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Restricted Jacobi Permutations

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Abstract: Jacobi permutations, introduced by Viennot in the context of Jacobi elliptic functions, are counted by the Euler numbers E_n appearing in the series expansion $\sec x + \tan x = \sum_{n=0}^{\infty} E_n x^n/n!$. We conduct a systematic study of pattern avoidance in Jacobi permutations, achieving a complete enumeration of Jacobi permutations avoiding a prescribed set of length 3 patterns. In the case of a single pattern restriction, we obtain refined enumerations with respect to several permutation statistics: the number of ascents (or descents), the number of left-to-right minima, and the last letter. Bijections involving certain subfamilies of binary trees and Dyck paths, as well as generating function techniques, play important roles in our proofs.

Keywords: Alternating permutations; Euler numbers; Jacobi permutations; Pattern avoidance; Permutation statistics
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1. Introduction

Let S be a finite set of positive integers. A *permutation* $\pi = \pi_1\pi_2\cdots\pi_n$ of S—where $n = |S|$—is a linear arrangement of the elements of S. For example, $\pi = 3174$ is a permutation of $S = \{1,3,4,7\}$. We think of permutations as words over the alphabet S without repetition, so the π_k are called *letters* of π and n the *length* of π. Let $|\pi|$ denote the length of π, so that $|\pi| = |S|$ whenever π is a permutation of S. If S is the empty set, then the only permutation of S is the empty word ε. The symmetric group of permutations of $[n] := \{1,2,\dots,n\}$ is denoted by $\mathfrak{S}_n$.

The *Euler numbers* E_n, defined as the coefficients in the series expansion

$$\sec x + \tan x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!},$$

have many combinatorial interpretations, including several interesting families of permutations. We say that π is *up-down alternating* (or simply *up-down*) if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Let $\mathfrak{U}_n$ be the set of up-down permutations in $\mathfrak{S}_n$. It is a classical result of André [1] that $|\mathfrak{U}_n| = E_n$ for all $n \geq 0$. By symmetry, E_n also counts *down-up* (*alternating*) permutations $\pi \in \mathfrak{S}_n$, which instead satisfy $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. Additionally, the Euler numbers are known to count André permutations [6], a variant of André permutations called simsun permutations [26], as well as the subject of the present paper: Jacobi permutations [28].

Given a permutation π of S and a letter x of π, let $\rho_\pi(x)$ be the maximal consecutive subword of π consisting of letters immediately to the right of x which are all larger than x. As an example, Table 1 shows $\rho_\pi(x)$ for each letter of $\pi = 417926$.

x	4	1	7	9	2	6
$\rho_\pi(x)$	ε	7926	9	ε	6	ε

Table 1: The subwords $\rho_\pi(x)$ for the permutation $\pi = 417926$.

We say that π is *Jacobi* if $|\rho_\pi(x)|$ is even for all $x \in S$. Observe that $\pi = 417926$ is not Jacobi because $|\rho_\pi(7)| = |\rho_\pi(2)| = 1$, but $\pi = 416972$ is Jacobi.

Jacobi permutations can also be defined recursively [28, Lemme 4] in the following way. First, the empty permutation is Jacobi. If S is nonempty, then let $y = \min S$ be the smallest element of S, and let us decompose a permutation π of S as $\pi = \alpha y \beta$; here, α is the subword of letters to the left of y and β is the subword of letters to the right of y. Then, π is Jacobi if both α and β are Jacobi and $|\beta|$ is even. We will use these two definitions of Jacobi permutations interchangeably.

Let $\mathfrak{J}_n$ denote the set of Jacobi permutations in $\mathfrak{S}_n$. See Table 2 for all permutations in $\mathfrak{J}_n$ up to $n = 5$.

$\mathfrak{J}_0$	$\mathfrak{J}_1$	$\mathfrak{J}_2$	$\mathfrak{J}_3$	$\mathfrak{J}_4$
ε	1	21	321 132	4321 2431 4132 3142 2143

$\mathfrak{J}_5$							
54321	35421	52431	42531	32541	54132	53142	43152
52143	42153	32154	15432	13542	15243	14253	13254

Table 2: Permutations in $\mathfrak{J}_n$ up to $n = 5$.

The *standardization* $\operatorname{std}(\pi)$ of a permutation π of length n is the permutation in $\mathfrak{S}_n$ obtained by replacing the smallest letter of π by 1, the second smallest by 2, and so on. For example, we have $\operatorname{std}(417926) = 315624$. Note that π is Jacobi if and only if its standardization $\operatorname{std}(\pi)$ is Jacobi, a fact that we will tacitly use throughout this paper.

Jacobi permutations were introduced by Viennot [28] in the context of Jacobi elliptic functions.* Viennot studied a sequence of polynomials $J_n(k^2)$ appearing in the series expansions of the functions sn, cn, and dn, giving them a combinatorial interpretation as encoding the distribution of a certain statistic over Jacobi permutations. He gave two proofs that Jacobi permutations are counted by Euler numbers, including one utilizing a bijection between Jacobi and alternating permutations. However, unlike alternating, André, and simsun permutations, very little seems to have been done with Jacobi permutations since their introduction.

Pattern avoidance in permutations has received widespread attention in recent decades, and a major theme within its study is the enumeration of special families of pattern-avoiding permutations. There is a sizable literature on pattern-avoiding alternating permutations—see, e.g., [2,12–15,29]—and pattern avoidance has also been studied in simsun permutations [3]. We pursue a variation of this theme by studying pattern avoidance in Jacobi permutations.

1.1 Pattern avoidance and permutation statistics

Given permutations σ and π, an *occurrence* of σ in π is a subword of π whose standardization is σ, and we say that π *avoids* σ (as a *pattern*)—and that π is *σ-avoiding*—if π does not contain an occurrence of σ. For example, 164 is an occurrence of $\sigma = 132$ in $\pi = 315624$, but $\pi = 315624$ avoids $\sigma = 321$.

Let $\mathfrak{S}_n(\sigma)$ denote the set of permutations in $\mathfrak{S}_n$ which avoid σ. As seen above, we have $315624 \in \mathfrak{S}_6(321)$. More generally, given a set Π of patterns, let $\mathfrak{S}_n(\Pi)$ denote the set of permutations in $\mathfrak{S}_n$ which avoid every pattern in Π. It is well known that for all $\sigma \in \mathfrak{S}_3$ and all $n \geq 0$, the number of permutations in $\mathfrak{S}_n(\sigma)$ is the nth Catalan number. The systematic study of pattern-avoiding permutations was initiated by Simion and Schmidt [19], who enumerated $\mathfrak{S}_n(\Pi)$ for all length 3 pattern sets $\Pi \subseteq \mathfrak{S}_3$.

Let $\mathfrak{J}_n(\Pi)$ denote the set of Jacobi permutations in $\mathfrak{J}_n$ which avoid every pattern in Π, and let $j_n(\Pi) := |\mathfrak{J}_n(\Pi)|$. We achieve a complete enumeration of the Jacobi avoidance classes $\mathfrak{J}_n(\Pi)$ where $\Pi \subseteq \mathfrak{S}_3$. A summary of these results is provided in Table 3.†

En passant, we find that all permutations π in $\mathfrak{J}_n(213)$, $\mathfrak{J}_n(231)$, and $\mathfrak{J}_n(312)$ are *doubly Jacobi*: both π and their inverse π^{-1} are Jacobi. This leads us to study doubly Jacobi permutations in the other single-pattern Jacobi avoidance classes. Notably, 123-avoiding doubly Jacobi permutations are counted by secondary structure numbers (see Theorem 6.5).

Furthermore, we study the distributions of several permutation statistics over Jacobi avoidance classes. For the following definitions, let π be a permutation of length n.

- We say that $k \in [n-1]$ is an *ascent* of π if $\pi_k < \pi_{k+1}$ and a *descent* of π if $\pi_k > \pi_{k+1}$. Let $\operatorname{asc}(\pi)$ denote the number of ascents of π, and $\operatorname{des}(\pi)$ the number of descents of π.‡
- We say that π_k is a *left-to-right minimum* of π if $\pi_k < \pi_i$ for all $i < k$. Let $\operatorname{lrmin}(\pi)$ denote the number of left-to-right minima of π.

*See also Viennot's survey [27].

†We exclude from Table 3 pattern sets Π for which $j_n(\Pi) \leq 1$, aside from the singleton $\Pi = \{132\}$.

‡Since $\operatorname{asc}(\pi) + \operatorname{des}(\pi) = n - 1$, the value of asc determines that of des and vice versa, but in some cases it will be easier to work with one statistic over the other.

- Let $\mathrm{last}(\pi) := \pi_n$, the *last letter* of π.

For example, if $\pi = 315624$, then the ascents of π are 2, 3, and 5, the descents of π are 1 and 4, the left-to-right minima of π are 3 and 1, and the last letter of π is 4. Thus, we have $\mathrm{asc}(\pi) = 3$, $\mathrm{des}(\pi) = 2$, $\mathrm{lrmin}(\pi) = 2$, and $\mathrm{last}(\pi) = 4$.

Π	$j_n(\Pi)$	Reference	OEIS [20]
$\{213\}$	$\begin{cases} \dfrac{1}{2n+1}\dbinom{3n}{n}, & \text{if } n \text{ is even,} \\ \dfrac{1}{2n+1}\dbinom{3n+1}{n+1}, & \text{if } n \text{ is odd.} \end{cases}$	Theorem 4.1	A047749
$\{312\}$			
$\{231\}$		Theorem 5.3	
$\{123\}$	$\displaystyle\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}\binom{n-k-1}{k}\binom{n}{2k}$	Theorem 6.1	A101785
$\{132\}$	1	Corollary 7.1	A000012
$\{321\}$	$C_{\lfloor n/2 \rfloor}$ (Catalan numbers)	Corollary 7.2	A000108
$\{123, 213\}$	f_{n-1} (Fibonacci numbers)	Theorem 8.1	A000045
$\{123, 231\}$	$1 + \left\lfloor \dfrac{(n-1)^2}{4} \right\rfloor$	Theorem 8.2	A033638
$\{123, 312\}$			
$\{213, 231\}$	$2^{\lfloor (n-1)/2 \rfloor}$	Theorem 8.3	A016116
$\{213, 312\}$			
$\{231, 312\}$		Theorem 8.4	
$\{123, 213, 231\}$	$\left\lceil \dfrac{n}{2} \right\rceil$	Theorem 8.5	A065033
$\{123, 213, 312\}$			
$\{123, 231, 312\}$		Theorem 8.6	
$\{213, 231, 312\}$	$\begin{cases} 1, & \text{if } n \geq 2 \text{ is even,} \\ 2, & \text{if } n \geq 3 \text{ is odd.} \end{cases}$	Theorem 8.7	A000034
$\{123, 213, 231, 312\}$			

Table 3: Summary of results on the enumeration of Jacobi avoidance classes.

We determine the distributions of asc/des, lrmin, and last over $\mathfrak{J}_n(\sigma)$ for each $\sigma \in \mathfrak{S}_3$, as well as over the full set $\mathfrak{J}_n$ of Jacobi permutations. (The only exception is the distribution of last over $\mathfrak{J}_n(123)$, which is left as an open problem.) See Table 4 for an abbreviated summary of our results in this direction. Most of these take the form of a formula for the numbers

$$j^{\mathrm{st}}_{n,k}(\sigma) := |\{\, \pi \in \mathfrak{J}_n(\sigma) : \mathrm{st}(\pi) = k \,\}| \quad \text{and} \quad j^{\mathrm{st}}_{n,k} := |\{\, \pi \in \mathfrak{J}_n : \mathrm{st}(\pi) = k \,\}|$$

(where st is one of the above statistics) or a generating function for these numbers.

The general techniques used here are also applicable to related statistics, such as the number of right-to-left minima, but we shall restrict our attention to the above statistics to keep the scope of this paper manageable.

1.2 Outline

The structure of this paper is as follows:

- We begin in Section 2 by establishing a collection of basic facts about Jacobi permutations which will be used later in this paper, and then we determine the distribution of asc, lrmin, and last over $\mathfrak{J}_n$.

<table>
<tr><th>σ</th><th>asc/des</th><th>lrmin</th><th>last</th></tr>
<tr><td>none</td><td>Theorem 2.1</td><td>Theorem 2.2</td><td>Theorem 2.3</td></tr>
<tr><td>213</td><td rowspan="2">Theorem 4.2</td><td rowspan="2">Theorem 4.3</td><td>Theorem 4.4</td></tr>
<tr><td>312</td><td>Theorem 4.5</td></tr>
<tr><td>231</td><td>Theorem 5.4</td><td>Theorem 5.5</td><td>Theorem 5.6</td></tr>
<tr><td>123</td><td>Theorem 6.4</td><td>Theorem 6.3</td><td>open</td></tr>
<tr><td>132</td><td>trivial</td><td>trivial</td><td>trivial</td></tr>
<tr><td>321</td><td>Corollary 7.3</td><td>Corollary 7.4</td><td>Theorem 7.2</td></tr>
</table>

Table 4: Summary of results on distributions of statistics over $\mathfrak{J}_n(\sigma)$.

- Permutations in $\mathfrak{S}_n(213)$, $\mathfrak{S}_n(312)$, and $\mathfrak{S}_n(231)$ are in bijection with binary trees. In Section 3, we introduce Jacobi trees and dual Jacobi trees, which are obtained by restricting these bijections to Jacobi permutations.
- We prove our results for the avoidance classes $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$ in Section 4, while we study $\mathfrak{J}_n(231)$ in Section 5. Jacobi and dual Jacobi trees are used throughout.
- Section 6 focuses on the avoidance class $\mathfrak{J}_n(123)$. Our main tool here is a variant of a bijection due to Krattenthaler [11] from 123-avoiding permutations to Dyck paths.
- The avoidance classes $\mathfrak{J}_n(132)$ and $\mathfrak{J}_n(321)$ are studied in Section 7, although these turn out to be less interesting: there is only one permutation in $\mathfrak{J}_n(132)$ for each $n \geq 0$, while the permutations in $\mathfrak{J}_n(321)$ are 321-avoiding down-up permutations when n is even, and are in bijection with 321-avoiding down-up permutations when n is odd.
- In Section 8, we enumerate all $\mathfrak{J}_n(\Pi)$ where $\Pi \subseteq \mathfrak{S}_3$ contains at least two patterns.
- We conclude in Section 9 by presenting open problems and conjectures for future study.

We give a few remarks on notation and terminology before continuing. Throughout this paper, S represents a finite set of positive integers. We will write $\mathfrak{S}_S$ for the set of permutations of S, and notations such as $\mathfrak{J}_S$, $\mathfrak{U}_S$, and $\mathfrak{S}_S(\Pi)$ are defined in the analogous way. We occasionally use the term "permutation" to refer only to standardized permutations—i.e., permutations in $\mathfrak{S}_n$—but at other times more generally to permutations of a set S; which one we mean should always be clear from context.

When decomposing permutations, we use lowercase Greek symbols for subwords and lowercase Latin symbols for letters. For example, if we write $\pi = \alpha x \tau y \beta$, then x and y are letters of π, while α is the subword of all letters appearing before x, τ is the subword of all letters appearing between x and y, and β is the subword of all letters appearing after y.

Finally, given a specific pattern set Π, we will omit the curly braces enclosing the elements of Π when writing out $\mathfrak{J}_n(\Pi)$ and similar notations. For example, we write $\mathfrak{J}_n(123, 213)$ as opposed to $\mathfrak{J}_n(\{123, 213\})$.

2. Jacobi permutations without restrictions

Before imposing any pattern avoidance restrictions, let us prove some basic facts about Jacobi permutations and study the distributions of the statistics asc, lrmin, and last over $\mathfrak{J}_n$.

2.1 Basic facts about Jacobi permutations

Lemma 2.1. *Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{J}_S$.*

(a) *Any suffix of π is Jacobi. That is, $\pi_k \pi_{k+1} \cdots \pi_n$ is Jacobi for each $k \in [n]$.*

(b) *If y is larger than every element of S, then $y\pi$ is Jacobi.*

(c) *If $\pi = y_1 \tau^{(1)} y_2 \tau^{(2)} \cdots y_m \tau^{(m)}$ where $y_1, y_2, \ldots, y_m$ are the left-to-right minima of π, then $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(m)}$ are Jacobi permutations of even length.*

(d) *If $n \geq 2$, then $\pi_{n-1} > \pi_n$.*

$n\backslash k$	0	1	2	3	4
0	1				
1	1				
2	1				
3	1	1			
4	1	4			
5	1	11	4		
6	1	26	34		
7	1	57	180	34	
8	1	120	768	496	
9	1	247	2904	4288	496

Table 5: The numbers $j^{\mathrm{asc}}_{n,k}$ up to $n = 9$.

By iterating Lemma 2.1 (b), we see that the decreasing permutation $n \cdots 21$ is Jacobi for every $n \geq 1$.

Proof. Let $\beta = \pi_k \pi_{k+1} \cdots \pi_n$. We have $\rho_\beta(\pi_i) = \rho_\pi(\pi_i)$ for each $k \leq i \leq n$, and since π is Jacobi, all of these $\rho_\pi(\pi_i)$ have even length. Therefore, β is Jacobi, which proves (a).

To prove (b), let $\tau = y\pi$. Then $\rho_\tau(y) = \varepsilon$ because y is larger than all letters of π, whereas $\rho_\tau(\pi_i) = \rho_\pi(\pi_i)$ for each $i \in [n]$. Again, all of these $\rho_\pi(\pi_i)$ have even length, so τ is Jacobi.

For (c), fix $k \in [m]$. Since $y_1, y_2, \ldots, y_m$ are the left-to-right minima of π, we have that $\rho_\pi(y_k) = \tau^{(k)}$ and $\rho_\pi(x) = \rho_{\tau^{(k)}}(x)$ for each letter x of $\tau^{(k)}$. Because π is Jacobi, the former implies that $\tau^{(k)}$ is of even length, while the latter implies that $\tau^{(k)}$ is Jacobi.

Finally, if $\pi_{n-1} < \pi_n$, then we would have $\rho_\pi(\pi_{n-1}) = \pi_n$ which is of length 1; this contradicts π being Jacobi. Thus, (d) is proven. □

Lemma 2.2. *Let $\pi = \alpha y \beta$ where y is a left-to-right minimum of π. If α and $y\beta$ are Jacobi, then π is Jacobi.*

Proof. Let x be an arbitrary letter of π. If x is in α, then $\rho_\pi(x) = \rho_\alpha(x)$ because y is a left-to-right minimum of π. Otherwise, $\rho_\pi(x) = \rho_{y\beta}(x)$. Since α and $y\beta$ are Jacobi, $\rho_\alpha(x)$ has even length for every letter x in α, and $\rho_{y\beta}(x)$ has even length for every letter x in $y\beta$. Hence, $\rho_\pi(x)$ has even length for every letter x of π. □

Mirroring the definition of left-to-right minimum, π_k is called a *right-to-left minimum* of π if $\pi_k < \pi_i$ for all $i > k$.

Lemma 2.3. *Let $\pi = \alpha y \beta$ where y is a right-to-left minimum of π. If αy and β are Jacobi and β is of even length, then π is Jacobi.*

Proof. Since αy and β are Jacobi, $\rho_{\alpha y}(x)$ has even length for every letter x in αy, and similarly with $\rho_\beta(x)$. Now, let x be an arbitrary letter of π. Consider the following cases:

- If x is in β, then $\rho_\pi(x) = \rho_\beta(x)$, which has even length.
- If $x = y$, then because y is a right-to-left minimum of π, we have $\rho_\pi(x) = \beta$, which is of even length.
- If $\rho_{\alpha y}(x)$ contains y, then $\rho_\pi(x) = \rho_{\alpha y}(x)\beta$ because y is a right-to-left minimum of π. Since $\rho_{\alpha y}(x)$ and β are both of even length, the same is true of $\rho_\pi(x)$.
- Otherwise, if x is a letter in α and $\rho_{\alpha y}(x)$ contains y, then $\rho_\pi(x) = \rho_{\alpha y}(x)$, which has even length.

Therefore, π is Jacobi. □

2.2 Ascents

We now study the distribution of several statistics over $\mathfrak{J}_n$, beginning with the number of ascents. Let

$$J^{\mathrm{asc}}(t, x) := \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{J}_n} t^{\mathrm{asc}(\pi)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} j^{\mathrm{asc}}_{n,k} t^k \frac{x^n}{n!}.$$

be the bivariate generating function for Jacobi permutations with respect to ascents. See Table 5 for the coefficients of this generating function up to $n = 9$.

Theorem 2.1. *We have $J^{\mathrm{asc}}(t, x) = 1 + \dfrac{2(e^{rx} - 1)}{1 + r + (r - 1)e^{rx}}$, where $r = \sqrt{1 - 2t}$.*

Proof. Let

$$J_{\mathrm{e}}^{\mathrm{asc}} = J_{\mathrm{e}}^{\mathrm{asc}}(t,x) := \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{J}_{2n}} t^{\mathrm{asc}(\pi)} \frac{x^n}{n!} \quad \text{and} \quad J_{\mathrm{o}}^{\mathrm{asc}} = J_{\mathrm{o}}^{\mathrm{asc}}(t,x) := \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{J}_{2n+1}} t^{\mathrm{asc}(\pi)} \frac{x^n}{n!}.$$

We claim that $J_{\mathrm{e}}^{\mathrm{asc}}$ and $J_{\mathrm{o}}^{\mathrm{asc}}$ satisfy the following system of differential equations:

$$\frac{\partial J_{\mathrm{e}}^{\mathrm{asc}}}{\partial x} = J_{\mathrm{o}}^{\mathrm{asc}} + tJ_{\mathrm{o}}^{\mathrm{asc}}(J_{\mathrm{e}}^{\mathrm{asc}} - 1) \tag{1}$$

$$\frac{\partial J_{\mathrm{o}}^{\mathrm{asc}}}{\partial x} = J_{\mathrm{e}}^{\mathrm{asc}} + tJ_{\mathrm{e}}^{\mathrm{asc}}(J_{\mathrm{e}}^{\mathrm{asc}} - 1). \tag{2}$$

To see why these equations hold, first note that every nonempty Jacobi permutation of even length can be written as $\pi = \alpha 1 \beta$ where α and β are Jacobi, α is of odd length, and β is of even length. When β is empty, we have $\mathrm{asc}(\pi) = \mathrm{asc}(\alpha)$, so this case contributes the term $J_{\mathrm{o}}^{\mathrm{asc}}$ to (1). When β is nonempty, we have $\mathrm{asc}(\pi) = \mathrm{asc}(\alpha) + \mathrm{asc}(\beta) + 1$, which results in the contribution $tJ_{\mathrm{o}}^{\mathrm{asc}}(J_{\mathrm{e}}^{\mathrm{asc}} - 1)$. Thus we have (1), and (2) is obtained using the same reasoning except that both α and β are of even length.

Solving the above system with the initial conditions $J_{\mathrm{e}}^{\mathrm{asc}}(t,0) = 1$ and $J_{\mathrm{o}}^{\mathrm{asc}}(t,0) = 0$, adding the solutions, and simplifying yields the desired expression for $J^{\mathrm{asc}}(t,x)$. ☐

The generating function $J^{\mathrm{asc}}(t,x)$ has appeared in an equivalent form in the study of André permutations and André polynomials. Following Foata and Han [7], *André permutations* (*of the first kind*) are defined recursively as follows. First, the empty permutation is André. If $\pi \in \mathfrak{S}_S$ where S is nonempty, then write $\pi = \alpha y \beta$ where $y = \min S$; then π is André if both α and β are André, and if the largest letter in the concatenation $\alpha\beta$ is a letter of β. Several equivalent definitions of André permutation can be found in the literature, e.g., in [7, Definitions 2.1 and 2.2]. Let $\mathfrak{A}_n$ be the set of André permutations in $\mathfrak{S}_n$.

Foata and Schützenberger [6] introduced a family $D_n(s,t)$ of polynomials—called *André polynomials*—whose $s = 1$ evaluation has exponential generating function

$$\sum_{n=0}^{\infty} D_n(1,t) \frac{x^n}{n!} = \frac{r(1 + we^{rx})}{1 - we^{rx}} \tag{3}$$

where $r = \sqrt{1-2t}$ and $w = (1-r)/(1+r)$. In particular, they showed that these polynomials count André permutations by descents:

$$D_n(1,t) = \sum_{\pi \in \mathfrak{A}_n} t^{\mathrm{des}(\pi)+1} \tag{4}$$

for all $n \geq 1$. In fact, because

$$\frac{1}{t}\left(\frac{r(1+we^{rx})}{1-we^{rx}} - 1\right) = \frac{2(e^{rx}-1)}{1+r+(r-1)e^{rx}},$$

it follows from Theorem 2.1, Equation (3), and Equation (4) that asc has the same distribution over $\mathfrak{J}_n$ as des does over $\mathfrak{A}_n$. We state this as a corollary below.

Corollary 2.1. *For all $n \geq 0$, the distribution of* asc *over $\mathfrak{J}_n$ is equal to the distribution of* des *over $\mathfrak{A}_n$.*

2.3 Left-to-right minima

Our next statistic of interest is the number of left-to-right minima. Let

$$J^{\mathrm{lrmin}}(t,x) := \sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{J}_n} t^{\mathrm{lrmin}(\pi)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} j_{n,k}^{\mathrm{lrmin}} t^k \frac{x^n}{n!};$$

see Table 6 for the first several coefficients. This generating function has a remarkably elegant expression which we state below.

Theorem 2.2. *We have $J^{\mathrm{lrmin}}(t,x) = (\sec x + \tan x)^t$.*

While this theorem can be proven by solving a system of differential equations as we did for ascents, we shall instead take a more direct approach utilizing the exponential formula [24, Section 5.1].

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	1	0	1						
4	0	0	4	0	1					
5	0	5	0	10	0	1				
6	0	0	40	0	20	0	1			
7	0	61	0	175	0	35	0	1		
8	0	0	768	0	560	0	56	0	1	
9	0	1385	0	4996	0	1470	0	84	0	1

Table 6: The numbers $j_{n,k}^{\mathrm{lrmin}}$ up to $n = 9$.

Proof. Recall from Lemma 2.1 (c) that every $\pi \in \mathfrak{J}_n$ can be written as

$$\pi = y_1\tau^{(1)}y_2\tau^{(2)}\cdots y_m\tau^{(m)}$$

where $y_1, y_2, \ldots, y_m$ are the left-to-right minima of π and $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(m)}$ are Jacobi permutations of even length. Conversely, we can uniquely obtain all permutations $\pi \in \mathfrak{J}_n$ with exactly m left-to-right minima in the following way:

1. Partition $[n]$ into subsets $B_1, B_2, \ldots, B_m$ of odd size, where the subsets are listed in decreasing order of their smallest element.

   For example, taking $n = 9$ and $m = 3$, we can take $B_1 = \{8\}$, $B_2 = \{3, 4, 6, 7, 9\}$, and $B_3 = \{1, 2, 5\}$.

2. For each $k \in [m]$, let y_k be the smallest element of B_k, and choose a Jacobi permutation $\tau^{(k)}$ on the remaining elements of B_k.

   Continuing the example, we have $y_1 = 8$, $y_2 = 3$, and $y_3 = 1$, and take $\tau^{(1)} = \varepsilon$, $\tau^{(2)} = 6974$, and $\tau^{(3)} = 52$.

3. Take $\pi = y_1\tau^{(1)}y_2\tau^{(2)}\cdots y_m\tau^{(m)}$.

   Continuing the example, we have $\pi = 836974152$.

By construction, the left-to-right minima of π are $y_1, y_2, \ldots, y_m$. Moreover, for all $k \in [m]$, we have $\rho_\pi(y_k) = \tau^{(k)}$ and $\rho_\pi(x) = \rho_{\tau^{(k)}}(x)$ for each letter x of $\tau^{(k)}$. Since the $\tau^{(k)}$ and $\rho_{\tau^{(k)}}(x)$ are all of even length, π is indeed Jacobi. This gives us a decomposition of permutations in $\mathfrak{J}_n$ into "irreducible" factors.

Because $\sec x$ is the exponential generating function for even-length Jacobi permutations, it follows that

$$\int_0^x \sec u \, du = \log(\sec x + \tan x)$$

is the exponential generating function for the irreducibles, so we have

$$J^{\mathrm{lrmin}}(t, x) = \exp(t \log(\sec x + \tan x)) = (\sec x + \tan x)^t$$

by the exponential formula. □

2.4 Last letter

The *Entringer number* $E_{n,k}$ [4] is defined to be the number of up-down alternating permutations $\pi \in \mathfrak{U}_n$ with $\pi_1 = k$. In other words, by introducing the statistic $\mathrm{first}(\pi) := \pi_1$, the Entringer numbers give the distribution of first over $\mathfrak{U}_n$. It can be shown bijectively that these numbers satisfy the recurrence

$$E_{n,k} = E_{n-1,n-k} + E_{n,k+1}$$

for all $n \geq 2$ and $k \geq 1$.

Entringer numbers capture the distribution of various other statistics over objects counted by Euler numbers. For example, $E_{n,k}$ is also the number of André permutations $\pi \in \mathfrak{A}_n$ with $\pi_1 = k$, as well as the number of $\pi \in \mathfrak{A}_n$ with $\pi_{n-1} = n - k$ [7, Theorem 1.1]. Our next result shows that last has the same distribution over $\mathfrak{J}_n$.

Theorem 2.3. *For all $n, k \geq 1$, we have $j_{n,k}^{\mathrm{last}} = E_{n,k}$. Consequently, the distribution of* last *over $\mathfrak{J}_n$ is equal to the distribution of* first *over $\mathfrak{U}_n$ or $\mathfrak{A}_n$.*

$n\backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1							
3	1	1						
4	2	2	1					
5	5	5	4	2				
6	16	16	14	10	5			
7	61	61	56	46	32	16		
8	272	272	256	224	178	122	61	
9	1385	1385	1324	1202	1024	800	544	272

Table 7: The Entringer numbers $E_{n,k}$ up to $n = 9$.

For the proof of this theorem, we will need the notion of the *complement* π^c of a permutation π, which is obtained by (simultaneously) replacing the ith smallest letter of π with the ith largest letter of π for all i. For example, given $\pi = 319754$, we have $\pi^c = 791345$. Note that π is up-down if and only if π^c is down-up.

Proof. We provide a bijection $\Phi\colon \mathfrak{J}_S \to \mathfrak{U}_S$—which is a variant of a map due to Viennot [28, Section 3]—such that, for all $\pi \in \mathfrak{J}_S$, the first letter of $\Phi(\pi)$ is equal to the last letter of π. (This condition is vacuously satisfied for $S = \emptyset$, since the empty permutation has no first or last letter.) Specializing to $S = [n]$ then implies the result.

We define Φ recursively as follows. First, Φ sends the empty permutation to itself. Otherwise, if π is nonempty, then

$$\Phi(\pi) := \Phi(\beta)y\Phi(\alpha)^c$$

where $\pi = \alpha y \beta$ and $y = \min S$. We use induction to show that Φ has the desired properties.

Assume that whenever $|S'| < n$, the map Φ is a bijection from $\mathfrak{J}_{S'}$ to $\mathfrak{U}_{S'}$ such that $\operatorname{first}(\Phi(\tau)) = \operatorname{last}(\tau)$ for all $\tau \in \mathfrak{J}_{S'}$. Now, let π be a Jacobi permutation of a set S of size n, and write $\pi = \alpha y \beta$ as described above. In particular, we know that β is of even length because π is Jacobi. Then by the induction hypothesis, both $\Phi(\alpha)$ and $\Phi(\beta)$ are up-down permutations on their respective letter sets—so $\Phi(\alpha)^c$ is down-up—and $\operatorname{first}(\Phi(\beta)) = \operatorname{last}(\beta)$ if β is nonempty. Since $\Phi(\beta)$ is up-down of even length, $\Phi(\alpha)^c$ is down-up, and y is smaller than all letters in $\Phi(\beta)$ and $\Phi(\alpha)^c$, it follows that the concatenation $\Phi(\pi) = \Phi(\beta)y\Phi(\alpha)^c$ is an up-down permutation on S. Also, we have

$$\operatorname{first}(\Phi(\pi)) = \operatorname{first}(\Phi(\beta)) = \operatorname{last}(\beta) = \operatorname{last}(\pi)$$

if β is nonempty, and

$$\operatorname{first}(\Phi(\pi)) = y = \operatorname{last}(\pi)$$

otherwise. Finally, if $\pi \in \mathfrak{U}_S$ with $\pi = \bar{\beta} y \bar{\alpha}$, then $\Phi^{-1}(\pi) = \Phi^{-1}(\bar{\alpha}^c)y\Phi^{-1}(\bar{\beta})$. Thus, $\Phi\colon \mathfrak{J}_S \to \mathfrak{U}_S$ is a bijection satisfying $\operatorname{first}(\Phi(\pi)) = \operatorname{last}(\pi)$ for all $\pi \in \mathfrak{J}_S$. ☐

3. Jacobi permutations and binary trees

Our study of 213-, 312-, and 231-avoiding Jacobi permutations will extensively use tree representations of these permutations, based on classical correspondences between permutations and increasing/decreasing binary trees. The purpose of this section is to recall these correspondences and their relevant properties, and to examine their restriction to Jacobi permutations.

3.1 Binary trees and traversals

A *binary tree* is a rooted tree in which each node has at most two children, which are distinguished as the *left child* and the *right child*. Each node of a binary tree has a *left subtree* and a *right subtree*, both of which are (possibly empty) binary trees. As such, binary trees have a simple recursive decomposition: a binary tree is either the empty tree, or it consists of a root with a left subtree and a right subtree. The number of nodes of a binary tree T is called its *size*, which we will denote by $|T|$.

A binary tree can either have labeled nodes or unlabeled nodes. A binary tree *on* S is a binary tree of size $|S|$ whose nodes are labeled with the elements of S. Such a binary tree is naturally associated with three permutations of S, each of which is obtained by a depth-first traversal of the tree but with nodes read in a

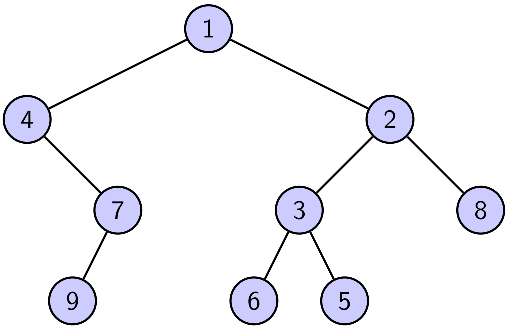

Figure 1: A binary tree T and its three associated permutations. Since T is an increasing binary tree, we have $T = \Theta(\pi)$ where $\pi = 497163528$.

different order. Each of the three traversals is defined recursively, starting at the root. In an *in-order traversal*, we do the following:

(1) Recursively traverse the current node's left subtree.

(2) Record the label of the current node. (The node is said to be *visited* at this step.)

(3) Recursively traverse the current node's right subtree.

In a *pre-order traversal*, the order of steps (1) and (2) are swapped, whereas in a *post-order traversal*, the order of steps (2) and (3) are swapped. See Figure 1 for an example of a binary tree on $S = [9]$ and the permutations obtained via each of these three traversals.

From this point forward, we will identify nodes with their labels when working with labeled binary trees. That is, if a node is labeled x, then we will refer to the node itself as x.

An *increasing binary tree* on S is a binary tree on S such that $x < y$ whenever y is a child of x. For example, the binary tree in Figure 1 is increasing. *Decreasing binary trees* are defined analogously.

The classical bijection Θ between permutations on S and increasing binary trees on S is defined as follows. If π is the empty permutation, then $\Theta(\pi)$ is the empty tree. Otherwise, let $y = \min S$, and write $\pi = \alpha y \beta$. Then we define $\Theta(\pi)$ as the increasing binary tree obtained by taking y as the root, attaching $\Theta(\alpha)$ as the left subtree of y, and attaching $\Theta(\beta)$ as the right subtree of y. Note that π is precisely the permutation obtained from $\Theta(\pi)$ upon performing an in-order traversal; see Figure 1 for an example.

3.2 Permutation statistics and tree statistics

Given a permutation statistic st, one can often find a statistic st′ on binary trees for which $\mathrm{st}(\pi) = \mathrm{st}'(\Theta(\pi))$. Here we examine such correspondences for the permutation statistics studied in this paper.

We define the *left branch* of a binary tree recursively as consisting of the root along with the left child of each node on the left branch; the *right branch* is defined in the analogous way. For example, the left branch of the tree in Figure 1 consists of the nodes labeled 1 and 4, whereas the right branch consists of the nodes labeled 1, 2, and 8. Observe that 1 and 4 are precisely the left-to-right minima of $\pi = 497163528$.

Proposition 3.1. *Let $\pi \in \mathfrak{S}_S$. Then $x \in S$ is on the left branch of $\Theta(\pi)$ if and only if x is a left-to-right minimum of π.*

Proof. Suppose that x is on the left branch of $\Theta(\pi)$. Observe that, in an in-order traversal of $\Theta(\pi)$, the nodes which are read before x are precisely those in the left subtree of x, so these are the only letters that appear before x in π. Because $\Theta(\pi)$ is an increasing tree, x is smaller than every node in its left subtree, so x is a left-to-right minimum of π.

For the converse, we will use the fact that every node of $\Theta(\pi)$ is either on the left branch of $\Theta(\pi)$ or is in the right subtree of a node on the left branch. So, assuming that x is not on the left branch of $\Theta(\pi)$, we have that x is in the right subtree of a node y. This means that the letter x appears after y in π, and also that $y < x$ because $\Theta(\pi)$ is an increasing tree. Therefore, x is not a left-to-right minimum of π. □

Let us call the letter π_k of a permutation π a *descent bottom* if $k - 1$ is a descent of π. (In other words, π_k is the "bottom" of a pair of letters corresponding to a descent of π.) For example, the descent bottoms of $\pi = 497163528$ are 7, 1, 3, and 2, which are precisely the nodes with left children in the tree $\Theta(\pi)$ displayed in Figure 1.

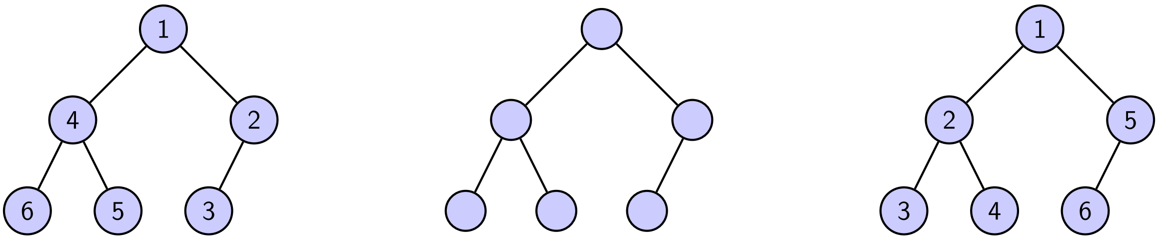

Figure 2: The increasing binary trees $\Theta(\pi)$ and $\Theta(\tau)$ for the 213-avoiding permutation $\pi = 645132$ (left) and the 312-avoiding permutation $\tau = 324165$ (right), along with their corresponding unlabeled binary tree T (middle).

Proposition 3.2. *Let $\pi \in \mathfrak{S}_S$. Then $x \in S$ has a left child in $\Theta(\pi)$ if and only if x is a descent bottom of π.*

Proof. Suppose that $x \in S$ has a left child in $\Theta(\pi)$, i.e., the left subtree of x is nonempty. Then, in an in-order traversal of $\Theta(\pi)$, the node read immediately before x is in the left subtree of x, and this is the letter which immediately precedes x in π. Because $\Theta(\pi)$ is an increasing tree, this letter is larger than x, so x is a descent bottom of π.

Conversely, suppose that $x \in S$ does not have a left child in $\Theta(\pi)$. If x is on the left branch of $\Theta(\pi)$, then this means that x is the first letter of π, in which case it is not a descent bottom. If x is not on the left branch of $\Theta(\pi)$, then the node read immediately before x in an in-order traversal is an ancestor of x, which is less than x by virtue of $\Theta(\pi)$ being an increasing tree. Since this is the letter immediately preceding x in π, it follows that x is not a descent bottom of π. □

Let $\mathrm{lbrch}(T)$ denote the number of nodes on the left branch of a binary tree T, and define $\mathrm{rbrch}(T)$ in the same way but for the right branch. Later, we'll refer to $\mathrm{lbrch}(T)$ as the *size* of the left branch of T, and similarly with $\mathrm{rbrch}(T)$. Also, let $\mathrm{lc}(T)$ denote the number of left children in T. Then the following is immediate from Propositions 3.1–3.2.

Corollary 3.1. *Let $\pi \in \mathfrak{S}_S$. Then*

(a) $\mathrm{lrmin}(\pi) = \mathrm{lbrch}(\Theta(\pi))$ *and*

(b) $\mathrm{des}(\pi) = \mathrm{lc}(\Theta(\pi))$.

While permutations in $\mathfrak{S}_S$ are in bijection with increasing binary trees on S, the 213-avoiding permutations in $\mathfrak{S}_S$ and the 312-avoiding permutations in $\mathfrak{S}_S$ are each in bijection with *unlabeled* binary trees on $|S|$ nodes. For example, notice that the permutation $\pi = 645132$ is 213-avoiding, and performing a post-order traversal of its corresponding tree $\Theta(\pi)$ yields the decreasing permutation 654321. Similarly, performing a pre-order traversal of the tree corresponding to the 312-avoiding permutation 324165 yields the increasing permutation 123456. (See Figure 2.) In fact, it can be verified that a permutation π of S is 213-avoiding if and only if a post-order traversal of $\Theta(\pi)$ yields the elements of S in decreasing order, and that π is 312-avoiding if and only if a pre-order traversal of $\Theta(\pi)$ yields the elements of S in increasing order. Thus, if we know in advance that π is 213- or 312-avoiding, then we can remove the labels of $\Theta(\pi)$ and still be able to recover π by traversing the tree appropriately.

The next proposition allows us to determine the last letter of a 213- or 312-avoiding permutation in $\mathfrak{S}_n$ from its tree. The result for 312-avoiding permutations π involves the number of nodes in a certain subtree of $\Theta(\pi)$ which we shall now define. Given a nonempty binary tree T, the *terminus* of T is the bottommost node of the right branch of T, and the *undergrowth* of T is the left subtree of its terminus. For example, the undergrowth of the tree in Figure 1 is the empty tree, whereas the undergrowth of the tree in Figure 2 consists of a single node, labeled 3 in $\Theta(\pi)$ and 6 in $\Theta(\tau)$.

Proposition 3.3. *Let $\pi \in \mathfrak{S}_n$.*

(a) *If π is 213-avoiding, then* $\mathrm{last}(\pi) = \mathrm{rbrch}(\Theta(\pi))$.

(b) *If π is 312-avoiding, then* $\mathrm{last}(\pi) = n - |U|$, *where U is the undergrowth of* $\Theta(\pi)$.

Proof. Suppose that $\pi \in \mathfrak{S}_n$ is 213-avoiding, and let $k = \mathrm{rbrch}(\Theta(\pi))$. The last letter of π is the last node visited in an in-order traversal of $\Theta(\pi)$, which is the terminus of $\Theta(\pi)$; call this node x. On the other hand, in a post-order traversal of $\Theta(\pi)$, the only nodes visited after x are the other nodes on the right branch; in other words, x is the k-to-last node visited. Recall that a post-order traversal of $\Theta(\pi)$ necessarily yields the decreasing permutation $n \cdots 21$. Therefore, we have $x = k$, which proves (a).

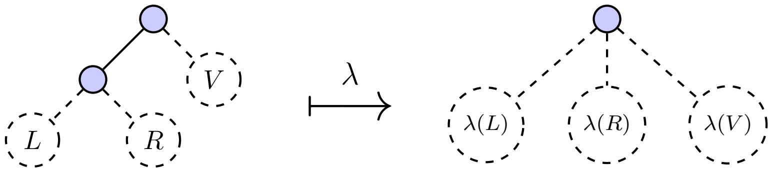

Figure 3: The bijection λ.

For (b), suppose that $\pi \in \mathfrak{S}_n$ is 312-avoiding, and let $k = |U|$. As above, let x be the terminus of $\Theta(\pi)$, which is the last letter of π. In a pre-order traversal of $\Theta(\pi)$, the only nodes visited after x are those in the undergrowth U, so x is the $(n-k)$th node visited. Because a pre-order traversal of $\Theta(\pi)$ yields $12\cdots n$, it follows that $x = n - k$, and we are done. $\square$

3.3 Jacobi trees

All of what has been discussed so far holds for permutations in general. Let us now describe the trees that correspond specifically to Jacobi permutations.

We say that an increasing binary tree T on S is a *labeled Jacobi tree* if $T = \Theta(\pi)$ for some Jacobi permutation π of S. An *unlabeled Jacobi tree* is an unlabeled binary tree obtained by taking a labeled Jacobi tree and removing the labels. We will usually be working with unlabeled Jacobi trees (as opposed to labeled ones), so we simply call these *Jacobi trees* from this point forward. Let $\mathcal{J}_n$ denote the set of Jacobi trees on n nodes. From the discussion earlier in this section, applying Θ and removing the labels gives us a bijection between $\mathfrak{J}_n(213)$ and $\mathcal{J}_n$, and between $\mathfrak{J}_n(312)$ and $\mathcal{J}_n$.

Observe that Jacobi trees are precisely unlabeled binary trees in which every right subtree is of even size. Equivalently, we have the following recursive decomposition for Jacobi trees: every nonempty Jacobi tree consists of a root whose left and right subtrees are both Jacobi trees and the right subtree is of even size.

The enumeration of Jacobi trees is closely related to that of (*unlabeled*) *ternary trees*, which are defined in the same way as (unlabeled) binary trees, except that each node has three subtrees, distinguished as the *left subtree*, *middle subtree*, and *right subtree*. It is well known [20, A001764 and A006013] that, for all $n \geq 0$, there are $\frac{1}{2n+1}\binom{3n}{n}$ ternary trees on n nodes and $\frac{1}{2n+1}\binom{3n+1}{n+1}$ pairs of ternary trees with n nodes in total. As shown below, these numbers also enumerate Jacobi trees of even and odd size, respectively.

Theorem 3.1. *For all $n \geq 0$, we have*

$$|\mathcal{J}_{2n}| = \frac{1}{2n+1}\binom{3n}{n} \quad \textit{and} \quad |\mathcal{J}_{2n+1}| = \frac{1}{2n+1}\binom{3n+1}{n+1}.$$

Of course, Theorem 3.1 directly translates to an enumeration for the Jacobi avoidance classes $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$; this will be stated explicitly in Theorem 4.1.

Proof. To prove the result for even-sized Jacobi trees, we give a recursive bijection λ between $\mathcal{J}_{2n}$ and ternary trees on n nodes. As the base case, λ sends the empty tree to the empty tree. Now, let T be a nonempty Jacobi tree of size $2n$. The root v of T must have a left child w, or else the right subtree of T would have odd size. Let L denote the left subtree of w, let R denote the right subtree of w, and let V be the right subtree of v. Note that L, R, and V all have even size; after all, R and V are right subtrees, and T would have an odd number of nodes if $|L|$ were odd. Thus, we may define $\lambda(T)$ to be the ternary tree whose root has left subtree $\lambda(L)$, middle subtree $\lambda(R)$, and right subtree $\lambda(V)$; see Figure 3 for an illustration. Since $\lambda(L)$, $\lambda(R)$, and $\lambda(V)$ have half as many nodes as L, R, and V, it follows that $\lambda(T)$ is a ternary tree on n nodes.

Now, let $T \in \mathcal{J}_{2n+1}$, and let L and R be the left and right subtrees of its root. By similar reasoning as above, both L and R must have even size, and $\lambda(L)$ and $\lambda(R)$ have n nodes in total. Then $T \mapsto (\lambda(L), \lambda(R))$ is a bijection between $\mathcal{J}_{2n+1}$ and pairs of ternary trees with n nodes in total; this yields the enumeration for odd-sized Jacobi trees. $\square$

Later, we will make use of another recursive decomposition for Jacobi trees that we call the *undergrowth decomposition*: every nonempty Jacobi tree T either consists of a root with a left subtree U and an empty right subtree, or can be obtained by taking a nonempty Jacobi tree T', attaching a new node as the right child of the terminus of T', and attaching a Jacobi tree U of odd size[§] as the left subtree of this new terminus. See Figure 4 for an illustration of this decomposition. We call this the "undergrowth decomposition" because the tree U in the decomposition is precisely the undergrowth of T.

[§]Note that U must have odd size, or else the terminus of T' would have a right subtree of odd size in T.

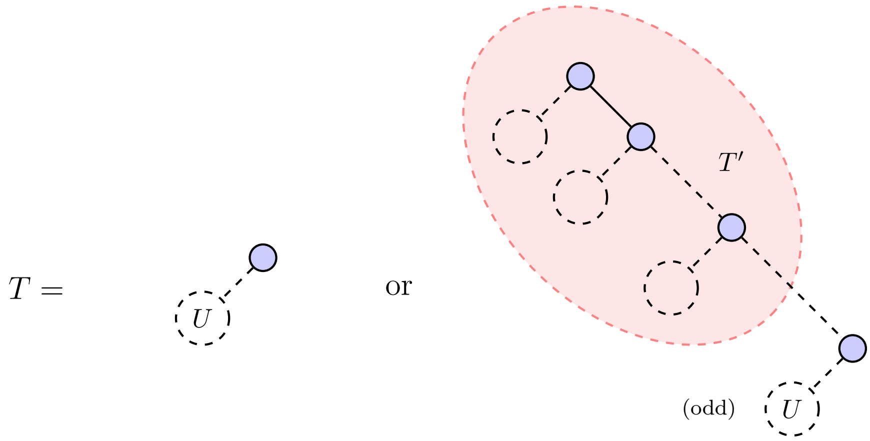

Figure 4: The undergrowth decomposition of a nonempty Jacobi tree T.

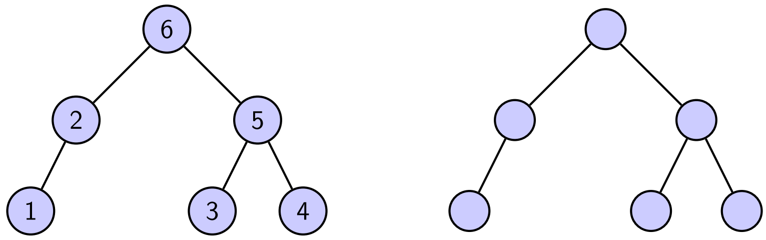

Figure 5: The decreasing binary tree $\tilde{\Theta}(\pi)$ for the 231-avoiding permutation $\pi = 126354$, along with the corresponding unlabeled binary tree T.

Recall that the proof of Theorem 2.2 utilized a unique factorization of Jacobi permutations into "irreducibles", whose first letters were the left-to-right minima. There is an analogous factorization where the last letters of the irreducibles are right-to-left minima, and all irreducibles other than the first are of even length; this factorization corresponds to the undergrowth decomposition defined above.

3.4 Dual Jacobi trees

While Jacobi trees are useful for studying 213- and 312-avoiding Jacobi permutations, we will need a different model for Jacobi permutations avoiding 231. To that end, let $\tilde{\Theta}$ be the bijection from permutations on S to decreasing binary trees on S, defined in the same way as Θ except that we take $y = \max S$ (as opposed to $y = \min S$). As with Θ, an in-order traversal of $\tilde{\Theta}(\pi)$ recovers the permutation π. It is readily checked that a permutation π of S is 231-avoiding if and only if a post-order traversal of $\tilde{\Theta}(\pi)$ yields the entries of S in increasing order, which implies that 231-avoiding permutations of S are in bijection with unlabeled binary trees of size $|S|$. See Figure 5 for an example.

The next proposition is proven similarly to Proposition 3.3; we will not give the details.

Proposition 3.4. *If* $\pi \in \mathfrak{S}_n(231)$, *then* $\mathrm{last}(\pi) = n - \mathrm{rbrch}(\tilde{\Theta}(\pi)) + 1$.

Let us now restrict to Jacobi permutations. We say that a decreasing binary tree T on S is a *labeled dual Jacobi tree* if $T = \tilde{\Theta}(\pi)$ for some $\pi \in \mathfrak{J}_S$. An unlabeled binary tree is called an (*unlabeled*) *dual Jacobi tree* if it can be obtained by taking a labeled dual Jacobi tree and removing the labels. Let $\tilde{\mathcal{J}}_n$ denote the set of dual Jacobi trees on n nodes. Then applying $\tilde{\Theta}$ and removing the labels gives a bijection between $\mathfrak{J}_n(231)$ and $\tilde{\mathcal{J}}_n$.

Later, we will show that $\mathfrak{J}_n(312)$ and $\mathfrak{J}_n(231)$ are in bijection, which induces a bijection between $\mathcal{J}_n$ and $\tilde{\mathcal{J}}_n$. This, along with Theorem 3.1, will imply the following.

Theorem 3.2. *For all* $n \geq 0$, *we have*

$$|\tilde{\mathcal{J}}_{2n}| = \frac{1}{2n+1}\binom{3n}{n} \quad \textit{and} \quad |\tilde{\mathcal{J}}_{2n+1}| = \frac{1}{2n+1}\binom{3n+1}{n+1}.$$

Our goal for the remainder of this section is to develop a recursive decomposition for dual Jacobi trees which we shall use when studying 231-avoiding Jacobi permutations. We begin with two lemmas.

Lemma 3.1. *Let $\pi \in \mathfrak{S}_S(231)$, let $x \in S$, and let $T = \tilde{\Theta}(\pi)$.*

(a) *If x has a right child in T, then $\rho_\pi(x)$ is empty.*

(b) *If x does not have a right child in $\tilde{\Theta}(\pi)$, then x is a right-to-left minimum of π, so $\rho_\pi(x)$ consists of all subsequent nodes visited in an in-order traversal of $\tilde{\Theta}(\pi)$.*

Proof. Suppose that x has a right child in $\tilde{\Theta}(\pi)$. Then, the letter immediately following x in π is a node in the right subtree of x, and this node must be smaller than x because $\tilde{\Theta}(\pi)$ is a decreasing binary tree. Therefore, $\rho_\pi(x)$ is empty, which proves (a).

For a contrapositive proof of (b), suppose that x is not a right-to-left minimum of π, so there exists a letter $y < x$ appearing after x in π. Due to their relative positions in π, we see that y is visited after x in an in-order traversal of $\tilde{\Theta}(\pi)$. Recall that a post-order traversal of $\tilde{\Theta}(\pi)$ where π is a 231-avoiding permutation returns the letters in increasing order; since $y < x$, this means that y is visited before x in a post-order traversal of $\tilde{\Theta}(\pi)$. The nodes visited after x in an in-order traversal but before x in a post-order traversal are precisely those in the right subtree of x. Thus, the right subtree of x is nonempty, so x has a right child in $\tilde{\Theta}(\pi)$. □

Lemma 3.2. *If T is a dual Jacobi tree, then the undergrowth of T is empty.*

Proof. Let T be a dual Jacobi tree and π the corresponding 231-avoiding Jacobi permutation. Suppose instead that the undergrowth U of T is nonempty, and let x be the terminus of U. By Lemma 3.1 (b), $\rho_\pi(x)$ consists of a single letter—the terminus of T—which contradicts π being a Jacobi permutation. □

Proposition 3.5. *Let T' be a nonempty dual Jacobi tree with terminus u, and let T'' be a dual Jacobi tree of odd size. Moreover, let T be the tree obtained from T' by attaching T'' as the left subtree of u, and attaching a new node v as the right child of u. Then T is a dual Jacobi tree.*

Note that we are implicitly using Lemma 3.2 in the statement of Proposition 3.5, as it guarantees that u does not have a left subtree prior to attaching T''.

Proof. For convenience, if π is the 231-avoiding permutation corresponding to T, then let us write ρ_T in place of ρ_π. We wish to show that $\rho_T(x)$ has even length for every node x in T, and we will do this through cases, appealing to Lemma 3.1 in each one.

- Note that $\rho_T(u)$ is empty because u has a right child, and that $\rho_T(v)$ is empty because it is the last node visited in an in-order traversal of T.
- If $x \neq u$ is a node in T' and $\rho_{T'}(x)$ is empty, then $\rho_T(x)$ is empty because we did not remove any right children in constructing T.
- If x is a node in T'' and $\rho_{T''}(x)$ is empty, then $\rho_T(x)$ is empty for the same reason as above.
- If x is a node in T' and $\rho_{T'}(x)$ is nonempty, then $\rho_T(x)$ contains every node in $\rho_{T'}(x)$ along with all nodes from T'' and v, as these are the nodes visited after x in an in-order traversal of T. It follows that
$$|\rho_T(x)| = |\rho_{T'}(x)| + |T''| + 1,$$
which is even because $|\rho_{T'}(x)|$ is even and $|T''|$ is odd.
- If x is a node in T'' and $\rho_{T''}(x)$ is nonempty, then for the same reason as above, $\rho_T(x)$ contains every node in $\rho_{T''}(x)$ along with u and v. So, we have
$$|\rho_T(x)| = |\rho_{T''}(x)| + 2,$$
which is even because $|\rho_{T''}(x)|$ is even. □

We will omit the proofs of the next two propositions as they are similar to that of Proposition 3.5.

Proposition 3.6. *Let T' be a dual Jacobi tree, let u be the terminus of T' (if T' is nonempty), and let T'' be a dual Jacobi tree of even size. Moreover, let T be the tree obtained in the following way:*

- *If T' is nonempty, attach a new node v as the right child of u, attach T'' as the left subtree of v, and attach a new node w as the right child of v.*
- *If T' is empty, let v be the root of T, attach T'' as the left subtree of v, and attach a new node w as the right child of v.*

Then T is a dual Jacobi tree.

Proposition 3.7. *Let T be a dual Jacobi tree with at least 2 nodes on its right branch, let v be the terminus of T, let u be the parent of v, and let T'' be the left subtree of u. Moreover, let T' be the tree obtained in the following way:*

- *If T'' is of even size, remove u, v, and T'' from T to form T'.*
- *If T'' is of odd size, remove v and T'' from T to form T'.*

Then T' and T'' are dual Jacobi trees.

Propositions 3.5–3.7 justify the following recursive decomposition for dual Jacobi trees: Every nonempty dual Jacobi tree either consists of a single node, consists of a root with a left dual Jacobi subtree of even size and whose right child is the terminus, or is built up from dual Jacobi trees in the way described in these propositions. See Figure 6 for an illustration. This decomposition can be thought of as an analogue of the undergrowth decomposition for (ordinary) Jacobi trees, although here T'' is not the undergrowth of T.[¶]

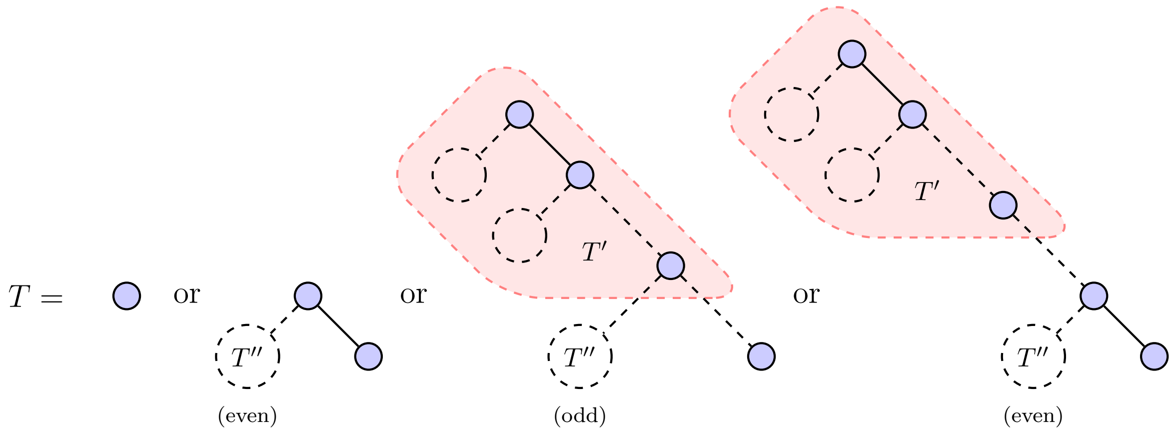

Figure 6: Recursive decomposition of a nonempty dual Jacobi tree T.

4. 213- and 312-avoiding Jacobi permutations

This section focuses on the Jacobi avoidance classes $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$. We begin by restating Theorem 3.1 as an enumeration for $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$.

Theorem 4.1. *For all $n \geq 0$, we have*

$$j_{2n}(213) = j_{2n}(312) = \frac{1}{2n+1}\binom{3n}{n} \quad \textit{and} \quad j_{2n+1}(213) = j_{2n+1}(312) = \frac{1}{2n+1}\binom{3n+1}{n+1}.$$

Using results from Section 3 relating certain permutation statistics to tree statistics, we can determine the distributions of these permutation statistics over $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$ by studying distributions of the corresponding statistics over $\mathcal{J}_n$. We shall take this approach in this section to count 213- and 312-avoiding Jacobi permutations by the number of descents, the number of left-to-right minima, and the value of the last letter.[‖]

n	0	1	2	3	4	5	6	7	8	9	10	11
$j_n(213)$	1	1	1	2	3	7	12	30	55	143	273	728

Table 8: The numbers $j_n(213) = j_n(312)$ up to $n = 11$.

[¶]One might instead call T'' the *penundergrowth* of T.

[‖]While it is possible to prove the results in this section by working directly with permutations, we found it easier to formulate our bijections using Jacobi trees. (Some of our bijections do not seem to have simple descriptions in terms of the permutations themselves.) Another advantage of this approach is that it allows us to prove most of our results simultaneously for $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$.

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1								
2	0	1							
3	0	1	1						
4	0	0	2	1					
5	0	0	2	4	1				
6	0	0	0	5	6	1			
7	0	0	0	5	15	9	1		
8	0	0	0	0	14	28	12	1	
9	0	0	0	0	14	56	56	16	1

Table 9: The numbers $j^{\mathrm{des}}_{n,k}(213) = j^{\mathrm{des}}_{n,k}(312)$ up to $n = 9$.

4.1 Descents

Counting Jacobi trees by the number of left children will allow us to achieve a refined enumeration of 213- and 312-avoiding Jacobi permutations by the number of descents. Our goal is to prove the following.

Theorem 4.2. *We have*

$$j^{\mathrm{des}}_{2n,k}(213) = j^{\mathrm{des}}_{2n,k}(312) = \frac{1}{n}\binom{n}{k-n}\binom{2n}{k+1} \qquad \text{for all } n \geq 1 \text{ and } k \geq 0;$$

$$j^{\mathrm{des}}_{2n+1,k}(213) = j^{\mathrm{des}}_{2n+1,k}(312) = \frac{1}{n+1}\binom{n+1}{k-n}\binom{2n}{k} \qquad \text{for all } n, k \geq 0.$$

Define the generating functions

$$F_{\mathrm{e}} = F_{\mathrm{e}}(t,x) := \sum_{n=0}^{\infty} \sum_{T \in \mathcal{J}_{2n}} t^{\mathrm{lc}(T)} x^{2n} \qquad \text{and} \qquad F_{\mathrm{o}} = F_{\mathrm{o}}(t,x) := \sum_{n=0}^{\infty} \sum_{T \in \mathcal{J}_{2n+1}} t^{\mathrm{lc}(T)} x^{2n+1},$$

which count even- and odd-sized Jacobi trees, respectively, by the number of left children. The next lemma provides several functional equations satisfied by these generating functions.

Lemma 4.1. *We have*

(a) $$F_{\mathrm{e}} = 1 + tx^2(1-t)F_{\mathrm{e}}^2 + t^2x^2F_{\mathrm{e}}^3,$$

(b) $$F_{\mathrm{e}} = \frac{1}{1 - txF_{\mathrm{o}}},$$

(c) $$F_{\mathrm{o}} = xF_{\mathrm{e}} + tx(F_{\mathrm{e}} - 1)F_{\mathrm{e}}, \quad \textit{and}$$

(d) $$F_{\mathrm{o}} = x + (t-1)tx^2F_{\mathrm{o}} + 2txF_{\mathrm{o}}^2 - t^2x^2F_{\mathrm{o}}^3.$$

Proof. We first prove (c). Let T be a Jacobi tree of odd size. Then T consists of a root whose left subtree L and right subtree R are both Jacobi trees of even size. The case where L is empty contributes the term xF_{e}, since $\mathrm{lc}(T) = \mathrm{lc}(R)$. If L is nonempty, then $\mathrm{lc}(T) = \mathrm{lc}(L) + \mathrm{lc}(R) + 1$, so this case contributes $tx(F_{\mathrm{e}} - 1)F_{\mathrm{e}}$. Summing these two terms yields (c).

To prove (a) and (b), now let T be a Jacobi tree of even size. The case where T is empty contributes the term 1. If T is nonempty, then T consists of a root whose left subtree L is a Jacobi tree of odd size and whose right subtree R is a Jacobi tree of even size. In particular, L is nonempty, so $\mathrm{lc}(T) = \mathrm{lc}(L) + \mathrm{lc}(R) + 1$. Thus, this case contributes $txF_{\mathrm{o}}F_{\mathrm{e}}$, and we have

$$F_{\mathrm{e}} = 1 + txF_{\mathrm{o}}F_{\mathrm{e}}. \tag{5}$$

Solving for F_{e} in (5) yields (b). Furthermore, substituting (c) into (5) and simplifying results in (a).

Finally, (d) can be obtained from substituting (b) into (c) and performing some routine algebraic manipulations. □

In order to obtain formulas for their coefficients, we will relate F_{e} and F_{o} to two generating functions which have previously appeared in the literature. Let $U = U(t,x)$ and $V = V(t,x)$ be the unique formal power series solutions to the functional equations

$$U = (1 + txU)(1 + xU)^2 \qquad \text{and} \tag{6}$$

$$V(1-xV)^2 = 1+(t-1)xV, \tag{7}$$

respectively. The coefficient of $t^k x^n$ in U counts ternary trees on n edges with k middle children—as can be seen directly from (6)—and these coefficients are given by the formula

$$[t^k x^n]\, U = \frac{1}{n+1}\binom{n+1}{k}\binom{2(n+1)}{n-k} \tag{8}$$

[20, A120986]. The coefficients of V, on the other hand, count a certain family of Feynman diagrams refined by the number of fermionic loops [17], and are given by

$$[t^k x^n]\, V = \frac{1}{n+1}\binom{n+1}{k}\binom{2n}{n+k} \tag{9}$$

[20, A286784].

Lemma 4.2. *We have*

(a) $$F_{\mathrm{e}}(t,x) = 1 + tx^2 U(t,tx^2) \quad \textit{and}$$

(b) $$F_{\mathrm{o}}(t,x) = xV(t,tx^2).$$

Proof. To prove (a), first we replace each instance of x with tx^2 in (6) to obtain

$$U(t,tx^2) = (1+t^2x^2U(t,tx^2))(1+tx^2U(t,tx^2))^2. \tag{10}$$

Let $\bar{U} = \bar{U}(t,x) := 1+tx^2U(t,tx^2)$. Substituting $U(t,tx^2) = (\bar{U}-1)/tx^2$ into (10) gives us

$$\frac{\bar{U}-1}{tx^2} = \left(1+t^2x^2\left(\frac{\bar{U}-1}{tx^2}\right)\right)\left(1+tx^2\left(\frac{\bar{U}-1}{tx^2}\right)\right)^2.$$

Multiplying both sides by tx^2, adding 1 to both sides, and simplifying yields

$$\bar{U} = 1 + tx^2(1-t)\bar{U}^2 + t^2x^2\bar{U}^3;$$

comparing with Lemma 4.1 (a), we conclude $F_{\mathrm{e}}(t,x) = \bar{U} = 1+tx^2U(t,tx^2)$.

Next, we prove (b). We make the substitution $x \mapsto tx^2$ in (7), giving us

$$V(t,tx^2)(1-tx^2V(t,tx^2))^2 = 1+(t-1)tx^2V(t,tx^2). \tag{11}$$

Let $\bar{V} = \bar{V}(t,x) := xV(t,tx^2)$, so that $V(t,tx^2) = x^{-1}\bar{V}$. Then (11) becomes

$$\bar{V}(1-tx\bar{V})^2 = x + (t-1)tx^2\bar{V}. \tag{12}$$

Expanding the left-hand side of (12) and rearranging yields

$$\bar{V} = x + (t-1)tx^2\bar{V} + 2tx\bar{V}^2 - t^2x^2\bar{V}^3,$$

which proves $F_{\mathrm{o}}(t,x) = \bar{V} = xV(t,tx^2)$ upon comparing with Lemma 4.1 (d). □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Using Lemma 4.2 (a) and Equation (8), we obtain

$$\begin{aligned}
F_{\mathrm{e}}(t,x) = 1+tx^2U(t,tx^2) &= 1 + \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k}\binom{2(n+1)}{n-k} t^{n+k+1}x^{2n+2} \\
&= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k}\binom{2n}{n-1-k} t^{n+k}x^{2n} \\
&= 1 + \sum_{n=1}^{\infty}\sum_{k=n}^{2n-1} \frac{1}{n}\binom{n}{k-n}\binom{2n}{2n-1-k} t^{k}x^{2n} \\
&= 1 + \sum_{n=1}^{\infty}\sum_{k=n}^{2n-1} \frac{1}{n}\binom{n}{k-n}\binom{2n}{k+1} t^{k}x^{2n}.
\end{aligned} \tag{13}$$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	1	0	1						
4	0	0	2	0	1					
5	0	3	0	3	0	1				
6	0	0	7	0	4	0	1			
7	0	12	0	12	0	5	0	1		
8	0	0	30	0	18	0	6	0	1	
9	0	55	0	55	0	25	0	7	0	1

Table 10: The numbers $j_{n,k}^{\mathrm{lrmin}}(213) = j_{n,k}^{\mathrm{lrmin}}(312)$ up to $n = 9$.

Similarly, by Lemma 4.2 (b) and Equation (9), we have

$$\begin{aligned} F_{\mathrm{o}}(t,x) = xV(t,tx^2) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k}\binom{2n}{n+k} t^{n+k}x^{2n+1} \\ &= \sum_{n=0}^{\infty}\sum_{k=n}^{2n} \frac{1}{n+1}\binom{n+1}{k-n}\binom{2n}{k} t^{k}x^{2n+1}. \end{aligned} \tag{14}$$

Moreover, we have

$$j_{2n,k}^{\mathrm{des}}(213) = j_{2n,k}^{\mathrm{des}}(312) = [t^k x^{2n}]\, F_{\mathrm{e}}(t,x)$$

and

$$j_{2n+1,k}^{\mathrm{des}}(213) = j_{2n+1,k}^{\mathrm{des}}(312) = [t^k x^{2n+1}]\, F_{\mathrm{e}}(t,x)$$

from Corollary 3.1 (b); combining these with (13) and (14) completes the proof. □

4.2 Left-to-right minima

Our next result counts permutations in $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$ by the number of left-to-right minima.

Theorem 4.3. *For all $n \geq k \geq 1$, we have*

$$j_{n,k}^{\mathrm{lrmin}}(213) = j_{n,k}^{\mathrm{lrmin}}(312) = \frac{2k}{3n-k}\binom{\frac{3n-k}{2}}{n}$$

if n and k have the same parity, and $j_{n,k}^{\mathrm{lrmin}}(213) = j_{n,k}^{\mathrm{lrmin}}(312) = 0$ otherwise.

From Table 10, the reader may notice a pattern captured by the following recurrence.

Proposition 4.1. *The numbers $j_{n,k}^{\mathrm{lrmin}}(213) = j_{n,k}^{\mathrm{lrmin}}(312)$ satisfy*

$$j_{n,k}^{\mathrm{lrmin}}(213) = j_{n-1,k-1}^{\mathrm{lrmin}}(213) + j_{n,k+2}^{\mathrm{lrmin}}(213)$$

for all $n \geq k \geq 1$.

In fact, Theorem 4.3 follows readily from an induction argument using Proposition 4.1. We will omit the details of this induction, and instead focus our attention on proving Proposition 4.1.

Proof. For convenience, let $\mathcal{J}_{n,k}$ denote the set of trees in $\mathcal{J}_n$ that have a left branch of size k. By Corollary 3.1 (a), it suffices to construct a bijection $\psi\colon \mathcal{J}_{n,k} \to \mathcal{J}_{n-1,k-1} \sqcup \mathcal{J}_{n,k+2}$.

Given $T \in \mathcal{J}_{n,k}$, let u denote the bottommost node in its left branch. If the right subtree of u is empty, then $\psi(T) \in \mathcal{J}_{n-1,k-1}$ is defined to be the Jacobi tree obtained from T upon removing u.

Now, suppose the right subtree of u is nonempty. Let v denote the right child of u. If v does not have a left child, then u would have a right subtree of odd size (consisting of v and the right subtree of v), a contradiction. So, let w denote the left child of v; also let L be the left subtree of w, let R be the right subtree of w, and let V be the right subtree of v. By a similar argument, L must have even size or else the right subtree of u would have odd size. We construct $\psi(T)$ from T in the following way:

1. Remove v and all its descendants from T.

2. Add a new node v' as the left child of u, and a new node w' as the left child of v'.
3. Attach L, R, and V as the right subtrees of w', v', and u, respectively.

See Figure 7 for an illustration. Observe that we have increased the size of the left branch by 2, and that $\psi(T)$ is a Jacobi tree because L, R, and V all have even size. Therefore, $\psi(T) \in \mathcal{J}_{n,k+2}$ in this case.

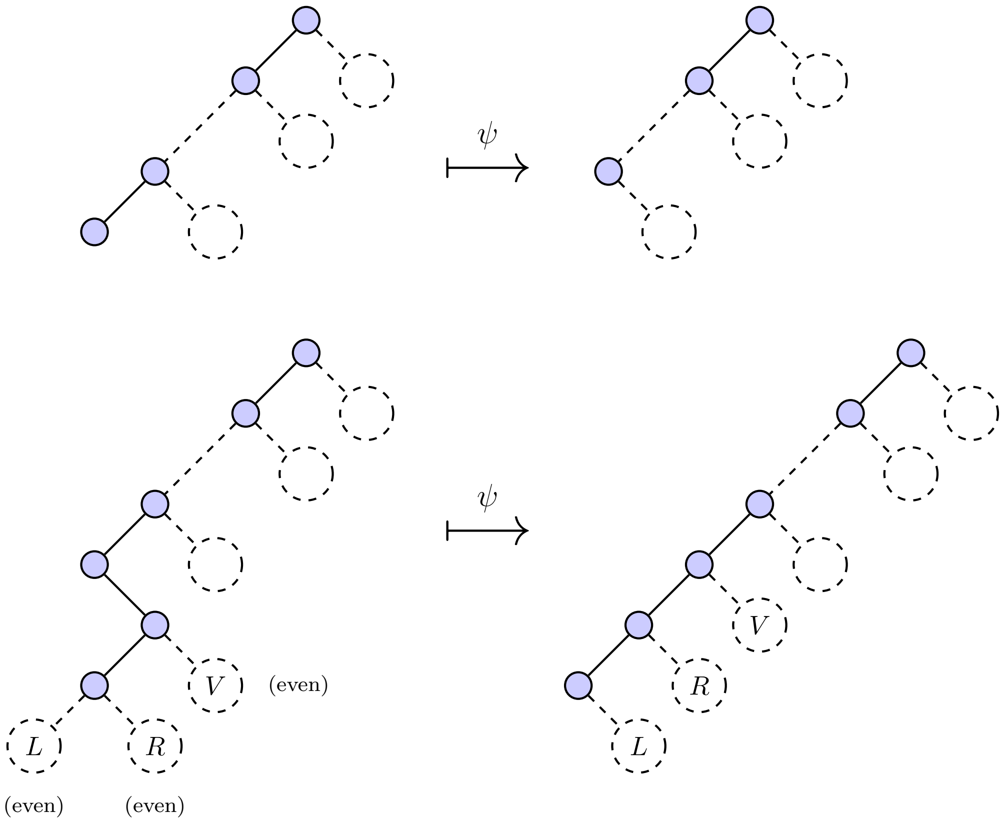

Figure 7: The bijection ψ.

Now, suppose that $T \in \mathcal{J}_{n-1,k-1} \sqcup \mathcal{J}_{n,k+2}$. We can recover the corresponding tree in $\mathcal{J}_{n,k}$ by either adding a node to the bottom of the left branch if $T \in \mathcal{J}_{n-1,k-1}$, or by reversing the steps in the above procedure for $T \in \mathcal{J}_{n,k+2}$. We conclude that ψ is a bijection. ☐

4.3 Last letter

Our last two theorems of this section give the distributions of last over $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$.

Theorem 4.4. *For all $n \geq k \geq 1$, we have*

$$j^{\mathrm{last}}_{2n,k}(213) = \frac{2k}{3n-k}\binom{3n-k}{2n} \qquad \textit{and} \qquad j^{\mathrm{last}}_{2n-1,k}(213) = \frac{2k-1}{3n-k-1}\binom{3n-k-1}{2n-1}.$$

Theorem 4.5. *For all $n \geq k \geq 1$, we have*

$$j^{\mathrm{last}}_{2n,2k-1}(312) = \frac{\binom{3k-3}{k-1}\binom{3n-3k+1}{n-k+1}}{(2k-1)(2n-2k+1)} \qquad \textit{and} \qquad j^{\mathrm{last}}_{2n+1,2k}(312) = \frac{\binom{3k-2}{k}\binom{3n-3k+1}{n-k+1}}{(2k-1)(2n-2k+1)}.$$

In addition, we have

$$j^{\mathrm{last}}_{2n+1,1}(312) = \frac{1}{2n+1}\binom{3n}{n}$$

for all $n \geq 0$ and, unless $k = 1$, $j^{\mathrm{last}}_{n,k}(312) = 0$ whenever n and k have the same parity.

See Table 11 for the numbers $j^{\mathrm{last}}_{n,k}(213)$ and $j^{\mathrm{last}}_{n,k}(312)$ up to $n = 9$. Comparing the $j^{\mathrm{last}}_{n,k}(213)$ with the $j^{\mathrm{lrmin}}_{n,k}(213) = j^{\mathrm{lrmin}}_{n,k}(312)$ from Table 10 reveals that they are shifts of each other, which we prove below.

$j^{\text{last}}_{n,k}(213)$

$n\backslash k$	1	2	3	4	5
1	1				
2	1				
3	1	1			
4	2	1			
5	3	3	1		
6	7	4	1		
7	12	12	5	1	
8	30	18	6	1	
9	55	55	25	7	1

$j^{\text{last}}_{n,k}(312)$

$n\backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1							
3	1	1						
4	2	0	1					
5	3	2	0	2				
6	7	0	2	0	3			
7	12	7	0	4	0	7		
8	30	0	7	0	6	0	12	
9	55	30	0	14	0	14	0	30

Table 11: The numbers $j^{\text{last}}_{n,k}(213)$ and $j^{\text{last}}_{n,k}(312)$ up to $n = 9$.

Proposition 4.2. *For all $n \geq k \geq 1$, we have*

$$j^{\text{last}}_{n,k}(213) = \begin{cases} j^{\text{lrmin}}_{n,2k}(213), & \textit{if } n \textit{ is even,} \\ j^{\text{lrmin}}_{n,2k-1}(213), & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. By Corollary 3.1 (a) and Proposition 3.3 (a), it suffices to construct a recursive bijection $\varphi\colon \mathcal{J}_n \to \mathcal{J}_n$ satisfying $\operatorname{lbrch}(T) = 2\operatorname{rbrch}(\varphi(T))$ when n is even and $\operatorname{lbrch}(T) = 2\operatorname{rbrch}(\varphi(T)) - 1$ when n is odd.

Let us first define φ for when n is even. As the base case, φ sends the empty tree to the empty tree. Now, let $T \in \mathcal{J}_n$ with $\operatorname{lbrch}(T) = 2k$ for some $k \geq 1$. Let u denote the bottommost node on the left branch of T, and let v denote its parent (the second bottommost node on that branch). Moreover, let U denote the right subtree of u and V the right subtree of v; both have an even number of nodes. Remove the nodes u and v (and their right subtrees) from T to form the Jacobi tree T', which has an even number of nodes because we have deleted an even number of nodes from T. Then $\varphi(T')$ is a Jacobi tree whose right branch has $k - 1$ nodes. We construct $\varphi(T)$ with the following steps:

1. Take the Jacobi tree $\varphi(T')$ and add a new node u' to its right branch.

2. Add a new node v' as the left child of u'.

3. Attach U as the left subtree of v', and V as the right subtree of v'.

Note that v' along with its subtrees U and V form the undergrowth of $\varphi(T)$ as defined in Section 3. See Figure 8 for an illustration.

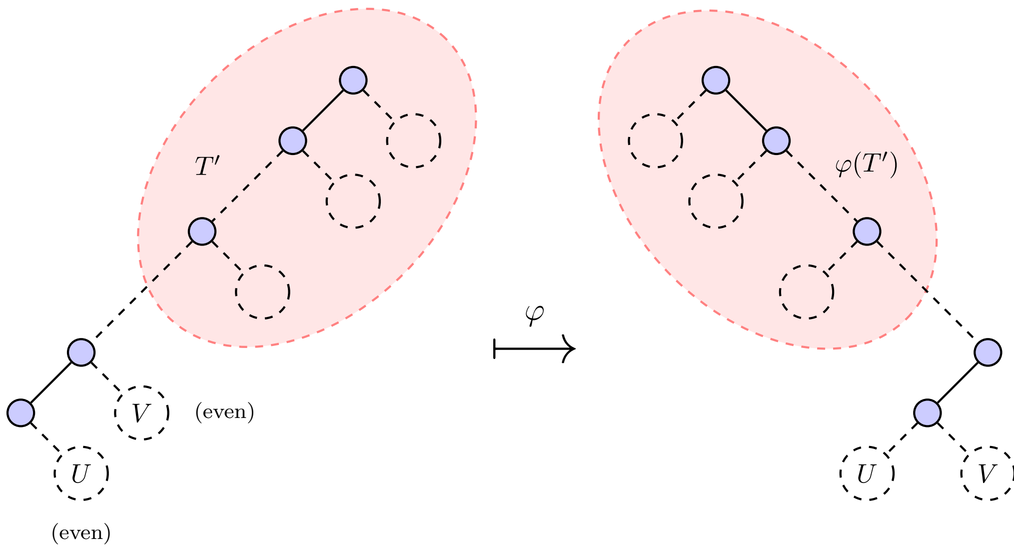

Figure 8: The bijection φ.

Given that $\varphi(T')$ is a Jacobi tree and that U and V each have an even number of nodes, it is evident that $\varphi(T)$ is a Jacobi tree as well. Furthermore, since the right branch of $\varphi(T')$ has $k-1$ nodes, the right branch of $\varphi(T)$ has k nodes, so $\operatorname{lbrch}(T) = 2\operatorname{rbrch}(\varphi(T))$ is satisfied. Finally, we see from the undergrowth decomposition that every nonempty Jacobi tree of even size is of the form depicted by the rightmost tree of Figure 8, with both U and V of even size. Thus, φ is a bijection when n is even.

Next, let us define φ for when n is odd. As the base case, let T be any tree in $\mathcal{J}_n$ whose left branch has a single node—namely, the root of T. Then T corresponds to a Jacobi permutation π that begins with the letter 1: $\pi = 1\pi_2\pi_3\cdots\pi_n$. We shall define $\varphi(T)$ so that it is the tree corresponding to $\pi_2\pi_3\cdots\pi_n 1$: remove the right subtree of the root of T, and attach it back as the left subtree. Then $\varphi(T)$ is a Jacobi tree because $\pi_2\pi_3\cdots\pi_n 1$ is a Jacobi permutation, and $\varphi(T)$ has a single node on its right branch as desired. For $T \in \mathcal{J}_n$ with $\operatorname{lbrch}(T) = 2k-1$ where $k \geq 2$, we define $\varphi(T)$ in the same way as the even case. Observe that φ restricts to a bijection between trees in $\mathcal{J}_n$ whose left branch has a single node and those whose right branch has a single node. Additionally, by similar reasoning as the even case, the Jacobi trees whose right branch has at least two nodes are all of the form of the rightmost tree of Figure 8, and it follows that φ is a bijection when n is odd. □

Theorem 4.4 is obtained by combining Theorem 4.3 with Proposition 4.2. Let us now move on to the proof of Theorem 4.5.

Proof of Theorem 4.5. Let

$$G_{\mathrm{e}}(x) := \sum_{n=0}^{\infty} |\mathcal{J}_{2n}|\, x^{2n} \qquad \text{and} \qquad G_{\mathrm{o}}(x) := \sum_{n=0}^{\infty} |\mathcal{J}_{2n+1}|\, x^{2n+1}$$

be the ordinary generating functions for Jacobi trees of even and odd size, respectively, and let $G(x) := G_{\mathrm{e}}(x) + G_{\mathrm{o}}(x)$. From Theorem 3.1, we have

$$G_{\mathrm{e}}(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{3n}{n} x^{2n} \qquad \text{and} \qquad G_{\mathrm{o}}(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{3n+1}{n+1} x^{2n+1}.$$

To determine the numbers $j_{n,k}^{\mathrm{last}}(312)$, we shall derive an expression for the generating function

$$J^{\mathrm{last}}(t,x;312) := \sum_{n=1}^{\infty} \sum_{\pi \in \mathfrak{J}_n(312)} t^{\mathrm{last}(\pi)} x^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} j_{n,k}^{\mathrm{last}}(312) t^k x^n.$$

By Proposition 3.3 (b), we seek a weighted count of Jacobi trees such that each node of the undergrowth contributes a weight of x and each other node contributes a weight of tx. Using the undergrowth decomposition, we have

$$\begin{aligned}
J^{\mathrm{last}}(t,x;312) &= txG(x) + tx(G(tx)-1)G_{\mathrm{o}}(x) \\
&= txG_{\mathrm{e}}(x) + txG_{\mathrm{o}}(x) + txG(tx)G_{\mathrm{o}}(x) - txG_{\mathrm{o}}(x) \\
&= txG_{\mathrm{e}}(x) + txG(tx)G_{\mathrm{o}}(x) \\
&= \sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{3n}{n} tx^{2n+1} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\binom{3k}{k}\binom{3(n-k)+1}{n-k+1}}{(2k+1)(2(n-k)+1)} t^{2k+1} x^{2n+2} \\
&\qquad + \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\binom{3k+1}{k+1}\binom{3(n-k)+1}{n-k+1}}{(2k+1)(2(n-k)+1)} t^{2k+2} x^{2n+3},
\end{aligned}$$

from which the desired result follows. □

5. 231-avoiding Jacobi permutations

Through Jacobi trees, we saw that $\mathfrak{J}_n(213)$ and $\mathfrak{J}_n(312)$ are in bijection, and that the statistics des and lrmin each has the same distribution over $\mathfrak{J}_n(213)$ as it does over $\mathfrak{J}_n(312)$. In this section, we shall show that $\mathfrak{J}_n(231)$ has the same enumeration and the same distributions of des and lrmin as these two Jacobi avoidance classes, and we will then use dual Jacobi trees to obtain a formula for counting permutations in $\mathfrak{J}_n(231)$ with respect to the last letter.

Inverses of permutations will play an essential role in this section, so let us start by recalling some relevant properties of the inverse.

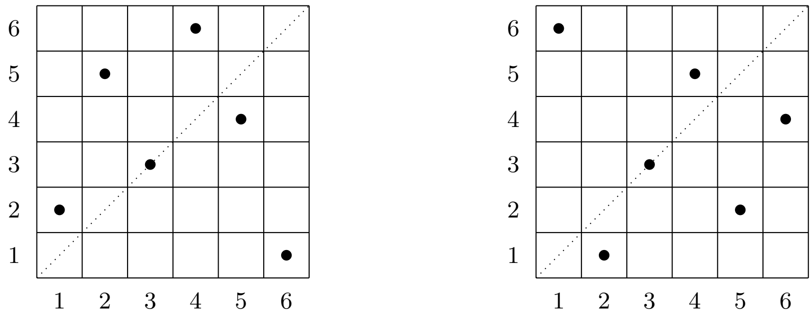

Figure 9: The arrays of $\pi = 253641$ (left) and its inverse $\pi^{-1} = 613524$ (right).

5.1 Permutation arrays and inverses

A permutation $\pi \in \mathfrak{S}_n$ is displayed diagrammatically by plotting each of the points (k, π_k) in an n-by-n grid, where the first coordinate indicates the column and the second indicates the row, with columns numbered 1 through n from left to right, and rows numbered 1 through n from bottom to top. The resulting diagram is called the *array* of π.

Let π^{-1} denote the inverse of $\pi \in \mathfrak{S}_n$. We adopt the convention that the inverse of the empty permutation is itself, and that if $\pi \in \mathfrak{S}_S$, then π^{-1} is the unique permutation of S whose standardization is equal to $\operatorname{std}(\pi)^{-1}$. For instance, we have $(72485)^{-1} = 45827$. Inverses of permutations in $\mathfrak{S}_n$ have a nice description in terms of their permutation arrays: the inverse of $\pi \in \mathfrak{S}_n$ is the permutation whose array is obtained by reflecting the array of π about the main diagonal. See Figure 9 for an example.

We will need a couple of facts about the inverses of certain kinds of permutations, which are most conveniently expressed using the terminology of direct and skew sums. Given $\alpha \in \mathfrak{S}_m$ and $\beta \in \mathfrak{S}_n$, the *direct sum* $\alpha \oplus \beta$ and the *skew sum* $\alpha \ominus \beta$ are defined to be the permutations in $\mathfrak{S}_{m+n}$ whose letters are given by

$$(\alpha \oplus \beta)_k := \begin{cases} \alpha_k, & \text{if } 1 \leq k \leq m, \\ \beta_{k-m} + m, & \text{if } m+1 \leq k \leq m+n, \end{cases}$$

and

$$(\alpha \ominus \beta)_k := \begin{cases} \alpha_k + n, & \text{if } 1 \leq k \leq m, \\ \beta_{k-m}, & \text{if } m+1 \leq k \leq m+n. \end{cases}$$

For example, if $\alpha = 132$ and $\beta = 21$, then $\alpha \oplus \beta = 13254$ and $\alpha \ominus \beta = 35421$. We note that $\alpha \oplus \beta$ can more intuitively be described as the permutation whose array is obtained by juxtaposing the arrays of α and β diagonally, and similarly with $\alpha \ominus \beta$ except that they are juxtaposed antidiagonally.

The next two lemmas, well known in the permutation patterns literature, are best explained using permutation arrays. We provide a visual depiction of these lemmas in Figure 10 in lieu of proofs.

Lemma 5.1. *If $\pi = \alpha \oplus \beta$, then $\pi^{-1} = \alpha^{-1} \oplus \beta^{-1}$.*

Lemma 5.2. *If $\pi = \alpha \ominus \beta$, then $\pi^{-1} = \beta^{-1} \ominus \alpha^{-1}$.*

The following two facts, also readily seen through permutation arrays, will be useful later.

Lemma 5.3. *A letter π_k is a left-to-right minimum of $\pi \in \mathfrak{S}_n$ if and only if k is a left-to-right minimum of π^{-1}.*

Lemma 5.4. *A permutation π avoids σ if and only if π^{-1} avoids σ^{-1}.*

Lastly, when a direct sum $\alpha \oplus \beta$ is Jacobi, the next lemma provides some information about its summands α and β.

Lemma 5.5. *Suppose that $\pi = \alpha \oplus \beta$ is Jacobi.*

(a) *If α is nonempty, then β has even length.*

(b) *The permutation α is Jacobi.*

Proof. Let β' denote the permutation obtained from β by adding $|\alpha|$ to every letter, so that π is the concatenation $\pi = \alpha\beta'$.

Note that (b) is trivial if α is empty, so assume that α is nonempty for both (a) and (b). Let y be the last letter of α. Since y is smaller than every letter of β', we have $\rho_\pi(y) = \beta'$. Then $\rho_\pi(y) = \beta'$ has even length because π is Jacobi, and we have shown (a).

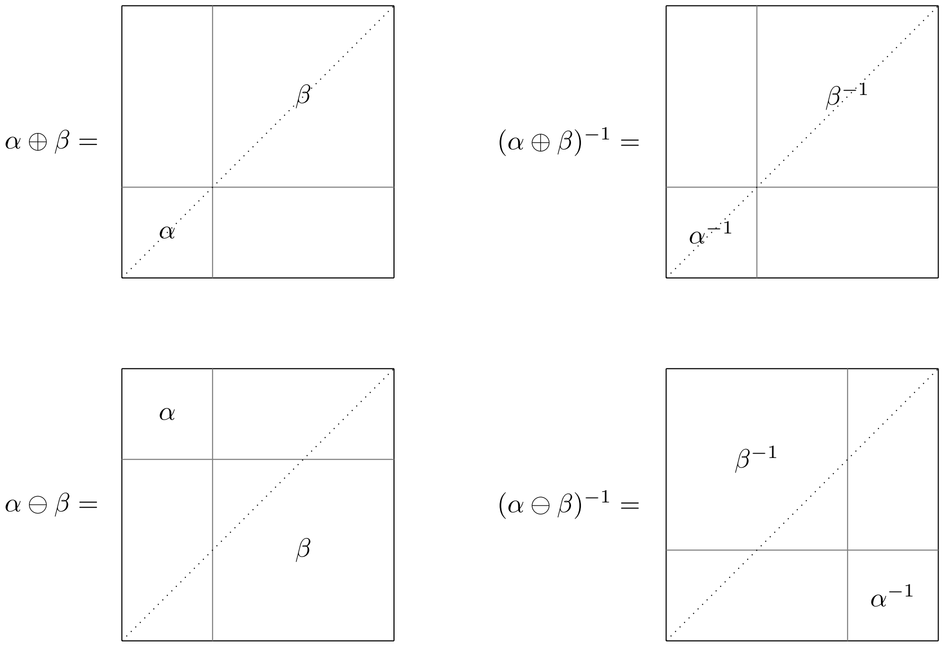

Figure 10: A direct sum, a skew sum, and their inverses.

Now, let x be an arbitrary letter of α other than the last letter y. Suppose that y is in $\rho_\alpha(x)$. Because every letter of β' is larger than x, we know that $\rho_\pi(x)$ is the concatenation $\rho_\pi(x) = \rho_\alpha(x)\beta'$. Since $\rho_\pi(x)$ and β' both have even length, it follows that $\rho_\alpha(x)$ has even length as well. In the case that $\rho_\alpha(x)$ does not contain y, we have $\rho_\alpha(x) = \rho_\pi(x)$, which is of even length. Thus, (b) is proven. $\square$

5.2 213-, 231-, and 312-avoiding Jacobi permutations are doubly Jacobi

Because 312 and 231 are inverses, we know from Lemma 5.4 that $\pi \in \mathfrak{S}_n$ avoids 312 if and only if π^{-1} avoids 231. Perhaps surprisingly, the inverse preserves the Jacobi property on these two avoidance classes.

Theorem 5.1. *For all $n \geq 0$, a permutation $\pi \in \mathfrak{S}_n(312)$ is Jacobi if and only if $\pi^{-1} \in \mathfrak{S}_n(231)$ is Jacobi. Consequently, $\pi \mapsto \pi^{-1}$ is a bijection from $\mathfrak{J}_n(312)$ to $\mathfrak{J}_n(231)$.*

Throughout this proof and the next one, we adopt the notation $\bar{\pi}$ for the standardization $\mathrm{std}(\pi)$.

Proof. We proceed by induction, with the base case ($n = 0$) being true by convention. Suppose that the result holds for all lengths less than n.

Let $\pi \in \mathfrak{J}_n(312)$. We write $\pi = \alpha 1 \beta$, so that α and β are 312-avoiding Jacobi permutations, β is of even length, and every letter of α is smaller than every letter of β. In fact, we can write π as the direct sum $\pi = (\alpha 1) \oplus \bar{\beta}$, where both summands $\alpha 1$ and $\bar{\beta}$ are Jacobi. Let us consider two cases:

- Suppose that $\bar{\beta}$ is nonempty, so that the lengths of $\alpha 1$ and $\bar{\beta}$ are both less than n. By the induction hypothesis, both $(\alpha 1)^{-1}$ and $\bar{\beta}^{-1}$ are Jacobi. Recall that $\pi^{-1} = (\alpha 1)^{-1} \oplus \bar{\beta}^{-1} = (\alpha 1)^{-1}\beta^{-1}$ by Lemma 5.1. Taking y to be the last letter of $(\alpha 1)^{-1}$, we see from Lemma 2.3 that π^{-1} is Jacobi.
- If β is empty, then $\pi = \bar{\alpha} \ominus 1$, so we have $\pi^{-1} = 1 \ominus \bar{\alpha}^{-1} = n\bar{\alpha}^{-1}$ by Lemma 5.2. Moreover, because $\alpha 1$ is Jacobi, α and thus $\bar{\alpha}$ are Jacobi as well. The induction hypothesis then tells us that $\bar{\alpha}^{-1}$ is Jacobi, so $\pi^{-1} = n\bar{\alpha}^{-1}$ is Jacobi by Lemma 2.1 (b).

Now, let $\pi \in \mathfrak{J}_n(231)$. Writing $\pi = \alpha n \beta$, we have that α and β are 231-avoiding permutations with every letter of α being smaller than every letter of β. Thus, we can write $\pi = \alpha \oplus \overline{n\beta}$. By Lemmas 2.1 (a) and 5.5 (b), $\overline{n\beta}$ and α are Jacobi. Dividing into cases again:

- If α is nonempty, then α and $\overline{n\beta}$ have lengths less than n, so the induction hypothesis tells us that α^{-1} and $(\overline{n\beta})^{-1}$ are Jacobi. Moreover, $n\beta$—and consequently $(n\beta)^{-1}$—have even length according to Lemma 5.5 (a). Similar to before, we have $\pi^{-1} = \alpha^{-1} \oplus (\overline{n\beta})^{-1} = \alpha^{-1}(n\beta)^{-1}$ by Lemma 5.1, and taking y to be the last letter of α^{-1}, Lemma 2.3 implies that $\pi^{-1} = \alpha^{-1}(n\beta)^{-1}$ is Jacobi.

- If α is empty, then we have $\pi = n\beta = 1 \ominus \beta$, so $\pi^{-1} = \beta^{-1} \ominus 1 = \beta^{-1}1$ by Lemma 5.2. We know that β is Jacobi because of Lemma 2.1 (a), so from the induction hypothesis, β^{-1} is Jacobi as well. Therefore, $\pi^{-1} = \beta^{-1}1$ is Jacobi. □

Recall that Jacobi trees are the unlabeled trees corresponding to the increasing binary trees of 312-avoiding Jacobi permutations, and that dual Jacobi trees are those corresponding to the decreasing binary trees of 231-avoiding Jacobi permutations. It is not obvious from their definitions that Jacobi and dual Jacobi trees are in bijection, but Theorem 5.1 shows that the inverse induces a bijection between $\mathcal{J}_n$ and $\tilde{\mathcal{J}}_n$, justifying Theorem 3.2.

In analogy to the notion of doubly alternating permutations—studied, e.g., in [9, 10, 18, 21]—let us call a permutation π *doubly Jacobi* if both π and π^{-1} are Jacobi. Then Theorem 5.1 implies that all permutations in $\mathfrak{J}_n(312)$ and $\mathfrak{J}_n(231)$ are doubly Jacobi, and the next result tells us that the same is true for $\mathfrak{J}_n(213)$.

Theorem 5.2. *For all $n \geq 0$, the permutation π^{-1} is Jacobi whenever $\pi \in \mathfrak{J}_n(213)$.*

Proof. We follow the same inductive approach as the proof of Theorem 5.1. Again, the base case is trivial, so assume that the result holds for all lengths less than n. Let $\pi \in \mathfrak{J}_n(213)$, and write $\pi = \alpha 1 \beta$ where α and β are 213-avoiding Jacobi permutations, β is of even length, and every letter of α is larger than every letter of β. In other words, π is the skew sum $\pi = \bar{\alpha} \ominus (1\beta)$. Note that 1β is Jacobi because β is Jacobi and has even length.

- If α is nonempty, then both $\bar{\alpha}$ and 1β have length less than n, so $\bar{\alpha}^{-1}$ and $(1\beta)^{-1}$ are Jacobi by the induction hypothesis. Because the first letter of $\bar{\alpha}^{-1}$ corresponds to a left-to-right minimum of $\pi^{-1} = (1\beta)^{-1} \ominus \bar{\alpha}^{-1}$, it follows from Lemma 2.2 that π^{-1} is Jacobi.

- If α is empty, then $\pi^{-1} = 1 \oplus \bar{\beta}^{-1} = 1\beta^{-1}$. Observe that β^{-1} is Jacobi due to the induction hypothesis, and because β^{-1} is of even length, $\pi^{-1} = 1\beta^{-1}$ is Jacobi. □

5.3 Enumeration, descents, and left-to-right minima

The fact that $\mathfrak{J}_n(312)$ and $\mathfrak{J}_n(231)$ are in bijection allows several of our results for $\mathfrak{J}_n(312)$ to directly translate over to $\mathfrak{J}_n(231)$.

Theorem 5.3. *For all $n \geq 0$, we have*

$$j_{2n}(231) = \frac{1}{2n+1}\binom{3n}{n} \quad \textit{and} \quad j_{2n+1}(231) = \frac{1}{2n+1}\binom{3n+1}{n+1}.$$

Proof. This follows immediately from Lemma 5.4 along with Theorems 4.1 and 5.1. □

Theorem 5.4. *We have*

$$j^{\mathrm{des}}_{2n,k}(231) = \frac{1}{n}\binom{n}{k-n}\binom{2n}{k+1} \qquad \textit{for all } n \geq 1 \textit{ and } k \geq 0;$$
$$j^{\mathrm{des}}_{2n+1,k}(231) = \frac{1}{n+1}\binom{n+1}{k-n}\binom{2n}{k} \qquad \textit{for all } n, k \geq 0.$$

Proof. It is known that $\mathrm{des}(\pi) = \mathrm{des}(\pi^{-1})$ for any $\pi \in \mathfrak{S}_n(312)$; see, e.g., [25, Corollary 3.2]. The desired result follows from this fact, Lemma 5.4, and Theorems 4.2 and 5.1. □

Theorem 5.5. *For all $n \geq k \geq 1$, we have*

$$j^{\mathrm{lrmin}}_{n,k}(231) = \frac{2k}{3n-k}\binom{\frac{3n-k}{2}}{n}$$

if n and k have the same parity, and $j^{\mathrm{lrmin}}_{n,k}(231) = 0$ otherwise.

Proof. Lemma 5.3 implies that $\mathrm{lrmin}(\pi) = \mathrm{lrmin}(\pi^{-1})$ for all $\pi \in \mathfrak{S}_n$. The desired result follows from this fact, Lemma 5.4, and Theorems 4.3 and 5.1. □

5.4 Last letter

Unlike des and lrmin, the distribution of last over $\mathfrak{J}_n(231)$ does not translate nicely from its distribution over $\mathfrak{J}_n(213)$ or $\mathfrak{J}_n(312)$. Instead, we will prove the next result using dual Jacobi trees, which we recall are in bijection with 231-avoiding Jacobi permutations.

$n\backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1							
3	1	1						
4	1	1	1					
5	1	2	2	2				
6	1	2	3	3	3			
7	1	3	5	7	7	7		
8	1	3	6	9	12	12	12	
9	1	4	9	16	23	30	30	30

Table 12: The numbers $j^{\mathrm{last}}_{n,k}(231)$ up to $n = 9$.

Theorem 5.6. *For all $n, k \geq 1$, we have*

$$j^{\mathrm{last}}_{2n,k}(231) = \sum_{m=0}^{n-1} \frac{n+k-3m-1}{n+k-1} \binom{n-m-1}{k-2m-1} \binom{n+k-1}{m} \quad \textit{and}$$
$$j^{\mathrm{last}}_{2n+1,k}(231) = \sum_{m=0}^{n-1} \frac{n+k-3m-1}{n+k-1} \binom{n-m}{k-2m-1} \binom{n+k-1}{m}.$$

Our first step in proving Theorem 5.6 is to establish the following expression for the generating function

$$J^{\mathrm{last}}(t, x; 231) := \sum_{n=1}^{\infty} \sum_{\pi \in \mathfrak{J}_n(231)} t^{\mathrm{last}(\pi)} x^n = \sum_{n=1}^{\infty} \sum_{k=1}^{n} j^{\mathrm{last}}_{n,k}(231) t^k x^n$$

in terms of the generating function for even-sized dual Jacobi trees.

Proposition 5.1. *We have*

$$J^{\mathrm{last}}(t, x; 231) = \frac{tx(1 + xG_{\mathrm{e}}(tx))}{1 - x^2 G_{\mathrm{e}}(tx)\,(1 + tG_{\mathrm{e}}(tx))}$$

where $G_{\mathrm{e}}(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1}\binom{3n}{n} x^{2n}$ is the generating function for dual Jacobi trees of even size (equivalently, Jacobi trees of even size, or 231-avoiding Jacobi permutations of even length).

Proof. Let

$$G_{\mathrm{e}}(x) := \sum_{n=0}^{\infty} |\tilde{\mathcal{J}}_{2n}| x^{2n} \qquad \text{and} \qquad G_{\mathrm{o}}(x) := \sum_{n=0}^{\infty} |\tilde{\mathcal{J}}_{2n+1}| x^{2n+1}$$

be the generating functions for even- and odd-sized dual Jacobi trees, respectively. By Proposition 3.4, we have

$$J^{\mathrm{last}}(t, x; 231) = \sum_{n=1}^{\infty} \sum_{T \in \tilde{\mathcal{J}}_n} t^{n-\mathrm{rbrch}(T)+1} x^n = \sum_{n=1}^{\infty} \sum_{T \in \tilde{\mathcal{J}}_n} x^{\mathrm{rbrch}(T)-1} (tx)^{n-\mathrm{rbrch}(T)+1};$$

we can interpret this as a weighted count of dual Jacobi trees where each node on the right branch, except the root, contributes x to the weight and all other nodes each contributes tx to the weight. From the recursive decomposition for dual Jacobi trees (see Figure 6), we obtain

$$J^{\mathrm{last}}(t, x; 231) = tx + tx^2 G_{\mathrm{e}}(tx) + xG_{\mathrm{o}}(tx) J^{\mathrm{last}}(t, x; 231) + x^2 G_{\mathrm{e}}(tx) J^{\mathrm{last}}(t, x; 231),$$

whence

$$J^{\mathrm{last}}(t, x; 231) = \frac{tx + tx^2 G_{\mathrm{e}}(tx)}{1 - xG_{\mathrm{o}}(tx) - x^2 G_{\mathrm{e}}(tx)}. \tag{15}$$

Since dual Jacobi trees are in bijection with (ordinary) Jacobi trees, let us now interpret $G_{\mathrm{e}}(x)$ and $G_{\mathrm{o}}(x)$ as being generating functions for the latter. Then $G_{\mathrm{o}}(x) = xG_{\mathrm{e}}(x)^2$, because every Jacobi tree of odd size consists of a root whose subtrees are both of even size. Substituting $G_{\mathrm{o}}(tx) = txG_{\mathrm{e}}(tx)^2$ into (15) yields the desired formula. □

We prove Theorem 5.6 from Proposition 5.1 by extracting coefficients from the generating function $J^{\mathrm{last}}(t,x;231)$.

Proof of Theorem 5.6. We first expand the expression for $J^{\mathrm{last}}(t,x;231)$ given in Proposition 5.1, with the help of the binomial theorem, to obtain

$$\begin{aligned}
J^{\mathrm{last}}(t,x;231) &= \frac{tx(1+xG_{\mathrm{e}}(tx))}{1-x^2G_{\mathrm{e}}(tx)\,(1+tG_{\mathrm{e}}(tx))} \\
&= tx(1+xG_{\mathrm{e}}(tx))\sum_{i=0}^{\infty} x^{2i}G_{\mathrm{e}}(tx)^i\,(1+tG_{\mathrm{e}}(tx))^i \\
&= tx(1+xG_{\mathrm{e}}(tx))\sum_{i,l=0}^{\infty}\binom{i}{l}G_{\mathrm{e}}(tx)^{i+l}t^lx^{2i} \\
&= \sum_{i,l=0}^{\infty}\binom{i}{l}G_{\mathrm{e}}(tx)^{i+l+1}t^{l+1}x^{2i+2} + \sum_{i,l=0}^{\infty}\binom{i}{l}G_{\mathrm{e}}(tx)^{i+l}t^{l+1}x^{2i+1}. \qquad (16)
\end{aligned}$$

It is well known that the k-fold convolution of the sequence $\left\{\frac{1}{2n+1}\binom{3n}{n}\right\}_{n\geq 0}$ is given by

$$\left\{\frac{k}{3n+k}\binom{3n+k}{n}\right\}_{n\geq 0};$$

see, e.g., [8, Section 3.3]. That is, we have

$$G_{\mathrm{e}}(tx)^k = \sum_{m=0}^{\infty}\frac{k}{3m+k}\binom{3m+k}{m}t^{2m}x^{2m}$$

for all $k\geq 1$. Substituting this expression, for the appropriate values of k, into (16) yields

$$\begin{aligned}
J^{\mathrm{last}}(t,x;231) = &\sum_{i,l,m=0}^{\infty}\binom{i}{l}\frac{i+l+1}{3m+i+l+1}\binom{3m+i+l+1}{m}t^{l+2m+1}x^{2i+2m+2} \\
&+ tx + \sum_{\substack{i,l,m=0\\ i+l\neq 0}}^{\infty}\binom{i}{l}\frac{i+l}{3m+i+l}\binom{3m+i+l}{m}t^{l+2m+1}x^{2i+2m+1}. \qquad (17)
\end{aligned}$$

We take the right-hand side of (17), make the substitutions $n=i+m+1$ and $k=l+2m+1$ in the first sum, and substitute $n=i+m$ and $k=l+2m+1$ in the second sum to get

$$\begin{aligned}
J^{\mathrm{last}}(t,x;231) = &\sum_{n,k=1}^{\infty}\sum_{m=0}^{n-1}\frac{n+k-3m-1}{n+k-1}\binom{n-m-1}{k-2m-1}\binom{n+k-1}{m}t^kx^{2n} \\
&+ tx + \sum_{n,k=1}^{\infty}\sum_{m=0}^{n-1}\frac{n+k-3m-1}{n+k-1}\binom{n-m}{k-2m-1}\binom{n+k-1}{m}t^kx^{2n+1}.
\end{aligned}$$

Extracting coefficients completes the proof. □

6. 123-avoiding Jacobi permutations

Next, we study 123-avoiding Jacobi permutations, whose enumeration is given by the following.

Theorem 6.1. *For all $n\geq 1$, we have*

$$j_n(123) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{1}{2k+1}\binom{n-k-1}{k}\binom{n}{2k}.$$

These numbers appear in [20, A101785], where it is stated that the entries also count Dyck paths of semilength n whose descents are all of odd length. We will prove Theorem 6.1 by showing that a certain bijection between 123-avoiding permutations and Dyck paths restricts to a bijection between 123-avoiding Jacobi permutations

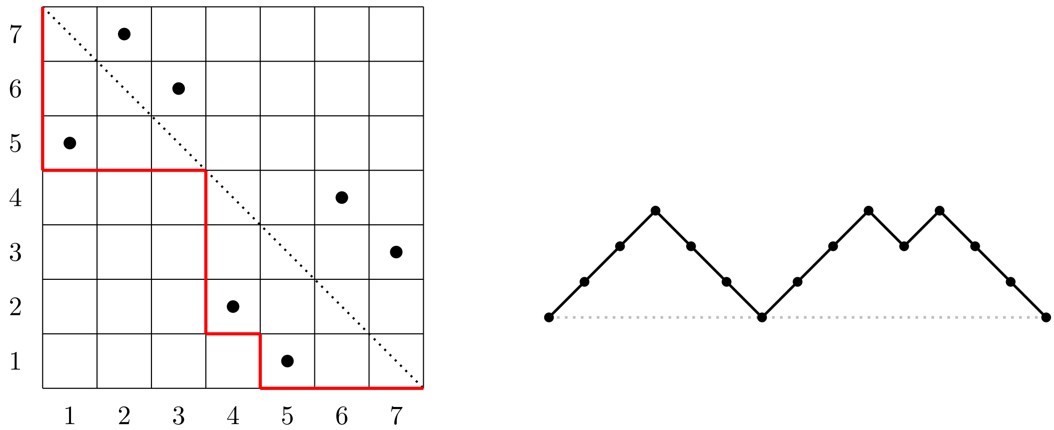

Figure 11: The Dyck path $\chi(5762143) = UUUDDDUUUDUDDD$, drawn in both ways.

and Dyck paths with all descents odd. Furthermore, we will determine the joint distribution of left-to-right minima and ascents over $\mathfrak{J}_n(123)$ by counting these Dyck paths with respect to certain subword statistics.

n	0	1	2	3	4	5	6	7	8	9	10	11
$j_n(123)$	1	1	1	2	5	12	30	79	213	584	1628	4600

Table 13: The numbers $j_n(123)$ up to $n = 11$.

6.1 Dyck paths and 123-avoiding permutations

A *Dyck path* of *semilength* n is a lattice path in $\mathbb{Z}^2$—consisting of $2n$ steps from the step set $\{(1,1), (1,-1)\}$—which starts at the origin $(0,0)$, ends at $(2n,0)$, and never traverses below the x-axis. The steps $(1,1)$ are denoted by U and called *up steps*, whereas the $(1,-1)$ are denoted by D and called *down steps*. Thus, Dyck paths can be represented as words on the alphabet $\{U, D\}$ subject to certain restrictions.

Let $\mathcal{D}_n$ denote the set of Dyck paths of semilength n. We shall define a bijection from $\mathfrak{S}_n(123)$ to $\mathcal{D}_n$ for which it will be convenient to illustrate Dyck paths as instead having steps from the step set $\{(0,-1),(1,0)\}$, beginning and ending on the antidiagonal line $y = -x$, and never traversing above this line. We can identify these paths with the Dyck paths defined earlier by swapping the "south steps" $(0,-1)$ with up steps, and the "east steps" $(1,0)$ with down steps. We will draw Dyck paths in this latter way (with south and east steps), while still thinking of them using the original definition (with up and down steps).

Define $\chi\colon \mathfrak{S}_n(123) \to \mathcal{D}_n$ in the following way. Given $\pi \in \mathfrak{S}_n(123)$, we first draw the array of π (as defined in Section 5.1). Then, $\chi(\pi)$ is the Dyck path drawn from the upper-left corner to the lower-right corner of the array which leaves all of the points (k, π_k) to its right but remains as close as possible to the antidiagonal. Equivalently, if $\pi_{i_1} > \pi_{i_2} > \cdots > \pi_{i_m}$ are the left-to-right minima of π, then

$$\chi(\pi) = U^{n+1-\pi_{i_1}} D^{i_2-i_1} U^{\pi_{i_1}-\pi_{i_2}} D^{i_3-i_2} \cdots U^{\pi_{i_{m-1}}-\pi_{i_m}} D^{n+1-i_m}. \tag{18}$$

See Figure 11 for an example.

A *factor* of a Dyck path refers to a factor (i.e., a consecutive subword) of the corresponding word. Given a Dyck path μ and a word α on the alphabet $\{U, D\}$, let $\operatorname{occ}_\alpha(\mu)$ denote the number of α-factors in μ, that is, occurrences of α as a factor of μ. For example, if μ is the Dyck path in Figure 11, then $\operatorname{occ}_{UD}(\mu) = 3$ and $\operatorname{occ}_{UDD}(\mu) = 2$.

To see why χ is a bijection, first note that every 123-avoiding permutation is a shuffle** of two decreasing subwords: one consisting of its left-to-right minima, and one consisting of all its other letters. The UD-factors of $\chi(\pi)$ correspond precisely to the left-to-right minima of π. So, given a Dyck path μ, we can recover the permutation π for which $\mu = \chi(\pi)$ by first identifying the left-to-right minima of π using the UD-factors of $\chi(\pi)$, which then determines the rest of π as the remaining letters must be placed in descending order.

The bijection χ is a slight modification of a bijection due to Krattenthaler [11]. Krattenthaler's original bijection is defined in the same way, except that the points (k, π_k) are to the left of the path (as opposed to the right), and the path never goes below (as opposed to above) the antidiagonal. As such, the UD-factors correspond to right-to-left maxima (as opposed to left-to-right minima).

**Given permutations α and β on disjoint sets of letters, we say that π is a *shuffle* of α and β if $|\pi| = |\alpha| + |\beta|$ and both α and β are subwords of π.

We use our bijection χ in lieu of Krattenthaler's because there is a nice characterization of the Dyck paths corresponding to Jacobi permutations under χ. A *descent* in a Dyck path is a maximal consecutive subword of down steps. For example, the Dyck path in Figure 11 has 3 descents; read from left to right, the descents have lengths 3, 1, and 3. Notice that the permutation in Figure 11 is Jacobi, which is not a coincidence.

For convenience, we say that a descent is *odd* if its length is odd.

Proposition 6.1. *Let $\pi \in \mathfrak{S}_n(123)$. Then π is Jacobi if and only if $\chi(\pi)$ has all descents odd.*

Proof. Because π avoids 123, it is a shuffle of two decreasing subwords, one consisting of the left-to-right minima of π, and one consisting of all other letters of π. Let $\pi_{i_1} > \pi_{i_2} > \cdots > \pi_{i_m}$ denote the left-to-right minima of π, and let $i_{m+1} = n+1$. Then $|\rho_\pi(\pi_{i_k})| = i_{k+1} - i_k - 1$ for all $k \in [m]$, and $|\rho_\pi(x)| = 0$ whenever x is not a left-to-right minimum of π. Thus, π is Jacobi if and only if $i_{k+1} - i_k - 1$ is even—or equivalently, $i_{k+1} - i_k$ is odd—for all $k \in [m]$, and we see from (18) that this condition is equivalent to $\chi(\pi)$ having all descents odd. $\square$

We are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Proposition 6.1 implies that the bijection χ restricts to a bijection between $\mathfrak{J}_n(123)$ and the subset of paths in $\mathcal{D}_n$ with all descents odd. Since the latter is counted by $\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}\binom{n-k-1}{k}\binom{n}{2k}$ [20, A101785], it follows that $j_n(123)$ is equal to this number. $\square$

The next lemma allows us to translate left-to-right minima and ascents in π to occurrences of UD- and UDD-factors of $\chi(\pi)$.

Lemma 6.1. *Let $\pi \in \mathfrak{S}_n(123)$. Then:*

(a) $\mathrm{lrmin}(\pi) = \mathrm{occ}_{UD}(\chi(\pi))$*, and*

(b) $\mathrm{asc}(\pi) = \mathrm{occ}_{UDD}(\chi(\pi))$.

Proof. We have already seen why (a) is true. For (b), notice that $\pi_i < \pi_{i+1}$ precisely when π_i is a left-to-right minimum of π and π_{i+1} is not a left-to-right minimum of π. The left-to-right minimum π_i corresponds to a UD-factor in $\chi(\pi)$, and the fact that the next letter is not a left-to-right minimum means that this UD-factor is followed by another down step. Hence, each ascent of π corresponds to a UDD-factor of $\chi(\pi)$. The converse—each UDD-factor of $\chi(\pi)$ corresponds to an ascent of π—is true by similar reasoning, and (b) follows. $\square$

One can also show that $\mathrm{des}(\pi) = \mathrm{occ}_{DU}(\chi(\pi)) + \mathrm{occ}_{DDD}(\chi(\pi))$, but it is easier to count Dyck paths by UDD-factors than by DU- and DDD-factors simultaneously. For this reason, we will work with ascents as opposed to descents in 123-avoiding permutations.

6.2 Left-to-right minima and ascents

So far, we have only studied distributions of individual statistics, but it turns out that we can find the joint distribution of lrmin and asc over $\mathfrak{J}_n(123)$ without much added difficulty. To that end, let

$$j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123) := |\{\, \pi \in \mathfrak{J}_n(123) : \mathrm{lrmin}(\pi) = i \text{ and } \mathrm{asc}(\pi) = k \,\}|\,.$$

Equivalently, $j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123)$ is the number of Dyck paths of semilength n with all descents odd and exactly i UD-factors and k UDD-factors.

Theorem 6.2. *For all n, i, and k satisfying $n \geq i \geq k \geq 0$ and $k \leq (n-i)/2$, we have*

$$j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123) = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1-k}{i-k}\binom{\frac{n-i}{2}-1}{k-1} \tag{19}$$

if n and i have the same parity, and $j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123) = 0$ otherwise.

The Dyck path interpretation of the numbers $j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123)$ will be essential to the proof of Theorem 6.2. We will also need to consider Dyck paths with last descent even and all other descents odd. Note that the empty path is excluded from this family of Dyck paths since the empty path does not have a last descent, but is vacuously a Dyck path with all descents odd.

Define the generating functions

$$P = P(s,t,x) := \sum_{n=0}^{\infty} \sum_{\mu \in \mathcal{D}_n^{(1)}} s^{\mathrm{occ}_{UD}(\mu)} t^{\mathrm{occ}_{UDD}(\mu)} x^n \qquad \text{and}$$

$$Q = Q(s,t,x) := \sum_{n=1}^{\infty} \sum_{\mu \in \mathcal{D}_n^{(2)}} s^{\mathrm{occ}_{UD}(\mu)} t^{\mathrm{occ}_{UDD}(\mu)} x^n,$$

where $\mathcal{D}_n^{(1)}$ is the set of Dyck paths of semilength n with all descents odd, and $\mathcal{D}_n^{(2)}$ is the set of Dyck paths of semilength n with last descent even and all other descents odd.

Lemma 6.2. *The generating function P satisfies the functional equation*

$$P = 1 + sxP - x^2P^2 + (x^2 + stx^3 - sx^3)P^3. \tag{20}$$

Proof. We first establish the system of functional equations

$$P = 1 + sxP + xPQ \tag{21}$$
$$Q = stx^2P^2 + xP(P - 1 - sxP). \tag{22}$$

To do so, we will use the *last return decomposition* of Dyck paths: every nonempty Dyck path μ can be uniquely written in the form $\mu = \nu U \eta D$ where ν and η are Dyck paths.

Let $\mu \in \mathcal{D}_n^{(1)}$. The empty path contributes 1 to (21), so now assume that μ is nonempty. Then $\mu = \nu U \eta D$ where ν has all descents odd, and η is either the empty path or has all descents odd except for its last one. The case where η is the empty path contributes sxP to (21) as

$$\mathrm{occ}_{UD}(\mu) = \mathrm{occ}_{UD}(\nu) + 1 \quad \text{and} \quad \mathrm{occ}_{UDD}(\mu) = \mathrm{occ}_{UDD}(\nu),$$

whereas if η is nonempty, then

$$\mathrm{occ}_{UD}(\mu) = \mathrm{occ}_{UD}(\nu) + \mathrm{occ}_{UD}(\eta) \quad \text{and} \quad \mathrm{occ}_{UDD}(\mu) = \mathrm{occ}_{UDD}(\nu) + \mathrm{occ}_{UDD}(\eta),$$

so this case contributes xPQ to (21). Hence, (21) holds.

To prove (22), let $\mu \in \mathcal{D}_n^{(2)}$. Then $\mu = \nu U \eta D$ where both ν and η have all descents odd and η is nonempty. If η ends with a UD-factor, then

$$\mathrm{occ}_{UD}(\mu) = \mathrm{occ}_{UD}(\nu) + \mathrm{occ}_{UD}(\eta) \quad \text{and} \quad \mathrm{occ}_{UDD}(\mu) = \mathrm{occ}_{UDD}(\nu) + \mathrm{occ}_{UDD}(\eta) + 1;$$

the generating function for these η is sxP, so this case provides the contribution stx^2P^2 to (22). On the other hand, if η does not end with a UD-factor, then

$$\mathrm{occ}_{UD}(\mu) = \mathrm{occ}_{UD}(\nu) + \mathrm{occ}_{UD}(\eta) \quad \text{and} \quad \mathrm{occ}_{UDD}(\mu) = \mathrm{occ}_{UDD}(\nu) + \mathrm{occ}_{UDD}(\eta);$$

these η have generating function $P - 1 - sxP$, thus accounting for the $x(P - 1 - sxP)$ term in (22). Therefore, (22) is proven.

The desired equation is obtained by substituting (22) into (21) and performing some algebraic manipulations. □

We now use Lemma 6.2 to prove Theorem 6.2.

Proof of Theorem 6.2. It follows from Lemma 6.1 that

$$j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123) = [s^i t^k x^n]\, P;$$

we will determine this coefficient using the Lagrange inversion formula [8]. In doing so, it will be helpful to define $\bar{P} := xP$, so that we seek the coefficient $[s^i t^k x^{n+1}]\, \bar{P}$. Multiplying both sides of (20) by x and performing some algebraic manipulations gives us

$$\bar{P} = x\left(1 + s\bar{P} + \frac{st\bar{P}^3}{1 - \bar{P}^2}\right),$$

so $\bar{P} = xR(\bar{P})$ if we take $R(u) := 1 + su + stu^3/(1 - u^2)$. Applying Lagrange inversion, along with the binomial theorem and the well-known identity

$$\left(\frac{1}{1-x}\right)^k = \sum_{l=0}^{\infty} \binom{l+k-1}{k-1} x^l,$$

yields

$$[s^i t^k x^{n+1}]\, \bar{P} = \frac{1}{n+1}\, [s^i t^k u^n] \left(1 + su + \frac{stu^3}{1-u^2}\right)^{n+1}$$

$$
\begin{aligned}
&= \frac{1}{n+1}\,[s^i t^k u^n] \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{stu^3}{1-u^2}\right)^k (1+su)^{n+1-k} \\
&= \frac{1}{n+1}\,[s^i t^k u^n] \sum_{k=0}^{n+1} \sum_{m=0}^{n+1-k} \binom{n+1}{k}\binom{n+1-k}{m} s^m u^m \sum_{l=0}^{\infty} s^k t^k u^{3k} \binom{l+k-1}{k-1} u^{2l} \\
&= \frac{1}{n+1}\,[s^i t^k u^n] \sum_{k=0}^{n+1} \sum_{m=0}^{n+1-k} \sum_{l=0}^{\infty} \binom{n+1}{k}\binom{n+1-k}{m}\binom{l+k-1}{k-1} s^{m+k} t^k u^{m+2l+3k} \\
&= \frac{1}{n+1}\,[s^i t^k u^n] \sum_{k=0}^{n+1} \sum_{i=k}^{n+1} \sum_{l=0}^{\infty} \binom{n+1}{k}\binom{n+1-k}{i-k}\binom{l+k-1}{k-1} s^i t^k u^{i+2l+2k}.
\end{aligned}
$$

We have $j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123) = 0$ whenever n and i have opposite parity, as in that case $i+2l+2k$ cannot equal n. Assuming that n and i have the same parity, note that $i+2l+2k=n$ is equivalent to $l=(n-i-2k)/2$, so

$$
j_{n,i,k}^{(\mathrm{lrmin},\mathrm{asc})}(123) = \frac{1}{n+1}\binom{n+1}{k}\binom{n+1-k}{i-k}\binom{\frac{n-i}{2}-1}{k-1}
$$

if $0 \le k \le i \le n$ and $(n-i-2k)/2 \ge 0$; the latter inequality is equivalent to $k \le (n-i)/2$. $\square$

The distributions of the individual statistics lrmin and asc over $\mathfrak{J}_n(123)$ are given by the following two theorems, which can be proven by either summing (19) over i or k, or by setting $s=1$ or $t=1$ in (20) prior to performing Lagrange inversion.

Theorem 6.3. *For all $n \ge k \ge 0$, we have*

$$
j_{n,k}^{\mathrm{lrmin}}(123) = \frac{1}{n+1}\binom{n+1}{k}\binom{\frac{n+k}{2}-1}{k-1}
$$

if n and k have the same parity, and $j_{n,k}^{\mathrm{lrmin}}(123) = 0$ otherwise.

Theorem 6.4. *For all $n \ge k \ge 0$, we have*

$$
j_{n,k}^{\mathrm{asc}}(123) = \frac{1}{n+1}\sum_{i=0}^{\lfloor (n-3k)/2 \rfloor} \binom{n+1}{k}\binom{n-k+1}{2i+2k+1}\binom{i+k-1}{k-1}.
$$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	0	1							
3	0	1	0	1						
4	0	0	4	0	1					
5	0	1	0	10	0	1				
6	0	0	9	0	20	0	1			
7	0	1	0	42	0	35	0	1		
8	0	0	16	0	140	0	56	0	1	
9	0	1	0	120	0	378	0	84	0	1

Table 14: The numbers $j_{n,k}^{\mathrm{lrmin}}(123)$ up to $n=9$.

Unfortunately, we do not have a formula for the distribution of the last letter statistic over $\mathfrak{J}_n(123)$. Unlike with left-to-right minima and ascents, which correspond to UD- and UDD-factors, the last letter of π does not seem to correspond nicely to a feature of the path $\chi(\pi)$. The value of the last letter does translate to the height of the last UD-factor under Krattenthaler's original bijection, which can be used to find the distribution of last over the full avoidance class $\mathfrak{S}_n(123)$. However, we were unable to adapt this approach to the Jacobi permutation setting, as we do not have a simple characterization of the Dyck paths corresponding to Jacobi permutations under Krattenthaler's bijection.

$n\backslash k$	0	1	2	3
0	1			
1	1			
2	1			
3	1	1		
4	1	4		
5	1	11		
6	1	26	3	
7	1	57	21	
8	1	120	92	
9	1	247	324	12

Table 15: The numbers $j^{\mathrm{asc}}_{n,k}(123)$ up to $n = 9$.

6.3 123-avoiding doubly Jacobi permutations

Recall from Section 5.2 that all permutations in $\mathfrak{J}_n(213)$, $\mathfrak{J}_n(231)$, and $\mathfrak{J}_n(312)$ are doubly Jacobi. The same is not true for $\mathfrak{J}_n(123)$, but the doubly Jacobi permutations in $\mathfrak{J}_n(123)$ have a nice enumeration.

Define the numbers s_n by

$$s_{n+1} = s_n + \sum_{k=1}^{n-1} s_k s_{n-1-k}$$

for all $n \geq 0$, with initial term $s_0 = 1$. The s_n have been called *secondary structure numbers* in the literature (e.g., in [3]), as they count certain graphs which are used to model secondary structures of RNA molecules.

n	0	1	2	3	4	5	6	7	8	9	10	11
s_n	1	1	1	2	4	8	17	37	82	185	423	978

Table 16: The secondary structure numbers s_n up to $n = 11$.

An *ascent* of a Dyck path is a maximal consecutive subword of up steps, and an ascent is *odd* if it has odd length. Another combinatorial interpretation for s_n, which we shall use for our proof of the next theorem, is the number of Dyck paths of semilength n with every ascent and descent odd [20, A004148].

Theorem 6.5. *For all $n \geq 0$, the number of doubly Jacobi permutations in $\mathfrak{J}_n(123)$ is equal to the secondary structure number s_n.*

Proof. Let $\pi \in \mathfrak{J}_n(123)$, so that $\chi(\pi)$ has all descents odd (Proposition 6.1). It suffices to prove that π^{-1} is Jacobi if and only if $\chi(\pi)$ has all ascents odd.

Let $\pi_{i_1} > \pi_{i_2} > \cdots > \pi_{i_m}$ be the left-to-right minima of π, so that $i_m > \cdots > i_2 > i_1$ are the left-to-right minima of π^{-1} by Lemma 5.3. Also, set $i_0 = 0$ and $\pi_0 = n + 1$. By the same reasoning as in the proof of Proposition 6.1, π^{-1} is Jacobi if and only if $\pi_{i_{k-1}} - \pi_{i_k}$ is odd for all $k \in [m]$, but observe from (18) that this is precisely the condition under which $\chi(\pi)$ has all ascents odd. □

One can study the distributions of lrmin and asc over 123-avoiding doubly Jacobi permutations by counting UD- and UDD-factors in Dyck paths with all ascents and descents odd, but we will not do this here.

7. 132- and 321-avoiding Jacobi permutations

The two remaining single-pattern Jacobi avoidance classes that we have yet to enumerate are $\mathfrak{J}_n(132)$ and $\mathfrak{J}_n(321)$. In this section, we shall find that $\mathfrak{J}_n(132) = \{n \cdots 21\}$ for all $n \geq 1$, and that the permutations in $\mathfrak{J}_n(321)$ are alternating.

7.1 132-avoidance

Proposition 7.1. *The only permutation in $\mathfrak{J}_S(132)$ is the decreasing permutation of S.*

Proof. Any decreasing permutation is clearly Jacobi and 132-avoiding, so it suffices to show that any nondecreasing Jacobi permutation has an occurrence of 132. Let π be a nondecreasing Jacobi permutation, and let k be its largest ascent. Then, by Lemma 2.1 (d), we know that π_{k+1} is not the last letter of π. Since k is the largest ascent of π, we have $\pi_{k+1} > \pi_{k+2}$. In addition, we have $\pi_k < \pi_{k+2}$ or else $\rho_\pi(\pi_k) = \pi_{k+1}$ which is of odd length. Therefore, $\pi_k\pi_{k+1}\pi_{k+2}$ is an occurrence of 132 in π. □

Corollary 7.1. *We have $j_n(132) = 1$ for all $n \geq 0$.*

7.2 321-avoidance: enumeration, ascents, left-to-right minima

The theorem below completely characterizes 321-avoiding Jacobi permutations.

Theorem 7.1. *Let $\pi \in \mathfrak{S}_S(321)$, $n = |S|$, and $y = \min S$.*

(a) *If n is even, then π is Jacobi if and only if π is down-up alternating.*

(b) *If n is odd, then π is Jacobi if and only if π is up-down alternating and $\pi_1 = y$.*

Our proof of Theorem 7.1 will require two lemmas. Together, these lemmas tell us that an even-length 321-avoiding Jacobi permutation must have its smallest letter in its second position and its largest letter in its penultimate position, and that an odd-length 321-avoiding Jacobi permutation must begin with its smallest letter.

Lemma 7.1. *If $\pi \in \mathfrak{J}_S(321)$ where $|S| = 2n > 0$, $y = \min S$, and $z = \max S$, then $\pi_2 = y$ and $\pi_{2n-1} = z$.*

Proof. Write $\pi = \alpha y \beta$, so that α and β are 321-avoiding Jacobi permutations and β is of even length. Observe that α cannot be empty, or else $\pi = y\beta$ would be of odd length. On the other hand, if $\alpha = \alpha_1\alpha_2\cdots\alpha_m$ where $m \geq 2$, then $\alpha_{m-1}\alpha_m y$ would be an occurrence of 321 in π; see Lemma 2.1 (d). Therefore, α must have length 1, which implies $\pi_2 = y$. The fact that $\pi_{2n-1} = z$ is also a consequence of Lemma 2.1 (d); π cannot end with z because π ends with a descent, and there cannot be more than one letter in π after z, or else z and the last two letters of π would form an occurrence of 321. □

We omit the proof of the next lemma because it is very similar to that of the above.

Lemma 7.2. *If $\pi \in \mathfrak{J}_S(321)$ where $|S| = 2n + 1$ and $y = \min S$, then $\pi_1 = y$.*

We now prove Theorem 7.1.

Proof of Theorem 7.1. We first prove the forward direction of (a) using induction, with the base case ($n = 0$) being trivial. Let π be a 321-avoiding Jacobi permutation of even length n, and suppose that the forward direction of (a) holds for all even lengths less than n. By Lemma 7.1, $\pi = \pi_1 y \beta$ where β is a 321-avoiding Jacobi permutation of even length. Then β is down-up according to the induction hypothesis, so $\pi = \pi_1 y \beta$ is down-up as desired.

Conversely, suppose that π is a 321-avoiding down-up permutation of even length. Because π is down-up, we have $|\rho_\pi(\pi_k)| = 0$ for all odd $k \in [n]$. On the other hand, if $k \in [n]$ is even, then π_k is a right-to-left minimum of π; otherwise, if there is a letter x appearing after π_k which is less than π_k, then $\pi_{k-1}\pi_k x$ would be an occurrence of 321 in π. Thus, $\rho_\pi(\pi_k) = \pi_{k+1}\pi_{k+2}\cdots\pi_n$ for all even $k \in [n]$, and these subwords are all of even length. It follows that π is Jacobi, and we have proven (a).

To prove the forward direction of (b), let π be a 321-avoiding Jacobi permutation of odd length n. Then, by Lemma 7.2, we have $\pi = y\beta$ where β is a 321-avoiding Jacobi permutation of even length. Then, applying part (a) of this theorem, we have that β is down-up. Hence, $\pi = y\beta$ is up-down.

Finally, let π be a 321-avoiding up-down permutation of odd length n with $\pi_1 = y$. Also, let $\beta = \pi_2\pi_3\cdots\pi_n$, so that $\pi = y\beta$. Observe that β is a 321-avoiding down-up permutation of even length, so β is Jacobi by part (a). Then $\rho_\pi(\pi_k) = \rho_\beta(\pi_k)$ is of even length for all $2 \leq k \leq n$, and $\rho_\pi(y) = \beta$ is of even length as well. We conclude that π is Jacobi. □

Let

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

denote the nth *Catalan number*. We have the following corollary to Theorem 7.1.

Corollary 7.2. *For all $n \geq 0$, we have $j_n(321) = C_{\lfloor n/2 \rfloor}$.*

Proof. It is known [23, Bijective Exercise 146] that C_n is the number of down-up permutations in $\mathfrak{S}_{2n}(321)$, so $j_{2n}(321) = C_n$. In addition, the up-down permutations in $\mathfrak{S}_{2n+1}(321)$ beginning with 1 are in bijection with the down-up permutations in $\mathfrak{S}_{2n}(321)$: simply remove the first letter and standardize the resulting permutation. Thus, $j_{2n+1}(321) = C_n$ as well. □

The next two corollaries are immediate consequences of Theorem 7.1.

$n\backslash k$	1	2	3	4	5	6	7	8
1	1							
2	1							
3	0	1						
4	0	1	1					
5	0	0	1	1				
6	0	0	1	2	2			
7	0	0	0	1	2	2		
8	0	0	0	1	3	5	5	
9	0	0	0	0	1	3	5	5

Table 17: The numbers $j^{\mathrm{last}}_{n,k}(321)$ up to $n = 9$.

Corollary 7.3. *For all $n \geq 1$ and $\pi \in \mathfrak{J}_{2n-1}(321) \sqcup \mathfrak{J}_{2n}(321)$, we have* $\mathrm{asc}(\pi) = n - 1$.

Corollary 7.4. *For all $n \geq 1$ and $\pi \in \mathfrak{J}_n(321)$, we have*

$$\mathrm{lrmin}(\pi) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

7.3 321-avoidance: last letter

While the distributions of asc and lrmin over $\mathfrak{J}_n(321)$ are not very interesting, it takes more work to prove the next theorem, which gives the distribution of last over $\mathfrak{J}_n(321)$.

Theorem 7.2. *For all $n \geq 1$, we have*

$$j^{\mathrm{last}}_{2n,k}(321) = \begin{cases} \dfrac{2n-k}{n}\dbinom{k-1}{n-1}, & \text{if } n \leq k \leq 2n-1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$j^{\mathrm{last}}_{2n+1,k}(321) = \begin{cases} \dfrac{2n-k+1}{n}\dbinom{k-2}{n-1}, & \text{if } n+1 \leq k \leq 2n, \\ 0, & \text{otherwise.} \end{cases}$$

One may notice from Table 17 that the $j^{\mathrm{last}}_{2n,k}(321)$ and $j^{\mathrm{last}}_{2n+1,k}(321)$ are each a shifted version of the *ballot numbers* $B_{n,k}$, defined by

$$B_{n,k} := \frac{n-k+1}{n+1}\binom{n+k}{n}$$

for all $n \geq k \geq 0$ [20, A009766].

We first state a few lemmas needed for the proof of Theorem 7.2.

Lemma 7.3. *Let $n \geq 1$.*

(a) *If $\pi \in \mathfrak{J}_{2n}(321)$, then $n \leq \pi_{2n} \leq 2n - 1$.*

(b) *If $\pi \in \mathfrak{J}_{2n+1}(321)$, then $n + 1 \leq \pi_{2n+1} \leq 2n$.*

Proof. Every 321-avoiding permutation is a shuffle of two increasing subwords, and because π is down-up, the odd-indexed letters form one increasing subword and the even-indexed letters form the other. If π has length $2n$, then we have $\pi_{2n} \geq n$ because π_{2n} is the nth term of an increasing subword of length n, and we also know from Lemma 2.1 (d) that $\pi_{2n} \neq 2n$ or else π would end with an ascent. The same reasoning holds when π has odd length. □

Lemma 7.4. *For all $n \geq 2$ and $n \leq k \leq 2n - 1$, we have*

$$j^{\mathrm{last}}_{2n,k}(321) = \sum_{l=n-1}^{k-1} j^{\mathrm{last}}_{2n-2,l}(321).$$

Proof. Let $\mathfrak{J}^{\text{last}}_{n,k}(321)$ denote the subset of permutations in $\mathfrak{J}_n(321)$ with $\text{last}(\pi) = k$. We provide a bijection $\delta\colon \bigsqcup_{l=n-1}^{k-1} \mathfrak{J}^{\text{last}}_{2n-2,l}(321) \to \mathfrak{J}^{\text{last}}_{2n,k}(321)$. Given $\pi \in \mathfrak{J}^{\text{last}}_{2n-2,l}(321)$, define $\delta(\pi)$ to be the permutation obtained from π by increasing every letter at least k by 1, and then appending the letters $2n$ and k to the result. For example, if $n = 4$ and $k = 5$, then $\delta(415263) = 41627385$.

To see why δ is well-defined, first recall from Theorem 7.1 (a) that π is down-up. By construction, $\delta(\pi)$ is necessarily a down-up permutation in $\mathfrak{S}_{2n}$ that ends with k. Suppose to the contrary that $\delta(\pi)$ has an occurrence of 321. Then the last letter k must correspond to the "1"—i.e., there exist two letters x and y in $\delta(\pi)$ for which xyk is an occurrence of 321. Because $x, y > k$, they correspond to the letters $x - 1$ and $y - 1$ in π, neither of which is the last letter l of π because $l < k$. It follows that $(x - 1)(y - 1)l$ is an occurrence of 321 in π, a contradiction. Thus, $\delta(\pi)$ is a 321-avoiding down-up permutation, so $\delta(\pi)$ is Jacobi by Theorem 7.1 (a), and therefore $\delta(\pi) \in \mathfrak{J}^{\text{last}}_{2n,k}(321)$.

The inverse of δ is as follows: given $\pi \in \mathfrak{J}^{\text{last}}_{2n,k}(321)$, we remove the last two letters of π, and then decrease every letter greater than k by 1. The result—call it π'—clearly remains a 321-avoiding down-up permutation. Also, because $2n$ is the penultimate letter of π (Lemma 7.1), it was removed and therefore π' is a permutation on the letters $1, 2, \ldots, 2n - 2$.

The last letter π'_{2n-2} of π' is at least $n-1$ by Lemma 7.3 (a); it remains to show that $\pi'_{2n-2} \leq k-1$. Suppose instead that $\pi'_{2n-2} \geq k$. Since π' is down-up and of even length, we have $\pi'_{2n-3} > \pi'_{2n-2}$, so $\pi_{2n-3} > \pi_{2n-2}$. Then $\pi_{2n-3}\pi_{2n-2}k$ would be an occurrence of 321 in π, a contradiction. Hence, $\pi' \in \mathfrak{J}^{\text{last}}_{2n-2,l}(321)$ for some $n - 1 \leq l \leq k - 1$. Since δ has a well-defined inverse, we conclude that it is a bijection. ☐

Lemma 7.5. *For all $n \geq 1$ and $n + 1 \leq k \leq 2n$, we have*

$$j^{\text{last}}_{2n+1,k}(321) = j^{\text{last}}_{2n,k-1}(321).$$

Proof. It is clear that the bijection from $\mathfrak{J}_{2n+1}(321) \to \mathfrak{J}_{2n}(321)$—removing the initial letter 1 and standardizing—decreases the value of the last letter by 1. ☐

We are now ready to prove Theorem 7.2.

Proof of Theorem 7.2. The ballot numbers $B_{n,k}$ satisfy the recurrence

$$B_{n,k} = \sum_{i=0}^{k} B_{n-1,i}$$

for all $n \geq 1$ and $0 \leq k \leq n$, along with initial term $B_{0,0} = 1$ [20, A009766]. Comparing this recurrence with that for $j^{\text{last}}_{2n,k}(321)$ given in Lemma 7.4, we find that

$$j^{\text{last}}_{2n,k}(321) = B_{n-1,k-n} = \frac{2n-k}{n}\binom{k-1}{n-1}$$

for all $n \geq 2$ and $n \leq k \leq 2n - 1$.[††] Moreover, we know from Lemma 7.3 (a) that $j^{\text{last}}_{2n,k}(321) = 0$ if k does not satisfy $n \leq k \leq 2n - 1$.

The formula for $j^{\text{last}}_{2n+1,k}(321)$ follows from that for $j^{\text{last}}_{2n,k}(321)$ along with Lemmas 7.3 (b) and 7.5. ☐

7.4 321-avoiding doubly Jacobi permutations

Finally, we prove that for all $n \geq 0$, there is exactly one doubly Jacobi permutation in $\mathfrak{J}_n(321)$.

Lemma 7.6. *Let $n \geq 1$. If $\pi \in \mathfrak{J}_{2n}(321)$ is doubly Jacobi, then $\pi_1 = 2$.*

Proof. The result clearly holds for $n = 1$, so assume $n \geq 2$. Let $\pi \in \mathfrak{J}_{2n}(321)$ be doubly Jacobi, and suppose for the sake of contradiction that $\pi_1 \neq 2$. By Lemma 7.1, we know that $\pi_1 \neq 1$, so $\pi_1 \geq 3$. This means that there are at least two letters preceding 1 in π^{-1}—i.e., if $\pi^{-1}_k = 1$, then $k \geq 3$. Notice that we must have $\pi^{-1}_{k-2} > \pi^{-1}_{k-1}$ because π^{-1} is Jacobi; otherwise, we would have $\rho_{\pi^{-1}}(\pi^{-1}_{k-2}) = \pi^{-1}_{k-1}$, which has odd length. But then $\pi^{-1}_{k-2}\pi^{-1}_{k-1}\pi^{-1}_k$ is an occurrence of 321 in π^{-1}, which is impossible because π being 321-avoiding means that π^{-1} is 321-avoiding as well (Lemma 5.4). Therefore, $\pi_1 = 2$. ☐

Proposition 7.2. *If $\pi \in \mathfrak{J}_{2n}(321)$ is doubly Jacobi, then*

$$\pi = \underbrace{21 \oplus 21 \oplus \cdots \oplus 21}_{n \text{ copies}} = 2143 \cdots (2n)(2n-1).$$

[††] In the case of $n = 1$, the formula for $j^{\text{last}}_{2n,k}(321)$ can be verified directly.

Proof. We induct on n, with the base case ($n = 0$) being trivial. Taking $n \geq 1$, suppose that the only doubly Jacobi permutation in $\mathfrak{J}_{2n-2}(321)$ is $\underbrace{21 \oplus 21 \oplus \cdots \oplus 21}_{n-1 \text{ copies}}$, and let $\pi \in \mathfrak{J}_{2n}(321)$ be doubly Jacobi. Lemmas 7.1 and 7.6 tell us that $\pi = 21 \oplus \beta$ for some $\beta \in \mathfrak{J}_{2n-2}(321)$, which implies $\pi^{-1} = 21 \oplus \beta^{-1}$ by Lemma 5.1. Then, because π^{-1} is Jacobi, we know from Lemma 2.1 (a) that β^{-1} is also Jacobi. Hence, β is doubly Jacobi, and we conclude from the induction hypothesis that

$$\pi = 21 \oplus \beta = 21 \oplus (\underbrace{21 \oplus 21 \oplus \cdots \oplus 21}_{n-1 \text{ copies}}) = \underbrace{21 \oplus 21 \oplus \cdots \oplus 21}_{n \text{ copies}}. \qquad \square$$

Proposition 7.3. *If $\pi \in \mathfrak{J}_{2n+1}(321)$ is doubly Jacobi, then*

$$\pi = 1 \oplus (\underbrace{21 \oplus 21 \oplus \cdots \oplus 21}_{n \text{ copies}}) = 13254 \cdots (2n+1)(2n).$$

Proof. We know from Lemma 7.2 that $\pi = 1 \oplus \beta$ for some $\beta \in \mathfrak{J}_{2n}(321)$. The same reasoning used in the previous proof shows that β is doubly Jacobi, and the result follows from Proposition 7.2. $\square$

Corollary 7.5. *For all $n \geq 0$, there exists exactly one doubly Jacobi permutation in $\mathfrak{J}_n(321)$.*

Notice that $2143 \cdots (2n)(2n-1)$ and $13254 \cdots (2n+1)(2n)$ are involutions, that is, they are self-inverse. In fact, $2143 \cdots (2n)(2n-1)$ is the only involution in $\mathfrak{J}_{2n}(321)$ and $13254 \cdots (2n+1)(2n)$ is the only involution in $\mathfrak{J}_{2n+1}(321)$, so $\pi \in \mathfrak{J}_n(321)$ is doubly Jacobi if and only if $\pi = \pi^{-1}$.

8. Multiple restrictions

We have, as of this point, enumerated each of the Jacobi avoidance classes $\mathfrak{J}_n(\sigma)$ where σ is a single pattern of length 3. In this section, we shall enumerate $\mathfrak{J}_n(\Pi)$ for all subsets $\Pi \subseteq \mathfrak{S}_3$ of size at least 2. Unlike before, we will only determine $j_n(\Pi)$ for each of these Π; we will not study the distribution of any statistics over these avoidance classes.

8.1 The trivial cases $132 \in \Pi$ and $321 \in \Pi$

If $\Pi \subseteq \mathfrak{S}_3$ with $|\Pi| \geq 2$ contains 132 or 321, then $j_n(\Pi) \leq 1$. We consider these trivial cases here before turning our attention to more interesting results.

Proposition 8.1. *If $132 \in \Pi \subseteq \mathfrak{S}_3$, then for all $n \geq 3$, we have*

$$j_n(\Pi) = \begin{cases} 0, & \text{if } 321 \in \Pi, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Immediate from Proposition 7.1. $\square$

Proposition 8.2. *Suppose $321 \in \Pi \subseteq \mathfrak{S}_3$ and $|\Pi| \geq 2$.*

(a) *If Π contains 123, 132, or 213, then $j_n(\Pi) = 0$ for all $n \geq 5$.*

(b) *Otherwise, $j_n(\Pi) = 1$ for each $n \geq 0$.*

Proof. The case $123 \in \Pi$ is an immediate consequence of the Erdős–Szekeres theorem [5], whereas the case $132 \in \Pi$ follows from Proposition 8.1.

Suppose that $213 \in \Pi$, and let $\pi \in \mathfrak{J}_n(\Pi)$ where $n \geq 4$. If n is even, then Lemma 7.1 implies that $\pi_2 = 1$, and in fact we must have $\pi_1 = n$ in order for π to avoid 213. But recall from Lemma 2.1 (d) that π ends with a descent, so $\pi_1\pi_{n-1}\pi_n$ is an occurrence of 321 in π. Hence, no such π can exist. If n is odd, then from Lemma 7.2, we have $\pi = 1 \oplus \beta$ where $\beta \in \mathfrak{J}_{n-1}(\Pi)$. However, we had just shown that there are no permutations in $\mathfrak{J}_{n-1}(\Pi)$, as $n-1$ is even. Therefore, (a) is proven.

Next, suppose that Π does not contain one of 123, 132, or 213. Then because $|\Pi| \geq 2$, we must have $231 \in \Pi$ or $312 \in \Pi$.

Assume that $231 \in \Pi$. Let $\pi \in \mathfrak{J}_{2n}(\Pi)$ where $n \geq 1$. Again, $\pi_2 = 1$, and to avoid 312 we must have $\pi_1 = 2$. Thus, $\pi = 21 \oplus \beta$ where $\beta \in \mathfrak{J}_{2n-2}(\Pi)$; this gives us a bijection between $\mathfrak{J}_{2n}(\Pi)$ and $\mathfrak{J}_{2n-2}(\Pi)$. On the other hand, when $\pi \in \mathfrak{J}_{2n-1}(\Pi)$, we have $\pi = 1 \oplus \beta$ where $\beta \in \mathfrak{J}_{2n-2}(\Pi)$. So, $\mathfrak{J}_{2n-1}(\Pi)$ and $\mathfrak{J}_{2n-2}(\Pi)$ are in bijection as well. Consequently, $j_{2n-2}(\Pi) = j_{2n-1}(\Pi) = j_{2n}(\Pi)$ for all $n \geq 1$, and because $j_0(\Pi) = 1$, we conclude that $j_n(\Pi) = 1$ for all $n \geq 0$.

The result for $312 \in \Pi$ follows from Lemma 5.4 and Theorem 5.1, thus completing the proof of (b). $\square$

8.2 Double restrictions

We will now determine $j_n(\Pi)$ for each $\Pi \subseteq \mathfrak{S}_3$ of size 2 (and where $132, 321 \notin \Pi$). Our first result here will feature an appearance of the *Fibonacci numbers* f_n, defined by $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$ along with $f_0 = f_1 = 1$. But first, we require a preliminary lemma.

Lemma 8.1. *Let $n \geq 2$. If $\pi \in \mathfrak{J}_n(123, 213)$, then either $\pi_1 = n$, or $\pi_2 = n$ and $\pi_3 = n-1$.*

Proof. Suppose that $\pi \in \mathfrak{J}_n(123, 213)$ and $\pi_1 \neq n$. Then $\pi_2 = n$, or else $\pi_1\pi_2 n$ would be an occurrence of 123 or 213 in π. Also, we cannot have $\pi_1 = n-1$, or else $\rho_\pi(\pi_1) = \pi_2$ which has odd length. And if $\pi_k = n-1$ for some $k \geq 4$, then $\pi_1\pi_3\pi_k$ would be an occurrence of 123 or 213 in π. □

Theorem 8.1. *For all $n \geq 1$, we have $j_n(123, 213) = f_{n-1}$.*

Proof. Define $\phi\colon \mathfrak{J}_n(123, 213) \to \mathfrak{J}_{n-1}(123, 213) \sqcup \mathfrak{J}_{n-2}(123, 213)$ by

$$\phi(\pi) = \begin{cases} \pi_2\pi_3 \cdots \pi_n, & \text{if } \pi_1 = n, \\ \pi_1\pi_4\pi_5 \cdots \pi_n, & \text{otherwise.} \end{cases}$$

For example, $\phi(7465132) = 465132$ and $\phi(5762431) = 52431$.

We claim that ϕ is a well-defined bijection. It is easy to see that if $\pi \in \mathfrak{J}_n(123, 213)$ and $\pi_1 = n$, then $\pi_2\pi_3 \cdots \pi_n \in \mathfrak{J}_{n-1}(123, 213)$. Now, suppose that $\pi \in \mathfrak{J}_n(123, 213)$ and $\pi_1 \neq n$. Recall from Lemma 8.1 that $\pi_2 = n$ and $\pi_3 = n-1$, so $\phi(\pi)$ is indeed a permutation of the set $[n-2]$. Moreover, we have $\rho_{\phi(\pi)}(\pi_k) = \rho_\pi(\pi_k)$ for all $4 \leq k \leq n$, as well as $\rho_\pi(\pi_1) = n(n-1)\rho_{\phi(\pi)}(\pi_1)$; and because each of these $\rho_\pi(\pi_k)$ has even length, the $\rho_{\phi(\pi)}(\pi_k)$ all have even length as well. Therefore, $\phi(\pi)$ is Jacobi (and clearly avoids 123 and 213).

The inverse of ϕ is defined by

$$\phi^{-1}(\pi) = \begin{cases} n\pi, & \text{if } \pi \in \mathfrak{J}_{n-1}(123, 213), \\ \pi_1 n(n-1)\pi_2\pi_3 \cdots \pi_n, & \text{if } \pi \in \mathfrak{J}_{n-2}(123, 213); \end{cases}$$

verifying that $\phi^{-1}(\pi) \in \mathfrak{J}_n(123, 213)$ can be done similarly.

Since ϕ is a bijection, it follows that $j_n(123, 213)$ satisfies the Fibonacci recurrence, and because $j_1(123, 213) = j_2(123, 213) = 1$, we conclude that $j_n(123, 213) = f_{n-1}$ for all $n \geq 1$. □

The next two Jacobi avoidance classes are counted by the quarter-squares plus 1.

Theorem 8.2. *For all $n \geq 0$ and $\sigma \in \{231, 312\}$, we have $j_n(123, \sigma) = 1 + \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.*

For the proof of this theorem, it will be helpful to have the notion of an *interval* of a permutation π, which is a consecutive subword of π consisting of consecutive integer letters. For example, 3412 is an interval of $\pi = 5734126$.

Proof. First, we prove the result for $\sigma = 231$. The formula clearly holds for $n = 0$, so assume that $n \geq 1$. Notice that any nonempty $\pi \in \mathfrak{J}_n(123, 231)$ is of the form $\pi = \alpha 1 \beta$ where β is a decreasing interval of even length and α is decreasing. The choice of β completely determines α, as α must consist of the remaining letters arranged in decreasing order; thus, it suffices to count the choices for β.

The length of β can be any nonnegative even integer $2k$ up to $n-1$, and assuming that $|\beta| = 2k > 0$, there are $n - 2k$ choices for the first letter β_1 of β. Because β is a decreasing interval, the choice of β_1 completely determines β, so there are $1 + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor}(n-2k)$ choices for β. A routine argument shows that

$$\sum_{k=1}^{\lfloor (n-1)/2 \rfloor}(n-2k) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor;$$

this completes the proof for $\sigma = 231$.

The result for $\sigma = 312$ follows from Lemma 5.4 and Theorem 5.1. □

Lastly, we consider three Jacobi avoidance classes which are enumerated by (repeated) powers of 2.

Theorem 8.3. *For all $n \geq 1$ and $\sigma \in \{231, 312\}$, we have $j_n(213, \sigma) = 2^{\lfloor (n-1)/2 \rfloor}$.*

Proof. We take $\sigma = 312$; the result for $\sigma = 231$ will follow from Lemma 5.4 and Theorem 5.1.

Let $\pi \in \mathfrak{J}_n(213, 312)$ be nonempty. Then 1 is either the first letter of π or the last letter of π, so either $\pi = \alpha \ominus 1$ or $\pi = 1 \oplus \beta$, where α and β are Jacobi permutations that avoid 213 and 312, and β is of even length. When $n \geq 3$ is odd, this gives us a 1-to-2 map from $\mathfrak{J}_{n-1}(213, 312)$ to $\mathfrak{J}_n(213, 312)$. On the other hand, we cannot have $\pi = 1 \oplus \beta$ when $n \geq 2$ is even, so in this case $\mathfrak{J}_{n-1}(213, 312)$ and $\mathfrak{J}_n(213, 312)$ are in bijection. In summary, we have

$$j_n(213, 312) = \begin{cases} 2j_{n-1}(213, 312), & \text{if } n \geq 3 \text{ is odd,} \\ j_{n-1}(213, 312), & \text{if } n \geq 2 \text{ is even,} \end{cases}$$

which along with $j_1(213, 312) = 1$ implies $j_n(213, 312) = 2^{\lfloor (n-1)/2 \rfloor}$ for all $n \geq 1$. ☐

Theorem 8.4. *For all $n \geq 1$, we have $j_n(231, 312) = 2^{\lfloor (n-1)/2 \rfloor}$.*

Proof. Let $\pi \in \mathfrak{J}_n(231, 312)$ be nonempty. Then we can write $\pi = \alpha 1 \beta$ where α is of the form $k(k-1)\cdots 2$ and β is a Jacobi permutation of even length on the letters $k+1, k+2, \ldots, n$ that avoids 231 and 312. Taking $J_{\mathrm{e}} := \sum_{n=0}^{\infty} j_{2n}(231, 312) x^{2n}$, we thus have the equation

$$J_{\mathrm{e}} = 1 + \frac{x^2}{1-x^2} J_{\mathrm{e}},$$

which leads to

$$J_{\mathrm{e}} = 1 + \frac{x^2}{1-2x^2} = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^{2n}.$$

Furthermore, if $J_{\mathrm{o}} := \sum_{n=1}^{\infty} j_{2n-1}(231, 312) x^{2n-1}$, then

$$J_{\mathrm{o}} = \frac{x}{1-x^2} J_{\mathrm{e}} = \frac{x}{1-2x^2} = \sum_{n=1}^{\infty} 2^{n-1} x^{2n-1}.$$

Adding these expressions for J_{e} and J_{o} gives us

$$\sum_{n=0}^{\infty} j_n(231, 312) x^n = 1 + \sum_{n=1}^{\infty} 2^{n-1} x^{2n} + \sum_{n=1}^{\infty} 2^{n-1} x^{2n-1} = 1 + \sum_{n=1}^{\infty} 2^{\lfloor (n-1)/2 \rfloor} x^n,$$

and comparing coefficients completes the proof. ☐

8.3 Triple and quadruple restrictions

We end this section by stating several results for $\mathfrak{J}_n(\Pi)$ where $|\Pi| = 3$ or $|\Pi| = 4$. Together with the results presented earlier, these complete the enumeration of $\mathfrak{J}_n(\Pi)$ over all subsets $\Pi \subseteq \mathfrak{S}_3$.

Theorem 8.5. *For all $n \geq 1$ and $\sigma \in \{231, 312\}$, we have $j_n(123, 213, \sigma) = \lceil n/2 \rceil$.*

Proof. We prove the result for $\sigma = 312$. As before, the result for $\sigma = 231$ will then follow from Lemma 5.4 and Theorem 5.1.

Let $\pi \in \mathfrak{J}_n(123, 213, 312)$ with $n \geq 2$. Then 1 is either the first letter of π or the last letter of π, so either $\pi = \alpha \ominus 1$ where $\alpha \in \mathfrak{J}_{n-1}(123, 213, 312)$, or $\pi = 1n(n-1)\cdots 2$; the latter is only possible when n is odd. Thus, we have

$$j_n(123, 213, 312) = \begin{cases} j_{n-1}(123, 213, 312), & \text{if } n \text{ is even,} \\ j_{n-1}(123, 213, 312) + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Because $j_1(123, 213, 312) = 1$, the result follows. ☐

Theorem 8.6. *For all $n \geq 1$, we have $j_n(123, 231, 312) = \lceil n/2 \rceil$.*

Proof. Let $\pi \in \mathfrak{J}_n(123, 231, 312)$ be nonempty. Recall from the proof of Theorem 8.2 that $\pi = \alpha 1 \beta$ where β is a decreasing interval of even length and α is decreasing, but since π also avoids 312 here, β must either be empty or begin with n. Adapting that proof accordingly, we have $1 + \lfloor (n-1)/2 \rfloor = \lceil n/2 \rceil$ choices for β, which again completely determines π. ☐

Theorem 8.7. *For all $n \geq 2$, we have*

$$j_n(213, 231, 312) = j_n(123, 213, 231, 312) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The proof is the same as that of Theorem 8.3, except that when n is even, π is forced to be $\pi = n \cdots 21$ as we are now requiring π to avoid 231. Hence, there is only one Jacobi permutation of length n avoiding the prescribed patterns for each even $n \geq 2$, and twice as many for each subsequent odd n. ☐

9. Future directions for research

To summarize, our work provides the first in-depth study of Jacobi permutations since their introduction by Viennot. We achieved a complete enumeration of the Jacobi avoidance classes $\mathfrak{J}_n(\Pi)$ where Π ranges over all subsets of $\mathfrak{S}_3$, and we have also studied the distribution of several statistics—the number of ascents/descents, the number of left-to-right minima, and the last letter—over the single-pattern avoidance classes and as well as the full set $\mathfrak{J}_n$.

Among those that we considered, the only distribution that eluded us was that of last over $\mathfrak{J}_n(123)$, so we leave this as an open problem.

Problem 9.1. *Determine the distribution of* last *over* $\mathfrak{J}_n(123)$.

Along the way, we discovered that all permutations in $\mathfrak{J}_n(213)$, $\mathfrak{J}_n(231)$, and $\mathfrak{J}_n(312)$ are doubly Jacobi, which led us to investigate doubly Jacobi permutations in the other single-pattern avoidance classes. However, we did not enumerate doubly Jacobi permutations in $\mathfrak{J}_n$ (without pattern avoidance restrictions).

Problem 9.2. *Enumerate doubly Jacobi permutations in* $\mathfrak{J}_n$.

Doubly alternating permutations, which motivated our notion of doubly Jacobi permutations, were also first studied with pattern avoidance restrictions [10, 18]. The enumeration of all doubly alternating permutations in $\mathfrak{S}_n$ was obtained later by Stanley [21] using symmetric function theory. Stanley's proof relies on the fact that whether a permutation is alternating is completely determined by its descent set. Since the Jacobi property is not determined by the descent set, Stanley's approach is inapplicable to our setting.

Given a permutation π of length at least 2, define $\mathrm{cpen}(\pi)$ to be the penultimate (second-to-last) letter of the complement of π. For example, the complement of $\pi = 263154$ is $\pi^c = 514623$, so $\mathrm{cpen}(\pi) = 2$. Recall from Theorem 2.3 that the distribution of last over $\mathfrak{J}_n$ is given by the Entringer numbers. We conjecture that the same is true for cpen.

Conjecture 9.1. *For all $n \geq 2$ and $k \geq 1$, we have $j^{\mathrm{cpen}}_{n,k} = E_{n,k}$.*

Furthermore, empirical evidence suggests the following conjecture about the joint distribution of cpen and last over $\mathfrak{J}_n$. Let

$$j^{(\mathrm{cpen},\mathrm{last})}_{n,i,k} := |\{\, \pi \in \mathfrak{J}_n : \mathrm{cpen}(\pi) = i \text{ and } \mathrm{last}(\pi) = k \,\}|\,.$$

Conjecture 9.2. *For all $n \geq 3$ and $i, k \geq 1$, we have $j^{(\mathrm{cpen},\mathrm{last})}_{n,i,k} = E_{n-1,n+1-i-k}$.*

In particular, Conjecture 9.2 implies that cpen and last have a symmetric joint distribution over $\mathfrak{J}_n$.

Recall from Corollary 2.1 and Theorem 2.3 that the distribution of asc over $\mathfrak{J}_n$ is equal to that of des over the set $\mathfrak{A}_n$ of André permutations, and also that the distribution of last over $\mathfrak{J}_n$ is equal to the distribution of first over $\mathfrak{A}_n$. Our next conjecture asserts that the joint distribution of $(\mathrm{asc}, \mathrm{last})$ over $\mathfrak{J}_n$ is equal to that of $(\mathrm{des}, \mathrm{first})$ over $\mathfrak{A}_n$. Let

$$\begin{aligned} j^{(\mathrm{asc},\mathrm{last})}_{n,i,k} &:= |\{\, \pi \in \mathfrak{J}_n : \mathrm{asc}(\pi) = i \text{ and } \mathrm{last}(\pi) = k \,\}| \qquad \text{and} \\ a^{(\mathrm{des},\mathrm{first})}_{n,i,k} &:= |\{\, \pi \in \mathfrak{A}_n : \mathrm{des}(\pi) = i \text{ and } \mathrm{first}(\pi) = k \,\}|\,. \end{aligned}$$

Conjecture 9.3. *For all $n \geq 1$, $i \geq 0$, and $k \geq 1$, we have $j^{(\mathrm{asc},\mathrm{last})}_{n,i,k} = a^{(\mathrm{des},\mathrm{first})}_{n,i,k}$.*

One might call this conjecturally-shared joint distribution the *André–Entringer distribution*, since it simultaneously refines the distributions given by the André polynomials and the Entringer numbers. To the best of our knowledge, the André–Entringer distribution has not appeared in the literature, which leads us to pose the following.

Problem 9.3. *Find a formula for computing the André–Entringer distribution.*

Finally, motivated by the various notions of "r-alternating permutation" in the literature,‡‡ we propose a notion of "r-Jacobi permutation". Given a positive integer r, we say that a permutation $\pi \in \mathfrak{S}_S$ is *r-Jacobi* if $|\rho_\pi(x)|$ is divisible by r for each $x \in S$. Notice that all permutations are 1-Jacobi, and 2-Jacobi permutations are precisely the Jacobi permutations.

Let $j^{(r)}_n$ denote the number of r-Jacobi permutations in $\mathfrak{S}_n$; see Table 18. The $j^{(r)}_n$ can be thought of as generalizing the Euler numbers. We believe that r-Jacobi permutations and the generalized Euler numbers $j^{(r)}_n$ are worthy of study.

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‡‡See, for example, Mendes–Remmel [16, p. 126] and Stanley [22, p. 17].

$r\backslash n$	1	2	3	4	5	6	7	8	9
1	1	2	6	24	120	720	5040	40320	362880
2	1	1	2	5	16	61	272	1385	7936
3	1	1	1	2	6	16	52	234	1018
4	1	1	1	1	2	7	22	57	184
5	1	1	1	1	1	2	8	29	85

Table 18: The numbers $j_n^{(r)}$ up to $r = 5$ and $n = 9$.

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