lambda_1\lambda_2\ldots\lambda_\ell = \mu_1\mu_2\ldots\mu_\ell$.

Note $\mathrm{pre}_{\ell-k}(\lambda)$ can be generated by taking the reciprocal of every element in $\mathrm{pre}_k(\lambda)$ and then multiplying every element by $\lambda_1\lambda_2\ldots\lambda_\ell$. Doing a similar operation for generating $\mathrm{pre}_{\ell-k}(\mu)$, by the assumption $\mathrm{pre}_k(\lambda) = \mathrm{pre}_k(\mu)$, we see that $\mathrm{pre}_{\ell-k}(\lambda) = \mathrm{pre}_{\ell-k}(\mu)$. Thus, pre_k being injective is equivalent to $\mathrm{pre}_{\ell-k}$ being injective, as desired. □

Corollary 3.1. *For positive integers k, the function pre_k is injective on partitions of length $k+1$ and length $k+2$.*

We now highlight the algorithmic approaches to injectivity. In their paper, Ballantine et al. [4] gave, in Remark 2.7, an algorithm to determine λ from $\nu = \mathrm{pre}_2(\lambda)$ for the class $\mathcal{S}_k$. We can also extend this to $k = 3$ through the cubic formula:

Example 3.2. *Assume $m_\lambda(1) \geq k = 3$. Recall the following equality:*

$$m_\nu(1) = \binom{m_\lambda(1)}{3}.$$

Letting $c = m_\nu(1)$, it follows that

$$m_\lambda(1)^3 - 3m_\lambda(1)^2 + 2m_\lambda(1) - 6c = 0.$$

Using the cubic formula, we obtain the following expression (which yields an integer if and only if $c = \binom{a}{3}$ for some integer a):

$$m_\lambda(1) = 1 + \left(3c + \sqrt{9c^2 - \frac{1}{27}}\right)^{\frac{1}{3}} + \left(3c - \sqrt{9c^2 - \frac{1}{27}}\right)^{\frac{1}{3}}.$$

Now, using a method analogous to Lemma 2.4 in [4], we can develop a recursive relation for $m_\lambda(y)$, allowing us to determine λ uniquely from ν.

For the case of $m_\lambda(1) = k - 1 = 2$ and $m_\lambda(p) \geq 1$ for some prime p, we can do a procedure analogous to the one above to determine λ from ν.

We can also show injectivity on the class of the m-ary partitions, $\mathcal{B}_m(n)$, for $m \geq k$. We will now prove Theorem 1.2.

Theorem 1.2. *For integers $m \geq k \geq 2$ and $n \geq 0$, the map pre_k is injective on the set of m-ary partitions with at least k parts.*

Convention 3.1. *Note, by the injectivity of pre_k on $\mathcal{S}_k$ [4], if $m_\lambda(1) \geq k$, then the pre-image of $\mathrm{pre}_k(\lambda)$ is $\{\lambda\}$. Thus, henceforth, assume $0 \leq m_\lambda(1) < k$.*

We prove this theorem by introducing two lemmas, after which an algorithmic approach, similar to the proof of injectivity for $\mathcal{S}_k$ in [4], finishes the proof.

Lemma 3.2. *Let $k \geq 2$, $m \geq k$, and $\lambda, \mu \in \mathcal{B}_m(n)$, with $\ell := \ell(\lambda)$ and $\ell' := \ell(\mu)$. Suppose $0 \leq m_\lambda(1) < k$ and $0 \leq m_\mu(1) < k$. Then $m_\lambda(1) = m_\mu(1)$.*

Proof. Suppose $\lambda = (m^{a'_1}, m^{a'_2}, \ldots, m^{a'_\ell})$ and $\mu = (m^{b'_1}, m^{b'_2}, \ldots, m^{b'_{\ell'}})$. Note the following equality:

$$n = m^{a'_1} + m^{a'_2} + \cdots + m^{a'_\ell} = m^{b'_1} + m^{b'_2} + \cdots + m^{b_{\ell'}}.$$

Taking the equality modulo m, we see that

$$n \equiv m_\lambda(1) \equiv m_\mu(1) \pmod{m}.$$

Thus, $m_\lambda(1) = m_\mu(1) + Cm$ for some integer constant C. However, as we assumed $0 \leq m_\lambda(1) \leq k - 1$ and as $m \geq k$, it follows that $C = 0$. Thus, $m_\lambda(1) = m_\mu(1)$, as desired. □

Let ν be $\mathrm{pre}_k(\lambda)$. Define the function $\varphi : \mathcal{P}(q) \to \mathbb{Z}$ such that $\varphi(\alpha) = \prod_{\text{distinct } a \in \alpha} \binom{m_\lambda(m^a)}{m_\alpha(a)}$, where in the indexing of the product we are identifying a partition with its multiset of parts. It follows that

$$m_\nu(m^q) = \sum_{\substack{\alpha \in \mathcal{P}(q) \\ k \geq \ell(\alpha) \geq k - m_\lambda(1)}} \varphi(\alpha) \cdot \binom{m_\lambda(1)}{k - \ell(\alpha)}. \tag{1}$$

To prove that ν determines λ, we proceed with induction on q. By Lemma 3.2, we know that $m_\lambda(1)$ is uniquely determined by ν and n.

Now, suppose we know what $m_\lambda(1), m_\lambda(m), \ldots, m_\lambda(m^{q-1})$ are for an integer $q \geq 1$. Let $a_i := m_\lambda(m^i)$ for $0 \leq i \leq q - 1$. Consider the following summation:

$$S_{q-1} = m_\lambda(1) + m_\lambda(m) + \cdots + m_\lambda(m^{q-1}).$$

Suppose $S_{q-1} < k$. Consider $m_\nu(m^{\sum_{i=0}^{q} i \cdot a_i})$ where $A_q := k - (a_0 + a_1 + \cdots + a_{q-1})$. Note that a potential nonzero term in its summation, given by Equation 1, would be as follows:

$$\binom{m_\lambda(m^q)}{A_q} \cdot \prod_{i=0}^{q-1} \binom{m_\lambda(m^i)}{a_i} = \binom{m_\lambda(m^q)}{A_q}.$$

We claim this is the only one.

Lemma 3.3. *Let $\lambda = (m^{a'_1}, m^{a'_2}, \ldots, m^{a'_\ell})$ be an m-ary partition, let q be a positive integer, and let*

$$\sum_{i<q} m_\lambda(m^i) < k.$$

Let $a_i := m_\lambda(m^i)$ for $0 \leq i \leq q - 1$ and let $A_q := k - (a_0 + a_1 + \cdots + a_{q-1})$. Let $\nu = \mathrm{pre}_k(\lambda)$. We have

$$m_\nu(m^{\sum_{i=0}^{q} i \cdot a_i}) = \binom{m_\lambda(m^q)}{A_q} \cdot \prod_{i=0}^{q-1} \binom{m_\lambda(m^i)}{a_i} = \binom{m_\lambda(m^q)}{A_q}.$$

Proof. Suppose for the sake of contradiction that in the summation for $m_\nu(m^{\sum_{i=0}^{q} i\cdot a_i})$, given by Equation 1, there exists another nonzero summand besides the one listed above. Suppose this other summand is of the following form:

$$\prod_{i=q}^{\infty}\binom{m_\lambda(m^i)}{b_i}\cdot\prod_{i=0}^{q-1}\binom{m_\lambda(m^i)}{b_i}$$

for some infinite sequence $(b_i)_{i\geq 0}$ with finitely many nonzero terms. Note $k = b_0 + b_1 + \dots$ and $b_i \leq a_i$ for all $0 \leq i \leq q-1$, with at least one inequality being strict. Since we are evaluating $m_\nu(m^{\sum_{i=0}^{q} i\cdot a_i})$, we have

$$\begin{aligned}
\sum_{i=0}^{q} i\cdot a_i &= \sum_{i=0}^{\infty} i\cdot b_i,\\
\sum_{i=q}^{\infty} i\cdot b_i &= q\cdot A_q + \sum_{i=0}^{q-1} i\cdot(a_i - b_i)\\
&= q\cdot(k - a_0 - a_1 - a_2 - \cdots - a_{q-1}) + \sum_{i=0}^{q-1} i\cdot(a_i - b_i). \qquad (2)
\end{aligned}$$

However, note that the left-hand side of (2) satisfies the following inequality:

$$\sum_{i=q}^{\infty} i\cdot b_i \geq q\cdot\sum_{i=q}^{\infty} b_i = q(k - b_0 - b_1 - b_2 - \cdots - b_{q-1}).$$

Thus, it follows that

$$\begin{aligned}
&q\cdot(k - a_0 - a_1 - a_2 - \cdots - a_{q-1}) + \sum_{i=0}^{q-1} i\cdot(a_i - b_i) \geq q(k - b_0 - b_1 - b_2 - \cdots - b_{q-1}),\\
&\sum_{i=0}^{q-1} i\cdot(a_i - b_i) \geq \sum_{i=0}^{q-1} q\cdot(a_i - b_i),\\
&\sum_{i=0}^{q-1}(q-i)\cdot(a_i - b_i) \leq 0.
\end{aligned}$$

Note that $a_i - b_i \geq 0$ for all i, with at least one being positive. Also, all of $q - i$ are positive. Therefore, the left-hand sum must be strictly positive, yielding a contradiction, as desired. □

We now finish the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 3.3, if $m_\nu(m^{\sum_{i=0}^{q} i\cdot a_i}) > 0$, then, because the binomial coefficient is increasing in its top parameter, we can determine $m_\lambda(m^q)$. On the other hand, if $m_\nu(m^{\sum_{i=0}^{q} i\cdot a_i}) = 0$, then $m_\lambda(m^q) < a_q \leq k \leq m$. However, note the following equality:

$$\begin{aligned}
n &= \sum_{i=0}^{\infty} m_\lambda(m^i)\cdot m^i\\
&\equiv m_\lambda(1) + m_\lambda(m)\cdot m + \cdots + m_\lambda(m^q)\cdot m^q \pmod{m^{q+1}}.
\end{aligned}$$

Therefore,

$$m_\lambda(m^q) \equiv \frac{n - m_\lambda(1) - m_\lambda(m)\cdot m - \cdots - m_\lambda(m^{q-1})\cdot m^{q-1}}{m^q} \pmod{m}.$$

As $m_\lambda(m^q) < m$, the congruence forces equality, allowing us to determine $m_\lambda(m^q)$.

Using this result, we can extract $m_\lambda(1), m_\lambda(m), \cdots$ until the summation S_q is at least k. Suppose this occurs at $m_\lambda(m^p)$.

$$\begin{aligned}
S_{p-1} &= m_\lambda(1) + m_\lambda(m) + \cdots + m_\lambda(m^{p-1}) < k,\\
S_p &= m_\lambda(1) + m_\lambda(m) + \cdots + m_\lambda(m^{p-1}) + m_\lambda(m^p) \geq k.
\end{aligned}$$

Thus, we know the values of $m_\lambda(1), m_\lambda(m), \cdots, m_\lambda(m^p)$. Now, consider $m_\nu(m^{1+\sum_{i=0}^{p} i\cdot a_i})$. Using a similar method as in the proof of Lemma 3.3, we can see that $m_\nu(m^{1+\sum_{i=0}^{p} i\cdot a_i})$ can be expressed solely in terms of $m_\lambda(1), m_\lambda(m), \cdots, m_\lambda(m^p)$, and $m_\lambda(m^{p+1})$. Furthermore, the only summands in $m_\nu(m^{1+\sum_{i=0}^{p} i\cdot a_i})$ that

contain $m_\lambda(m^{p+1})$ would be of the form $\binom{m_\lambda(m^{p+1})}{j} \cdot X$ for some positive integer $j \leq m_\lambda(m^{p+1})$ and some positive integer X. Thus, because the binomial coefficient is increasing in its top parameter, it follows that $m_\nu(m^{1+\sum_{i=0}^{p} i \cdot a_i})$ would be an increasing function of $m_\lambda(m^{p+1})$. Thus, if $m_\nu(m^{1+\sum_{i=0}^{p} i \cdot a_i})$ is nonzero, we can determine $m_\lambda(m^{p+1})$.

Now, suppose $m_\nu(m^{1+\sum_{i=0}^{p} i \cdot a_i}) = 0$. Note that an element of the summation, given by Equation 1, is as follows:

$$m_\nu(m^{1+\sum_{i=0}^{p} i \cdot a_i}) \geq m_\lambda(m^{p+1}) \cdot \binom{m_\lambda(m^p)}{A_p - 1} \cdot \prod_{i=0}^{p-1} \binom{m_\lambda(m^i)}{A_i} \geq m_\lambda(m^{p+1}).$$

Thus, it follows $m_\lambda(m^{p+1}) = 0$.

We can repeat this process inductively. Namely, we can write $m_\nu(m^{g+\sum_{i=0}^{p} i \cdot a_i})$, which, by Equation 1 can be expressed in terms of solely $m_\lambda(1), m_\lambda(m), \cdots, m_\lambda(m^{p+g-1})$, and $m_\lambda(m^{p+g})$. Thus, if $m_\nu(m^{g+\sum_{i=0}^{p} i \cdot a_i}) \neq 0$, we can determine $m_\lambda(m^{p+g})$. If it were 0, then

$$m_\nu(m^{g+\sum_{i=0}^{p} i \cdot a_i}) \geq m_\lambda(m^{p+g}) \cdot \binom{m_\lambda(m^p)}{A_p - 1} \cdot \prod_{i=0}^{p-1} \binom{m_\lambda(m^i)}{A_i} \geq m_\lambda(m^{p+g}).$$

Thus, $m_\lambda(m^{p+g})$ is 0. By induction on $g \in \mathbb{Z}^+$, it follows that we can determine all of λ from ν, thus proving injectivity on the m-ary partitions. □

Example 3.3. *To illustrate how we can determine λ from ν for an m-ary partition, consider $n = 28$, $m = k = 3$, and $\nu = 243^3 81^7 27^7 9^3$. As $m_\nu(1) = 0$, we can determine $m_\lambda(1)$ to be the residue of n modulo 3, which is 1. Now, we have $3 = m_\nu(9) = \binom{m_\lambda(3)}{2}$. Thus $m_\lambda(3) = 3$. Then, we have $7 = m_\nu(27) = m_\lambda(9) \cdot m_\lambda(3) + \binom{m_\lambda(3)}{3}$. Thus, $m_\lambda(9) = 2$. We can stop here as the components of our partition sum to 28, making the unique λ be $(9, 9, 3, 3, 3, 1)$.*

Remark 3.5. *One may ask whether this result can be extended to the $m < k$ case. However, if $m < k$, the map pre_k is not injective on the set of m-ary partitions with k parts, for all integers $m \geq k \geq 2$ and $n \geq 0$. For instance, consider $k = 5$ and the binary partitions $\lambda = (8, 8, 8, 1, 1)$ and $\mu = (16, 4, 2, 2, 2)$ of 26. Note $\mathrm{pre}_5(\lambda) = 512 = \mathrm{pre}_5(\mu)$, so injectivity fails.*

We end this section on pre_k by answering Question 1.2, as well as a generalization of it. Note that Garg [8] also proved the following result, but we provide it anyway for completeness and also because it was independently derived.

Theorem 3.3 ([8], independent proof here)**.** *The only positive integers n for which $\mathrm{pre}_2(n) = 1$ are $n = 1, 2$, and 4.*

Proof. For $n = 1, 2$ or 4, we can easily verify that $\mathrm{pre}_2(n) = 1$. We may also verify that if $\lambda = (n, 1)$, then $\mathrm{pre}_2(\lambda) = (n)$, so it suffices to show that for each $n \notin \{1, 2, 4\}$, there exists μ such that $\mathrm{pre}_2(\mu)$ is a partition of n not equal to (n). To show $\mathrm{pre}_2(n) > 1$ for $n \notin \{1, 2, 4\}$, we perform casework on n mod 4.

If $n \geq 3$ is odd, we may write $n = 2x + 1$ for $x \in \mathbb{Z}_+$, and take $\mu = (x, 1, 1)$. Then $\mathrm{pre}_2(\mu) = (x, x, 1)$ which is a partition of n not equal to (n).

If $n \geq 6$ is even but not divisible by 4, we can write $n = 4x + 6$ for $x \in \mathbb{Z}_{\geq 0}$. If $x = 0$, take $\mu = (1, 1, 1, 1)$; otherwise, take $\mu = (x, 1, 1, 1, 1)$. If $x = 0$, $\mathrm{pre}_2(\mu) = 1^6$; otherwise, $\mathrm{pre}_2(\mu) = (x, x, x, x, 1, 1, 1, 1, 1, 1)$, which is a partition of n not equal to (n).

If $n = 8$, take $\mu = (2, 2, 1)$. Otherwise, if $n = 2^r$ for $r \geq 4$, take $\mu = (2^{r-2} - 1, 2, 2)$.

Finally, if there is μ of length ℓ such that $\mathrm{pre}_2(\mu)$ is a partition of n not equal to (n), then $\mathrm{pre}_2(2\mu)$ is a partition of $4n$ not equal to $(4n)$, where $2\mu := (2\mu_1, \cdots, 2\mu_\ell)$.

We can thus show that if 4 divides n and $n > 4$, then $\mathrm{pre}_2(n) > 1$. □

4. Complete homogeneous polynomials

We first prove general injectivity for prh_k. Note that Hadelyn et al. [10] proved general injectivity as well, but we offer our own proof, independently derived, in order to highlight the relevance of this methodology.

Theorem 4.1. *For $k \geq 2$, the function prh_k is injective on the set of all partitions.*

Proof. We use a "multiset" argument analogous to Lemma 2.1 in [11].

In particular, take two partitions λ, μ such that $\mathrm{prh}_k(\lambda) = \mathrm{prh}_k(\mu)$. Note the following equality:

$$\ell(\mathrm{prh}_k(\lambda)) = \binom{k + \ell(\lambda) - 1}{k} = \binom{k + \ell(\mu) - 1}{k} = \ell(\mathrm{prh}_k(\mu)).$$

Thus, as the binomial coefficient is strictly increasing in its top parameter, it follows that $\ell(\lambda) = \ell(\mu)$.

Furthermore, note $\lambda_1^k = \mu_1^k$, as they are both maximum elements (there can exist multiple maximum elements) in $\mathrm{prh}_k(\lambda)$ and $\mathrm{prh}_k(\mu)$, respectively. Thus, $\lambda_1 = \mu_1$.

We now proceed by induction on i, the index of the partitions. Our base case is $\lambda_1 = \mu_1$. Now, suppose $\lambda_i = \mu_i$ for all $1 \leq i \leq k'$. Consider the multiset $M_{k,k'}(\lambda) := \{\{\lambda_1^{p_1}\lambda_2^{p_2}\cdots\lambda_{k'}^{p_{k'}} : p_1 + \cdots + p_{k'} = k, p_i \geq 0\ \forall i\}\}$. Thus, by the inductive hypothesis, $M_{k,k'}(\lambda) = M_{k,k'}(\mu)$.

Thus, it follows that

$$\mathrm{prh}_k(\lambda) \setminus M_{k,k'}(\lambda) = \mathrm{prh}_k(\mu) \setminus M_{k,k'}(\mu)$$

where $\setminus$ denotes multiset difference. Hence, the maximum elements in both of these sets are equal. One can easily see that a maximum element in both sets is $\lambda_1^{k-1}\lambda_{k'+1}$ and $\mu_1^{k-1}\mu_{k'+1}$, respectively. Therefore, $\lambda_{k'+1} = \mu_{k'+1}$, as desired.

Therefore, by the principle of strong induction, we conclude $\lambda_i = \mu_i$ for all $1 \leq i \leq \ell$, so $\lambda = \mu$. Thus, prh_k is injective, as desired. □

Remark 4.1. *One may note the relative ease of this proof compared to those about* pre_k. *Particularly, as* h_k *has monomials of one variable, it is much easier to extract information about* λ *from* $\mathrm{prh}_k(\lambda)$ *compared to* $\mathrm{pre}_k(\lambda)$.

To highlight this, we demonstrate an algorithmic approach when $m_\lambda(1) > 0$.

Proposition 4.1. *For any partition* λ *such that* $m_\lambda(1) > 0$, *we can extract* λ *from* $\nu = \mathrm{prh}_k(\lambda)$.

Proof. Using a "stars and bars" argument, it follows that

$$m_\nu(1) = \binom{m_\lambda(1) + k - 1}{k}.$$

Thus, as $m_\lambda(1) > 0$, and due to the strictly increasing nature of the binomial coefficient, it follows that we can determine $m_\lambda(1)$ from $m_\nu(1)$.

Now, consider a prime p. By logic similar to that used above, it follows

$$m_\nu(p) = m_\lambda(p) \cdot \binom{m_\lambda(1) + k - 2}{k - 1}.$$

Thus, we can extract $m_\lambda(p)$ from $m_\nu(p)$ for all prime p.

We now proceed by induction. The base case is that we can retrieve $m_\lambda(1)$ from $m_\nu(1)$ and $m_\lambda(p)$ from $m_\nu(p)$ for all primes p, as demonstrated before. Now, fix a composite y, and assume that for all $x < y$, we can determine $m_\lambda(x)$ from ν. Analogous to the argument made in Ballantine et al. [4], let D_y be the collection of tuples $(d_1^{b_1}, d_2^{b_2}, \ldots, d_t^{b_t})$ satisfying the following properties:

- $1 < d_1 < d_2 < \cdots < d_t < y$
- $b_1, \ldots, b_t > 0$
- $b_1 + \cdots + b_t \leq k$
- $y = d_1^{b_1} d_2^{b_2} \ldots d_t^{b_t}$

We obtain the following recursive relation:

$$\begin{aligned} m_\nu(y) = m_\lambda(y) \cdot &\binom{m_\lambda(1) + k - 2}{k - 1} \\ &+ \sum_{(d_1^{b_1}, \ldots, d_t^{b_t}) \in D_y} \binom{m_\lambda(1) + k - 1 - (b_1 + b_2 + \cdots + b_t)}{m_\lambda(1) - 1} \prod_{i=1}^{t} \binom{m_\lambda(d_i) + b_i - 1}{b_i}. \end{aligned}$$

Hence, we can determine $m_\lambda(y)$ from $m_\nu(y)$. Thus, by the principle of strong induction, we can determine $m_\lambda(i)$ from ν for all positive integers i. Therefore, we can construct λ from $\nu = \mathrm{prh}_k(\lambda)$, as desired. □

Remark 4.2. *If* $m_\lambda(1) = 0$, *by Theorem 1.3, there does exist an algorithm as well; compare with [10]. However, it would be of a different nature than the one above, as* $m_\lambda(1) > 0$ *is a necessary condition for it to work.*

5. Schur polynomials

5.1 Main results

A natural question that arises is why we choose to consider injectivity for $\mathcal{P}(n)$ rather than the general case of $\mathcal{P}$. To illustrate this necessity, we give rexamples for which $\mathrm{prs}_{\lambda'}$ is not injective on $\mathcal{P}$.

Theorem 5.1. *Consider a partition λ' whose Young diagram is a rectangle with at least 2 rows. Then $\mathrm{prs}_{\lambda'}$ is not injective on $\mathcal{P}$. It follows that $\mathrm{prs}_{\lambda'}$ is not injective on $\mathcal{P}$ for infinitely many partitions λ'.*

Proof. Let $\lambda' = k^n$ where k, n are positive integers and $n \geq 2$. Using the Jacobi-Trudi identities, it follows that

$$s_{\lambda'}(x_1, \ldots, x_n) = (x_1 \cdots x_n)^k.$$

Hence, consider the partitions $\lambda = (4, 1, \ldots, 1)$ and $\mu = (2, 2, 1, \ldots, 1)$, where λ has $(n-1)$ 1's and μ has $(n-2)$ 1's. Thus, $s_{\lambda'}(\lambda) = 4^k = 2^k \cdot 2^k = s_{\lambda'}(\mu)$, so $\mathrm{prs}_{\lambda'}$ is not injective, as desired. □

Note that $e_k = s_{(1^k)}$ is indexed by a rectangular partition with at least 2 rows if $k \geq 2$, so Theorem 5.1 applies to the setting of pre_k. This is why our pre_k injectivity results, such as Theorem 1.2, are stated over $\mathcal{P}(n)$ rather than just $\mathcal{P}$.

Generalizing the idea from Theorem 3.1, we can show the necessity of working over partitions of length strictly greater than $\ell(\lambda')$:

Proposition 5.1. *Consider a partition λ' whose Young diagram is a rectangle with at least 3 rows. Let $\ell \geq 3$ be the length of λ' and let n be an integer greater than $\ell + 15$. Then $\mathrm{prs}_{\lambda'}$ is noninjective on $\mathcal{P}_\ell(n)$.*

Proof. Let $\lambda' = k^\ell$ where k, ℓ are positive integers with $\ell \geq 3$. Recall $s_{\lambda'}(x_1, \ldots, x_\ell) = (x_1 \cdots x_\ell)^k$. Let $n > \ell + 15$. By Theorem 3.1, there exists distinct $\lambda, \mu \in \mathcal{P}(n)$ of length ℓ such that $\mathrm{pre}_\ell(\lambda) = \mathrm{pre}_\ell(\mu)$.

Thus $\lambda_1 \lambda_2 \cdots \lambda_\ell = \mu_1 \mu_2 \ldots \mu_\ell$. Therefore, it follows that

$$s_{\lambda'}(\lambda) = (\lambda_1 \cdots \lambda_\ell)^k = (\mu_1 \cdots \mu_\ell)^k = s_{\lambda'}(\mu).$$

Hence, $\mathrm{prs}_{\lambda'}$ is not injective, as desired. □

We now focus on proving Theorem 1.4, using an argument similar to [11].

Theorem 1.4. *For all positive integers $n \geq 2$, the map $prs_{(2,1)}$ is injective on the set of partitions of n with at least two parts.*

We first prove the following lemma, analogous to Lemma 2.1 in [11]:

Lemma 5.1. *Let $n \geq 2$ and let λ, μ be partitions of n with at least two parts. If $\mathrm{prs}_{(2,1)}(\lambda) = \mathrm{prs}_{(2,1)}(\mu)$ and $\lambda_1 = \mu_1$, then $\lambda = \mu$.*

Proof. Recall from Example 2.4 that

$$\ell(\mathrm{prs}_{(2,1)}(\lambda)) = 2 \cdot \binom{\ell(\lambda)}{2} + 2 \cdot \binom{\ell(\lambda)}{3} = 2 \cdot \binom{\ell(\mu)}{2} + 2 \cdot \binom{\ell(\mu)}{3} = \ell(\mathrm{prs}_{(2,1)}(\mu)).$$

Thus, due to the strictly increasing nature of the binomial coefficient in its top parameter, it follows that $\ell(\lambda) = \ell(\mu)$.

Using an inductive approach, it suffices to show that if $\lambda_i = \mu_i$ for $1 \leq i \leq k-1$, then $\lambda_k = \mu_k$ for $k \geq 2$.

By Example 2.4, that the Schur polynomial of $(2, 1)$ over ℓ variables can be rewritten as the following:

$$s_{(2,1)}(x_1, x_2, \ldots, x_\ell) = \sum_{1 \leq i < j \leq \ell} (x_i^2 x_j + x_i x_j^2) + 2 \sum_{1 \leq i < j < k \leq \ell} x_i x_j x_k.$$

By the inductive hypothesis, the summands of $s_{(2,1)}(\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$ and $s_{(2,1)}(\mu_1, \mu_2, \ldots, \mu_{k-1})$ are the same.

Thus, consider the summands of the polynomial

$$s_{(2,1)}(x_1, x_2, \ldots, x_\ell) - s_{(2,1)}(x_1, x_2, \ldots, x_{k-1})$$

for $k \geq 3$. When evaluated at λ and μ, we can see that maximum elements would be $\lambda_1^2 \lambda_k$ and $\mu_1^2 \mu_k$. Thus, as $\mathrm{prs}_{(2,1)}(\lambda) = \mathrm{prs}_{(2,1)}(\mu)$ and by the inductive hypothesis, this means $\lambda_k = \mu_k$ if $k \geq 3$, as desired.

Thus, it suffices to show $\lambda_1 = \mu_1$ implies $\lambda_2 = \mu_2$. Note that maximum elements of $\mathrm{prs}_{(2,1)}(\lambda)$ and $\mathrm{prs}_{(2,1)}(\mu)$ are $\lambda_1^2 \lambda_2$ and $\mu_1^2 \mu_2$, respectively. Thus, as these two are equal, if $\lambda_1 = \mu_1$, it follows that $\lambda_2 = \mu_2$, as desired. □

We now finish the main proof of Theorem 1.4.

Proof of Theorem 1.4. Recall the following fact [11]:

$$\lambda_1 = \lim_{p\to\infty} \left(\sum_{i=1}^{\ell} \lambda_i^p \right)^{\frac{1}{p}}. \tag{3}$$

We claim that $\sum_{i=1}^{\ell} \lambda_i^{3^a} = \sum_{i=1}^{\ell} \mu_i^{3^a}$ for all $a \geq 0$, which would, by Equation 3, imply $\lambda_1 = \mu_1$. We do so through induction on a. Our base case is $a = 0$, where we have $\sum_{i=1}^{\ell} \lambda_i = \sum_{i=1}^{\ell} \mu_i = n$, which is true. Now, for the inductive step, assume $\sum_{i=1}^{\ell} \lambda_i^{3^a} = \sum_{i=1}^{\ell} \mu_i^{3^a}$. Therefore,

$$\begin{aligned}\sum_{1\leq i\leq \ell} \lambda_i^{3^{a+1}} &= \left(\sum_{1\leq i\leq \ell} \lambda_i^{3^a} \right)^3 - 3 \sum_{1\leq i\neq j\leq \ell} (\lambda_i^2 \lambda_j)^{3^a} - 6 \sum_{1\leq i<j<k\leq \ell} (\lambda_i \lambda_j \lambda_k)^{3^a} \\ &= \left(\sum_{1\leq i\leq \ell} \mu_i^{3^a} \right)^3 - 3 \sum_{1\leq i\neq j\leq \ell} (\mu_i^2 \mu_j)^{3^a} - 6 \sum_{1\leq i<j<k\leq \ell} (\mu_i \mu_j \mu_k)^{3^a} = \sum_{1\leq i\leq \ell} \mu_i^{3^{a+1}}.\end{aligned}$$

Therefore, by induction, we conclude that $\lambda_1 = \mu_1$, so $\mathrm{prs}_{(2,1)}$ is injective on $\mathcal{P}(n)$. □

Remark 5.1. *The reason this argument works so well is that $s_{(2,1)}$ can be written solely in terms of power sums that are powers of 3:*

$$s_{(2,1)} = \frac{1}{3}(p_1^3 - p_3).$$

Thus, this inductive method works perfectly, similar to the one used in [11]. However, the method does not readily extend to general λ'.

We now provide a lemma similar in nature to Theorem 3.2. In this particular proof, n is not being used as the sum of the parts of λ, but rather as an integer greater than or equal to λ_1.

Lemma 5.2. *For any partition λ and integer $n \geq \lambda_1$, we have*

$$(x_1 x_2 \dots x_m)^n \cdot s_\lambda(x_1^{-1}, \dots, x_m^{-1}) = s_{f_n(\lambda)}(x_1, \dots, x_m)$$

where $f_n(\lambda)$ is the partition whose ith part is $f_n(\lambda)_i := n - \lambda_{m+1-i}$. We throw out parts of $f_n(\lambda)$ of size zero. If the length of λ is less than m, we pad λ with zeroes until we get a length m tuple.

Proof. We can use the Jacobi bialternant formula $s_\lambda = \frac{\det(x_j^{\lambda_i+m-i})_{i,j=1}^m}{\det(x_j^{m-i})_{i,j=1}^m}$. We look at the numerator and denominator of this expression separately. Note, in the denominator, if we were to replace x_i with its reciprocal, it would become $\det(x_j^{i-m})_{i,j=1}^m$. Thus, it follows that

$$(x_1 \dots x_m)^{m-1} \cdot \det(x_j^{i-m})_{i,j=1}^m = \det(x_j^{i-1})_{i,j=1}^m = (-1)^{\binom{m}{2}} \cdot \det(x_j^{m-i})_{i,j=1}^m.$$

Likewise, if we replaced x_i with its reciprocal in the numerator and multiplied it by $(x_1 \dots x_m)^{n+m-1}$, we would get the following equality:

$$\begin{aligned}(x_1 \dots x_m)^{n+m-1} \det(x_j^{-\lambda_i-m+i})_{i,j=1}^m &= \det(x_j^{n-\lambda_i+i-1})_{i,j=1}^m \\ &= \det(x_j^{f_n(\lambda)_{m+1-i}+i-1})_{i,j=1}^m \\ &= (-1)^{\binom{m}{2}} \det(x_j^{f_n(\lambda)_i+m-i})_{i,j=1}^m.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}(x_1 \dots x_m)^n \cdot s_\lambda(x_1^{-1}, \dots, x_m^{-1}) &= \frac{(x_1 \dots x_m)^{n+m-1} \det(x_j^{-\lambda_i-m+i})_{i,j=1}^m}{(x_1 \dots x_m)^{m-1} \det(x_j^{-m+i})_{i,j=1}^m} \\ &= \frac{(-1)^{\binom{m}{2}} \det(x_j^{f_n(\lambda)_i+m-i})_{i,j=1}^m}{(-1)^{\binom{m}{2}} \det(x_j^{m-i})_{i,j=1}^m} \\ &= s_{f_n(\lambda)}(x_1, \dots, x_m),\end{aligned}$$

as desired. □

Note, by the symmetry of $s_{\lambda'}$, if $\mathrm{prs}_{\lambda'}(\lambda) = \mathrm{prs}_{\lambda'}(\mu)$, then $\prod \lambda_i = \prod \mu_i$. Therefore, it follows that $s_{f_n(\lambda')}(\lambda) = s_{f_n(\lambda')}(\mu)$, potentially allowing us to replace certain partitions by easier-to-access partitions. Hence, by Lemma 5.2, we can make the following statement:

Corollary 5.1. *Let $m \geq 3$ be a positive integer. If $\mathrm{prs}_{(m,m-1,\ldots,1)}(\lambda) = \mathrm{prs}_{(m,m-1,\ldots,1)}(\mu)$ where λ, μ are each of length m, then $\mathrm{prs}_{(m-1,\ldots,1)}(\lambda) = \mathrm{prs}_{(m-1,\ldots,1)}(\mu)$.*

So far, we have been thinking about Schur polynomials from the perspective of the Jacobi-Trudi identities. However, it is helpful to think about it visually, from the perspective of their semi-standard Young tableaux. To demonstrate this, we prove the following theorem, a generalization of Lemma 5.1:

Theorem 5.2. *Let n be a positive integer, and let $\lambda, \mu \in \mathcal{P}(n)$ be partitions of the same length ℓ. Let λ' be a nonempty partition for which $s_{\lambda'}(\lambda)$ and $s_{\lambda'}(\mu)$ are defined and nonzero, and let k be the length of λ'. Assume $\mathrm{prs}_{\lambda'}(\lambda) = \mathrm{prs}_{\lambda'}(\mu)$ and $\lambda_i = \mu_i$ for all $1 \leq i \leq k-1$. Then $\lambda = \mu$.*

Proof. Consider any integer k' with $k \leq k' \leq \ell$, which we let be our inductive index. Using an inductive approach, it suffices to show if $\lambda_i = \mu_i$ for $1 \leq i \leq k'-1$, then $\lambda_{k'} = \mu_{k'}$.

First, note that a maximum element of $\mathrm{prs}_{\lambda'}(\lambda)$ is generated by the SSYT of λ' such that the first row consists of only 1's, the second row consists of only 2's, and so on with the kth row consisting of only k's. Thus, a maximum element of $\mathrm{prs}_{\lambda'}(\lambda)$ is $\lambda_1^{\lambda'_1}\lambda_2^{\lambda'_2}\cdots\lambda_{k-1}^{\lambda'_{k-1}}\lambda_k^{\lambda'_k}$. Likewise, a maximum element of $\mathrm{prs}_{\lambda'}(\mu)$ is $\mu_1^{\lambda'_1}\mu_2^{\lambda'_2}\cdots\mu_{k-1}^{\lambda'_{k-1}}\mu_k^{\lambda'_k}$. Thus, as both of these expressions are equal, and by the hypothesis that for all $1 \leq i \leq k-1$, we have $\lambda_i = \mu_i$, we can determine that $\lambda_k = \mu_k$.

Now, by the inductive hypothesis, it follows that the summands of

$$s_{\lambda'}(\lambda_1, \lambda_2, \ldots, \lambda_{k'-1}, 0, \ldots, 0)$$

and

$$s_{\lambda'}(\mu_1, \mu_2, \ldots, \mu_{k'-1}, 0, \ldots, 0)$$

are the same. Thus, consider the summands of the polynomial

$$s_{\lambda'}(x_1, x_2, \ldots, x_\ell) - s_{\lambda'}(x_1, x_2, \ldots, x_{k'-1}, 0, \ldots, 0).$$

Consider the SSYT of λ' such that the first row consists of only 1's, the second row consists of only 2's, and so on, except with the kth row consisting of $\lambda'_k - 1$ k's and then one k'. It follows that this particular SSYT represents a maximum element of $s_{\lambda'}(x_1, x_2, \ldots, x_\ell) - s_{\lambda'}(x_1, x_2, \ldots, x_{k'-1}, 0, \ldots, 0)$. In particular, when evaluated at λ and μ, these maximum elements are equal:

$$\lambda_1^{\lambda'_1}\lambda_2^{\lambda'_2}\cdots\lambda_{k-1}^{\lambda'_{k-1}}\lambda_k^{\lambda'_k-1}\lambda_{k'} = \mu_1^{\lambda'_1}\mu_2^{\lambda'_2}\cdots\mu_{k-1}^{\lambda'_{k-1}}\mu_k^{\lambda'_k-1}\mu_{k'}.$$

Thus, by the inductive hypothesis, it follows that $\lambda_k^{\lambda'_k-1}\lambda_{k'} = \mu_k^{\lambda'_k-1}\mu_{k'}$. As $\lambda_k = \mu_k$, it then follows that $\lambda_{k'} = \mu_{k'}$, as desired. □

As shown by the above proof, the geometry of an ordinary partition can be quite restrictive, particularly by the column-strict condition in the definition of a SSYT. Therefore, we are motivated to look beyond ordinary partitions into skew partitions.

5.2 Skew partitions

Theorem 1.5. *For any nonempty skew partition λ'/μ' in which there is at most one square in each column and at most one square in each row, $prs_{\lambda'/\mu'}$ is injective on $P(n)$ where n is at least the number of squares in λ'/μ'.*

Proof. Let λ'/μ' be a skew partition in which there is at most one square in each column and at most one square in each row. Suppose λ'/μ' is of size m. Thus, over $k \geq 1$ variables, it follows that

$$s_{\lambda'/\mu'}(x_1, x_2, \ldots, x_k) = (x_1 + x_2 + \cdots + x_k)^m.$$

Due to the similarity in structure of this polynomial with the complete homogeneous polynomials, we can readily apply the same argument used in Theorem 1.3, allowing us to prove the injectivity of $\mathrm{prs}_{\lambda'/\mu'}$. □

Corollary 5.2. *For all positive integers n and all $m \geq 3$, the function $\mathrm{prs}_{(m,m-1,\ldots,1)/(m-1,m-2,\ldots,1)}$ is injective on $\mathcal{P}(n)$.*

As shown by the above arguments, the combinatorial and determinantal formulas of s_λ tell us a lot about $\mathrm{prs}_{\lambda'}$.

5.3 Applications

Given symmetric functions f, g where $g = \sum_i x^{a^i}$ where each a^i is a degree vector and repeated monomials appear multiple times, the plethysm $f[g]$ is defined to be $f(x^{a_1}, x^{a_2}, ...)$. Li's injectivity result for pre_2 [11] shows that the family of plethysms $\{e_r[e_2](\lambda)\}_{r\geq 1}$, which determines the multiset $\mathrm{pre}_2(\lambda)$, also in fact determines λ and hence $\{e_r(\lambda)\}_{r\geq 1}$ as observed in [4]. However, due to the failure of injectivity for pre_k for $k \geq 3$, the family of plethysms $\{e_r[e_k]\}$ evaluated at the parts of a partition λ does not necessarily determine the entire family $\{e_r\}$ evaluated at λ.

Furthermore, based on the result that $\mathrm{prs}_{(2,1)}$ is injective on $\mathcal{P}(n)$, as shown in Theorem 1.4, it follows that the family of plethysms $\{s_\nu[s_{(2,1)}]\}_{\nu\in\mathcal{P}}$ evaluated at λ a partition of n determines λ and hence the original Schur polynomials evaluated at λ.

The problem of checking whether prs_μ is injective on the set $\mathcal{P}(n)$ is equivalent to building a diagonal matrix $M_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_\ell)$, and asking whether M_λ can be retrieved (up to ordering of eigenvalues) from the spectrum, regarded as a multiset, of $\mathbb{S}_\mu(M_\lambda)$, where $\mathbb{S}_\mu(-)$ is the Schur functor, with the additional constraint that the diagonal entries of M_λ are positive integers with sum n.

6. Open Questions

Even though the answer to Question 1.1 is "almost none," there are still many interesting open questions to consider.

Our work and [10] lead us to the following refinement of Question 1.1.

Conjecture 6.1. *For each integer $k \geq 3$, there exists some constant M_k such that* pre_k *fails to be injective on $\mathcal{P}_{>k}(n)$ for all $n \geq M_k$.*

The result of [8], proven independently by us, that $\mathrm{pre}_2(n) = 1$ if and only if $n \in \{1, 2, 4\}$ motivates the following question.

Question 6.1. *For what positive integers a do there exist n and k such that* $\mathrm{pre}_k(n) = a$*?*

Another fruitful direction concerns $\mathrm{prs}_{\lambda'}$. We refine Question 5.3 of [10] as follows.

Conjecture 6.2. *For every two-row partition λ', the function* $\mathrm{prs}_{\lambda'}$ *is injective on $\mathcal{P}(n)$ for every positive integer n on which* $\mathrm{prs}_{\lambda'}$ *is defined.*

The case $\lambda' = (1, 1)$ was shown in [11], but the general proof will probably not use exactly the same techniques. However, proving this conjecture would have a nice plethysm implication analogous to that of $\mathrm{prs}_{(2,1)}$ in Section 5.3, except with a two-row partition instead of $(2, 1)$. It would also be interesting to consider the case of λ' a staircase or hook partition.

For reference, the code for this project can be found in the GitHub repository linked [†].

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