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Duality and a Canonical Sheaf in Periodic Riemann Functions

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Abstract: Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a Riemann function whose weight W is a perfect matching. Then there is a family of sheaves of k-vector spaces $\{\mathcal{M}_{W,\mathbf{d}}\}_{\mathbf{d}\in\mathbb{Z}^2}$ on a five-point topological space that models f in that $f(\mathbf{d}) = b^0(\mathcal{M}_{W,\mathbf{d}})$ and that

$$b^1(\mathcal{M}_{W,\mathbf{d}}) = f^{\wedge}_{\mathbf{K}}(\mathbf{d}-\mathbf{K})$$

for any $\mathbf{K}\in\mathbb{Z}^2$. Hence a Riemann-Roch formula for f is equivalent to an Euler characteristic computation of $\mathcal{M}_{W,\mathbf{d}}$.

If f and W are r-periodic, then the sheaves $\mathcal{M}_{W,\mathbf{d}}$ become $\mathcal{O}_r$-modules of finite type for a natural sheaf of rings $\mathcal{O}=\mathcal{O}_r$. We show that in this case there is a "canonical $\mathcal{O}$-module" $\omega=\omega_W$ and a pairing for $i=0,1$,

$$H^i(\mathcal{M}_{W,\mathbf{0}}\otimes\mathcal{F}) \times \mathrm{Ext}^{1-i}(\mathcal{F}, \mathcal{M}_{W^{\wedge}_{\mathbf{L}},\mathbf{K}}) \to H^1(\omega)\simeq k$$

that is perfect when $\mathbf{L}=\mathbf{K}+\mathbf{1}$ and $\mathcal{F}$ is a certain type of line bundle or a certain type of skyscraper sheaf. In particular, when $\mathcal{F}$ is a line bundle, we realize the above formula for $b^1(\mathcal{M}_{W,\mathbf{d}})$ as a duality theorem akin to Serre duality.

We show that canonical $\mathcal{O}$-module ω_W is a rather exceptional element in a family of tensor products of two modules $\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}'$, where $\mathcal{M}$ and $\mathcal{M}'$ vary over $\mathcal{O}_r$-modules of the form $\mathcal{M}_{W',\mathbf{d}}$.

This article doesn't assume any background in sheaf theory; rather we describe all our sheaves as a "diagrams of vector spaces," where each diagram is essentially a sheaf of vector spaces on a fixed topological space of five points.

Keywords: Betti numbers; Cohomology; Duality; Euler characteristic; Graph Riemann-Roch theorem; Riemann function; Riemann's theorem
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1. Introduction

In [4], Baker and Norine proved what they called a "graph Riemann-Roch" formula, which is an equality that resembles the classical Riemann-Roch formula. There is a large literature on this formula [3, 7, 9, 11, 24] and related formulas, e.g., [1–3, 6, 16, 19, 21, 23].

The classical Riemann-Roch formula has a modern proof where the formula represents an expression for the Euler characteristic of a sheaf, namely a line bundle over the appropriate Riemann surface. Baker and Norine [4] asked whether their formula could be viewed as an "Euler characteristic." Folinsbee and Friedman [12] gave a positive answer with a few caveats.

Let us provide some details. We emphasize that we do not assume the reader has seen sheaf theory: all sheaves are described explicitly as a certain "diagrams" of vector spaces in Section 2.

Earlier, Folinsbee and Friedman [11] defined a *Riemann function* to be any function $f\colon \mathbb{Z}^n\to\mathbb{Z}$ such that

1. $f(\mathbf{d})=0$ whenever $\deg(\mathbf{d}) \stackrel{\text{def}}{=} d_1+\cdots+d_n$ is sufficiently small, and
2. for some $C\in\mathbb{Z}$ — called the *offset of* f — we have $f(\mathbf{d})=\deg(\mathbf{d})+C$ whenever $\deg(\mathbf{d})$ is sufficiently large.

*No AI was used (in any part) of this article.

A Baker-Norine *rank function* [4] is an example, provided that we add one to the function. As such Riemann functions are vast generalizations — with far less structure — of the Baker-Norine rank functions and related functions. The classical *Riemann's formula* (and *Riemann-Roch formula*) for algebraic curves also give rise to Riemann functions.

Given a Riemann function, f, as above, for any $\mathbf{K} \in \mathbb{Z}^n$, define $f^\wedge_{\mathbf{K}} \colon \mathbb{Z}^n \to \mathbb{Z}$ via the formula

$$f(\mathbf{d}) - f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C \tag{1}$$

where C is the offset of f; we easily see that

$$f^\wedge_{\mathbf{K}}(\mathbf{d}) = f(\mathbf{K} - \mathbf{d}) - C - \deg(\mathbf{K} - \mathbf{d})$$

and that $f^\wedge_{\mathbf{K}}$ is also a Riemann function. We refer to (1) as a *Riemann-Roch formula*.

The Baker-Norine formula (and classical Riemann-Roch formula) implies that under certain conditions there exists a $\mathbf{K} \in \mathbb{Z}^n$ such that $f^\wedge_{\mathbf{K}} = f$. Folinsbee-Friedman [11] wrote this equivalently as follows: one easily sees that for any Riemann function $f \colon \mathbb{Z}^n \to \mathbb{Z}$ there is a unique function $W \colon \mathbb{Z}^n \to \mathbb{Z}$ such that

$$f(\mathbf{d}) = \sum_{\mathbf{d}' \le \mathbf{d}} W(\mathbf{d}').$$

We call W the *weight function* (or simply *weight*) of f. It turns out that $f^\wedge_{\mathbf{K}} = f$ iff for $\mathbf{L} = \mathbf{K} + \mathbf{1}$ (here $\mathbf{1} = (1, \dots, 1)$) we have $W^*_{\mathbf{L}} = (-1)^n W$ where $W^*_{\mathbf{L}}$ is defined by $W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L} - \mathbf{d})$. Given the relationship between $f^\wedge_{\mathbf{K}}$ and f, it is simpler to write $W^*_{\mathbf{L}} = (-1)^n W$. Folinsbee-Friedman [11] also showed that if G is a complete graph, then the weight functions associated to the Baker-Norine rank functions have a very simple form. This yielded a (second) simple formula for the Baker-Norine rank function on complete graphs, G, as an alternative to the earlier formula of Cori and Le Borne formula [7, 8].

Returning to (1), we ask whether there is a "sheaf" — we leave this vague for now — $\mathcal{M} = \mathcal{M}_{\mathbf{d}} = \mathcal{M}_{\mathbf{d},k}$, such that if $b^i(\mathcal{M})$ is the i-th Betti number of $\mathcal{M}$, then

$$b^0(\mathcal{M}_{\mathbf{d}}) = f(\mathbf{d}), \quad b^1(\mathcal{M}_{\mathbf{d}}) = f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}),$$

and $b^i(\mathcal{M}_{\mathbf{d}}) = 0$ if $i \neq 0, 1$. If so, then the left-hand-side of (1) equals the *Euler characteristic* of $\mathcal{M}_{\mathbf{d}}$, and the right-hand-side is a simple expression of $\mathbf{d}$.

Folinsbee-Friedman [12] showed that if W is a *perfect matching*, meaning a non-negative weight function of a Riemann function $f \colon \mathbb{Z}^2 \to \mathbb{Z}$, then such $\mathcal{M}_{\mathbf{d}} = \mathcal{M}_{W,\mathbf{d},k}$ exist, and can be built from W. In fact, for every field, k, one can build sheaves of k-vector spaces (on a five-point topological space[1]), $\{\mathcal{M}_{W,\mathbf{d},k}\}_{\mathbf{d} \in \mathbb{Z}^2}$. (Since the sheaves $\mathcal{M}_{\mathbf{d},k}$ depend on k in a simple manner, much like in [14], we tend to suppress the field k.) Moreover, the $\mathcal{M}_{\mathbf{d}} = \mathcal{M}_{W,\mathbf{d},k}$ fit together into certain *short exact sequences* involving skyscraper sheaves. Finally, there is a duality theorem that describes an isomorphism

$$H^i(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-d})' \to H^{1-i}(\mathcal{M}_{W,\mathbf{d}}) \tag{2}$$

where $'$ denotes the dual k-vector space.

Moreover, Folinsbee and Friedman [12] showed that all Riemann functions $\mathbb{Z}^n \to \mathbb{Z}$ and resulting Riemann-Roch formulas express the Euler characteristic of a family of sheaves that can be pieced together from those involving perfect matchings, provided that one is willing to work with *virtual sheaves*, meaning formal differences of sheaves. This is a fairly technical undertaking: there are many choices involved in building general sheaves for Riemann functions $\mathbb{Z}^n \to \mathbb{Z}$ from those involving perfect matchings, and one has to show that the resulting virtual sheaves are independent of all choices.

Hence Riemann functions whose weights are perfect matchings are the building blocks that can express the Riemann-Roch formulas associated to any Riemann function $\mathbb{Z}^n \to \mathbb{Z}$ as an Euler characteristic equation of virtual sheaves.

One ultimate goal of writing Baker-Norine formulas as expressing the Euler characteristics of sheaves would be to get a "simpler" proof of their Riemann-Roch formula. It is unclear if the sheaf theory of [12] will be able to do this: one reason is that its "duality theorem" for perfect matchings (akin to Serre duality) gives an isomorphism

$$H^1(\mathcal{M}) \simeq \mathrm{Hom}(\mathcal{M}, \omega)'.$$

where the "canonical sheaf," ω, is rather simple and independent of the perfect matching. Hence ω does not encode anything interesting about the Riemann function; so in the case of Baker-Norine formulas, ω doesn't encode anything interesting about the underlying graph.

[1] A sheaf of vector spaces on a five-point topological space is equivalent to the category of a certain type of diagram of five vector spaces with certain linear maps between these spaces. Hence these sheaves are quite simple in nature.

The point of this article is to give a more remarkable duality theorem, one that more closely resembles Serre duality. We restrict our attention to Riemann functions $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ whose weights $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ are perfect matchings; in addition, we assume that f is *r-periodic* in the sense of [11]: namely, $f(\mathbf{d}+(r,-r)) = f(\mathbf{d})$ for all $\mathbf{d}$, or equivalently $W(\mathbf{d}+(r,-r)) = W(\mathbf{d})$. In this case the sheaves $\mathcal{M}_{W,\mathbf{d}} = \mathcal{M}_{W,\mathbf{d},k}$ turn out to be $\mathcal{O}_r = \mathcal{O}_{r,k}$-modules of finite type for a simple sheaf of rings $\mathcal{O}_{r,k}$.[2]

Let us briefly describe our duality theorem: first, we show that if W is an r-periodic perfect matching, then for any $\mathcal{O}_r$-module $\mathcal{F}$, and fixed $\mathbf{K} \in \mathbb{Z}^2$ and $\mathbf{L} = \mathbf{K} + \mathbf{1}$, for $i = 0, 1$ we have a pairing

$$H^i(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Ext}^{1-i}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to H^1(\omega_{W,\mathbf{K}}) \tag{3}$$

where

$$\omega_{W,\mathbf{K}} \stackrel{\text{def}}{=} \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}.$$

This is constructed from:

1. a canonical map

$$\mathrm{Ext}^{1-i}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to \mathrm{Ext}^{1-i}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) = \mathrm{Ext}^{1-i}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega_{W,\mathbf{K}});$$

2. the Yoneda pairing

$$H^i(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Ext}^{1-i}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega_{W,\mathbf{K}}) \to H^1(\omega_{W,\mathbf{K}});$$

   and

3. the fact that $H^1(\omega_{W,\mathbf{K}}) \simeq k$.

We will then show that for certain values of $\mathcal{F}$, namely certain line bundles, $\mathcal{L}_{\mathbf{d}} = \mathcal{L}_{\mathbf{d},r,k}$ and certain skyscraper sheaves, (3) is a perfect pairing. Applying this to $\mathcal{F} = \mathcal{L}_{\mathbf{d}}$ and using

$$\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{L}_{\mathbf{d}} \simeq \mathcal{M}_{W,\mathbf{d}}$$

and

$$\mathrm{Ext}^{1-i}(\mathcal{L}_{\mathbf{d}}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \simeq \mathrm{Ext}^{1-i}(\mathcal{O}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \simeq H^{1-i}(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})$$

gives

$$H^i(\mathcal{M}_{W,\mathbf{d}}) \simeq H^{1-i}(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})',$$

which again shows (2), although it is now realized as a very special case of one choice of $\mathcal{F}$ for which (3) is a perfect pairing.

We wish to make a number of remarks on the above.

Remark 1.1. *We say that an $\mathcal{O}_r$-module, $\mathcal{F}$, satisfies* strong duality *if for $i = 0, 1$, (3) is a perfect pairing. There is a "two out of three principle," namely if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence, and two of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ satisfy strong duality, then so does the third one. This can be used to give a class of $\mathcal{O}_r$-modules that satisfy strong duality larger than merely $\mathcal{L}_{\mathbf{d},r}$ and certain skyscraper sheaves. Hence strong duality seems to hold for a rich set of examples $\mathcal{F}$.*

Remark 1.2. *If W, W' are perfect matchings, and $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$, then*

$$\mathcal{M}_{W,\mathbf{d}} \otimes \mathcal{M}_{W',\mathbf{d}}$$

has infinite zeroth Betti number, b^0; moreover it has vanishing first Betti number b^1 unless $W' = W^*_{\mathbf{L}}$ for some $\mathbf{L} \in \mathbb{Z}^2$. Hence the canonical sheaf

$$\omega_{W,\mathbf{K}} = \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}$$

has a property that is exceptional.

Remark 1.3. *If W is an r-periodic matching, then $\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}$ is independent — up to isomorphism — of $\mathbf{K}, \mathbf{L}$ with $\mathbf{L} = \mathbf{K} + \mathbf{1}$. For this reason $\omega_{W,\mathbf{K}}$ depends only W up to isomorphism. However, as W varies, it seems likely that the ω_W take on non-isomorphic values. In Subsection 4.6 we discuss this further.*

Remark 1.4. *Theorem 3.1 of [12] shows that the any r-periodic Riemann function $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ can be modeled by a virtual sheaf built from sheaves $\mathcal{M}_{W_i,\mathbf{d}}$ where the W_i are r-periodic perfect matchings. We don't know if the virtual sheaf one gets for f is unique up to isomorphism as an $\mathcal{O}_r$-module. (This uniqueness is proven in [12] for a general f not assumed to be p-periodic, as diagrams of k-vector spaces.) Similarly for Riemann functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$. We hope to address this in a future article.*

[2] By contrast, the sheaves $\{\mathcal{M}_{W,d}\}$ are not of finite type as k-diagrams.

Again, we emphasize that in this article we assume no prior knowledge of sheaves and no knowledge of algebraic geometry or topology. We explain everything we do in terms of "diagrams" of vector spaces, of rings, and of algebras that we define from scratch (as was done in [12]). We include some optional sections where we explain how our diagrams relate to modern sheaf theory.

The rest of this article is organized as follows. In Section 2 we state our main results. In Section 3 we give the diagrams (sheaves) of rings, $\mathcal{O}_r = \mathcal{O}_{r,k}$, which are of interest to us, and give some basic facts about $\mathcal{O}$-modules. In Section 4 we prove various properties of the canonical diagram $\omega_{W,\mathbf{K}}$. In Section 5 we prove a duality theorem that proves that for $\mathcal{F} = \mathcal{L}_{\mathbf{d}}$, (3) is a perfect pairing for $i = 1$, by a long computation that assumes that the Yoneda pairing

$$H^1(\mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{G}) \to H^1(\mathcal{G})$$

is given by a certain formula. Section 6 formulates "strong duality," which is a stronger property than the duality in Section 5. Section 6 then proves strong duality for $\mathcal{F}$ being $\mathcal{L}_{\mathbf{d}}$ and certain skyscraper diagrams, as well as proving the "two out of three principle" for strong duality. Section 6 requires significantly more background on the homological algebra of $\mathcal{O}$-modules than the previous sections; some of this background is relegated to Appendix A.

We remark that some of the foundations that we develop in this article are based on the unpublished manuscript [10].

2. Main Results

In this section, we make our notation precise and state our main theorems.

2.1 Basic Notation

We use $\mathbb{Z}$ to denote the integers, and $\mathbb{N} = \mathbb{Z}_{\geq 1} = \{1, 2, \ldots\}$ for the natural numbers. For $n \in \mathbb{N}$ we use $[n]$ to denote $\{1, \ldots, n\}$. For $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, the *degree* of $\mathbf{d}$ is defined as $\deg(\mathbf{d}) = d_1 + \cdots + d_n$, and endow $\mathbb{Z}^n$ with its usual partial order, writing $\mathbf{d}' \leq \mathbf{d}$ to mean $d'_i \leq d_i$ for all $i \in [n]$.

2.1.1 Direct Sums and Maps

If S is a set and k a field, then, as usual, $k^{\oplus S}$ denotes the set of maps $f\colon S \to k$ such that $f(s) = 0$ for all but finitely many s in S; we call $k^{\oplus S}$ the *direct sum of S copies of k*; for $s \in S$, we use $\mathbf{e}_s \in k^{\oplus S}$ to denote the element that is 1 on s and 0 elsewhere; we refer to $\mathbf{e}_s$ as the *standard basis vector at s* (with $k^{\oplus S}$ understood)[3]. If $\pi\colon S \to T$ a map of sets, then we use $k^{\oplus \pi}$ to denote the unique linear map $k^{\oplus S} \to k^{\oplus T}$ taking $\mathbf{e}_s$ to $\mathbf{e}_{\pi(s)}$.

2.2 Riemann Functions

We refer the reader to Section 2 of [12] and Section 2 of [11] for proofs of the statements in this subsection and for examples of Riemann functions.

Definition 2.1. *By a* Riemann function *we mean a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ such that:*

1. *$f(\mathbf{d}) = 0$ for $\deg(\mathbf{d})$ sufficiently small, and*
2. *for some $C \in \mathbb{Z}$—called the* offset of f*—we have $f(\mathbf{d}) = \deg(\mathbf{d}) + C$ for $\deg(\mathbf{d})$ sufficiently large.*

Definition 2.2. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function of offset C, and $\mathbf{K} \in \mathbb{Z}^n$. We define the* $\mathbf{K}$-dual of f *to be the function $f^\wedge_{\mathbf{K}}\colon \mathbb{Z}^n \to \mathbb{Z}$ given by*

$$f^\wedge_{\mathbf{K}}(\mathbf{d}) \stackrel{def}{=} f(\mathbf{K} - \mathbf{d}) - C - \deg(\mathbf{K} - \mathbf{d}).$$

We easily see that $f^\wedge_{\mathbf{K}}$ is a Riemann function with offset $-\deg(\mathbf{K}) - C$, and that

$$\forall \mathbf{d} \in \mathbb{Z}^n, \quad f(\mathbf{d}) - f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C. \tag{4}$$

We refer to (4) as a *Riemann-Roch formula for f*. Hence $f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d})$ is independent of $\mathbf{K}$.

Definition 2.3. *We say that a Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is* self-dual *if for some $\mathbf{K}$ we have $f^\wedge_{\mathbf{K}} = f$.*

[3] If $s \in S$ and $S \subset S'$, then $\mathbf{e}_s$ can refer to an element of $k^{\oplus S}$ or $k^{\oplus S'}$; so we must always be clear on what is the ambient set containing s.

The point of articles such as [2, 4] is to study certain Riemann functions of interest, f, and determine if such f are self-dual; more precisely, these articles define "rank functions" $r\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $f = 1 + r$ is a Riemann function. In our approach, we do not require a Riemann function to be self-dual; hence we consider a much wider class of functions $\mathbb{Z}^n \to \mathbb{Z}$ with far less structure.

Our motivation for the term *Riemann function* is the classical *Riemann's theorem* for curves.

For examples of Riemann functions, we refer the reader to Section 2.3 of [12] and Sections 2.5 and 2.6 of [11].

2.3 Weights of Riemann Functions

For details and proofs of the material in this subsection, we refer the reader to Section 3 of [11].

We say that a function $g\colon \mathbb{Z}^n \to \mathbb{Z}$ is *initially zero* if $g(\mathbf{d}) = 0$ when $\deg(\mathbf{d})$ is sufficiently small; e.g., a Riemann function is initially zero. If $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is any initially zero function, then there is a unique initially zero function $W\colon \mathbb{Z}^n \to \mathbb{Z}$ such that

$$f(\mathbf{d}) = \sum_{\mathbf{d}' \le \mathbf{d}} W(\mathbf{d}');$$

we call W the *weight of* f.

More formally, for any initially zero function $W\colon \mathbb{Z}^n \to \mathbb{Z}$ we define the function $\mathfrak{s}W\colon \mathbb{Z}^n \to \mathbb{Z}$ via

$$(\mathfrak{s}W)(\mathbf{d}) = \sum_{\mathbf{d}' \le \mathbf{d}} W(\mathbf{d}');$$

for any function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ we define $\mathfrak{m}f\colon \mathbb{Z}^n \to \mathbb{Z}$ via

$$(\mathfrak{m}f)(\mathbf{d}) = \sum_{I \subset [n]} (-1)^{|I|} f(\mathbf{d} - \mathbf{e}_I)$$

where $\mathbf{e}_I$ is the vector of 1's and 0's that is 1 on the components in I (see Proposition 17 [11]). Proposition 17 of [11] says that if f and W are initially zero, then $f = \mathfrak{s}W$ iff $W = \mathfrak{m}f$. We will later use Theorem 30 of [11] which states that for all Riemann functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$ and $\mathbf{L}, \mathbf{K} \in \mathbb{Z}^n$ such that $\mathbf{L} = \mathbf{K} + \mathbf{1}$ ($\mathbf{1}$ is the all 1's vector),

$$\mathfrak{m}f^{\wedge}_{\mathbf{K}} = (-1)^n W^*_{\mathbf{L}},$$

where $W^*_{\mathbf{L}}$ from $\mathbb{Z}^n \to \mathbb{Z}$ is given by

$$W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L} - \mathbf{d}).$$

2.4 Periodic Functions

Definition 2.4. *For $r \in \mathbb{N}$, we say that $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is* r-periodic *if for all $i, j \in [n]$ we have*

$$\forall \mathbf{d} \in \mathbb{Z}^n,\ i, j \in [n], \quad f(\mathbf{d}) = f(\mathbf{d} + r\mathbf{e}_i - r\mathbf{e}_j).$$

Proposition 2.1. *A function f that is initially zero is r-periodic iff its weight, W is.*

Proof. See Proposition 2.3 [12]; the basic idea is that f and W have the same set of translations, since any translation commutes with $\mathfrak{s}$ and with $\mathfrak{m}$. □

Remark 2.1. *Riemann functions based on the classical Riemann's Theorem (see Section 2.6 of [11]) are not generally r-periodic. By contrast, the Baker-Norine rank functions and related functions (see Section 2.5 of [11]) are r-periodic.*

Remark 2.2. *If $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is a Riemann function, then for any $\mathbf{K} \in \mathbb{Z}^n$ and $\mathbf{L} = \mathbf{K} + \mathbf{1}$ (where $\mathbf{1}$ is the all 1's vector), we have $f^{\wedge}_{\mathbf{K}} = f$ iff $W^*_{\mathbf{L}} = (-1)^n W$, where $W^*_{\mathbf{L}}$ is the function defined by*

$$W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L} - \mathbf{d})$$

(see Theorem 30 of [11]). Of course, $(W^*_{\mathbf{L}})^*_{\mathbf{L}} = W$. Section 6 of [11] shows that the Baker-Norine rank function for complete graphs has a remarkably simple weight function.

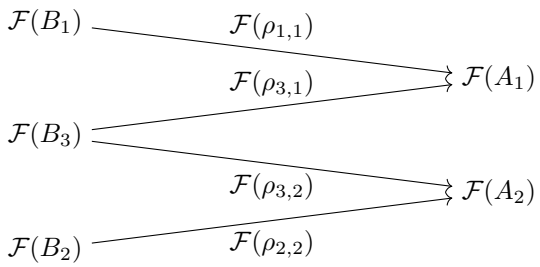


Figure 1: Our Diagrams

2.5 Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ that are Perfect Matchings

We model Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ by starting with a particularly simple case of functions, related to what we call *perfect matchings*.

Definition 2.5. *We say that $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is* of bounded support *if for some $C \in \mathbb{N}$ we have $W(\mathbf{d}) = 0$ whenever $|\deg(\mathbf{d})| > C$. We say that $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is a* perfect matching *if (1) W is of bounded support, and (2) there is a bijection $\pi\colon \mathbb{Z} \to \mathbb{Z}$ such that*

$$W(i,j) = \begin{cases} 1 & \text{if } j = \pi(i), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If so, we call π the bijection associated to W*, and W the* weight function *(or merely* weight*)* associated to π*.*

We easily see that in the above definition, W is r-periodic iff π satisfies $\pi(i+r) = \pi(i) - r$ for all $i \in \mathbb{Z}$. We also see that if W is a perfect matching, then $\pi(a) + a$ is bounded above and below (above by C and below by $-C$ for the same C in Definition 2.5).

Remark 2.3. *Proposition 2.4 of [12] characterizes the Riemann functions $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ such that $W = \mathfrak{m}f$ is a perfect matching. It also shows that f is a Riemann function, then $W = \mathfrak{m}f$ is everywhere non-negative iff W is a perfect matching.*

Remark 2.4. *If G is a graph consisting of two vertices joined by r edges, and f is one plus the Baker-Norine rank function, then f is r-periodic. Moreover, $W = \mathfrak{m}f$ is a perfect matching whose associated bijection π is given uniquely by $\pi(i) = i$ for $i = 0, 1, \dots, r-1$ and $\pi(i+r) = \pi(i) - r$ for all $i \in \mathbb{Z}$. This was the motivation for a lot of the theory developed in [10].*

2.6 k-Diagrams and Perfect Matchings

We now introduce the main structure that is the way we speak of Betti numbers, skyscrapers, etc. The following definition assumes no experience with sheaf theory, although we will soon explain this connection.

We begin by reviewing some of the definitions and main results in Section 4 of [12]; see also Section 4 of [10].

2.6.1 k-Diagrams

Definition 2.6. *Let k be a field. By a* diagram of k-vector spaces*, or simply a* k-diagram *we mean a collection, $\mathcal{F}$, of data consisting of:*

1. *five k-vector spaces,*
$$\mathcal{F}(B_1), \mathcal{F}(B_2), \mathcal{F}(B_3), \mathcal{F}(A_1), \mathcal{F}(A_2)$$
called the values *of $\mathcal{F}$; and*
2. *k-linear maps $\mathcal{F}(\rho_{i,j})\colon \mathcal{F}(B_i) \to \mathcal{F}(A_j)$ for the pairs (i,j) where*
$$(i,j) \in \{(1,1), (2,2), (3,1), (3,2)\} \tag{5}$$
(i.e., $\mathcal{F}(\rho_{1,2})$ and $\mathcal{F}(\rho_{2,1})$ don't exist); we call the $\mathcal{F}(\rho_{ij})$ the restriction maps *of $\mathcal{F}$.*

We similarly define a diagram of rings *replacing "k-vector spaces" above by "rings," and "k-linear maps" above by "morphisms of rings." We similarly define a* diagram of k-algebras*,* of abelian groups*, etc. To a diagram of k-vector spaces, $\mathcal{F}$, we associate the k-vector spaces*

$$\mathcal{F}(B) = \mathcal{F}(B_1) \oplus \mathcal{F}(B_2) \oplus \mathcal{F}(B_3), \quad \mathcal{F}(A) = \mathcal{F}(A_1) \oplus \mathcal{F}(A_2),$$

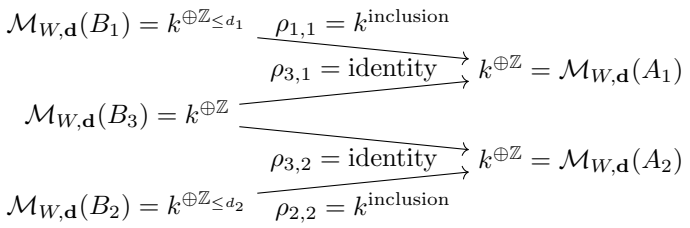


Figure 2: The k-Diagram $\mathcal{M}_{W,\mathbf{d}}$. Notice that this description of the values and restrictions of $\mathcal{M}_{W,\mathbf{d}}$ as a k-diagram is simplest. However, when we work with $\mathcal{O}_r$-modules, it will be easier to slightly rename these values and the restriction maps. Compare with Figure 5.

and the linear transformation $\mathcal{F}(\partial)\colon \mathcal{F}(B) \to \mathcal{F}(A)$*, called the* differential of $\mathcal{F}$*, given by*

$$\mathcal{F}(\partial)(b_1, b_2, b_3) = \big(\mathcal{F}(\rho_{1,1})(b_1) - \mathcal{F}(\rho_{3,1})(b_3), \mathcal{F}(\rho_{2,2})(b_2) - \mathcal{F}(\rho_{3,2})(b_3)\big), \tag{6}$$

and define the zeroth and first cohomology groups of $\mathcal{F}$ *to be, respectively*

$$H^0(\mathcal{F}) \stackrel{def}{=} \ker\big(\mathcal{F}(\partial)\big), \quad H^1(\mathcal{F}) \stackrel{def}{=} \operatorname{coker}\big(\mathcal{F}(\partial)\big),$$

i.e., the kernel and cokernel of $\mathcal{F}(\partial)$*. By a* global section of $\mathcal{F}$ *we mean any tuple* $(b_1, b_2, b_3, a_1, a_2)$ *such that* $\mathcal{F}(\rho_{ij})b_i = a_j$ *whenever* $\mathcal{F}(\rho_{ij})$ *is defined; we use* $\Gamma(\mathcal{F})$ *to denote the set of global sections of* $\mathcal{F}$*. We easily see that the map* $(b_1, b_2, b_3, a_1, a_2) \mapsto (b_1, b_2, b_3)$ *gives an isomorphism* $\Gamma(\mathcal{F}) \to H^0(\mathcal{F})$*. We define the* zeroth Betti number *and* first Betti number of $\mathcal{F}$ *to be, respectively*

$$b^0(\mathcal{F}) = \dim\big(H^0(\mathcal{F})\big), \quad b^1(\mathcal{F}) = \dim\big(H^1(\mathcal{F})\big),$$

and the Euler characteristic of $\mathcal{F}$ *to be*

$$\chi(\mathcal{F}) \stackrel{def}{=} b^0(\mathcal{F}) - b^1(\mathcal{F}).$$

Convention 2.1. *When we speak of a* k-vector space *or a* k-diagram *without prior reference to* k*, we understand* k *to be an arbitrary field.*

Convention 2.2. *Let* $\mathcal{F}$ *be a* k*-diagram. We say that* $\rho_{i,j}$ *and* $\mathcal{F}(\rho_{i,j})$ exist *if* (5) *holds; otherwise they* do not exist*. At times we refer to* $\mathcal{F}(\rho_{i,j})$ *as the* restriction $\rho_{i,j}$ *when* $\mathcal{F}$ *is clear. At times we drop the comma between* i *and* j *in* $\rho_{i,j}$ *and* $\mathcal{F}(\rho_{i,j})$*.*

At this point we have given the abstract definition, with no motivation. The motivation is that we can model perfect matchings with k-diagrams, as we now see.

2.6.2 k-Diagrams in Perfect Matchings

Recall the definitions and notation in Subsubsection 2.1.1.

Definition 2.7. *Let* $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ *be a perfect matching. For each* $\mathbf{d} \in \mathbb{Z}^2$ *we use* $\mathcal{M}_{W,\mathbf{d}}$ *to denote the following* k*-diagram:*

1. *for* $i = 1, 2$*,* $\mathcal{M}_{W,\mathbf{d}}(B_3) = \mathcal{M}_{W,\mathbf{d}}(A_i) = k^{\oplus\mathbb{Z}}$*, and* $\mathcal{M}_{W,\mathbf{d}}(B_i) = k^{\oplus\mathbb{Z}_{\le d_i}}$ *where* $\mathbb{Z}_{\le d}$ *denotes the set of integers* $\le d$*.*
2. *For* $i = 1, 2$*, if* ι_i *denotes the inclusion* $\mathbb{Z}_{\le d_i} \to \mathbb{Z}$*, then* $\rho_{i,i} = k^{\oplus \iota_i}$*.*
3. *For* $i = 1, 2$*,* $\rho_{3,i}$ *is the identity map.*

We depict these k-diagrams in Figure 2.

Remark 2.5. *Figure 2 is the simplest way to understand the* $\mathcal{M}_{W,\mathbf{d}}$ *as* k*-diagrams. However, when we think of* $\mathcal{M}_{W,\mathbf{d}}$ *is an* $\mathcal{O}_r$*-module, it will be convenient to modify the values of* $\mathcal{M}_{W,\mathbf{d}}$ *to reflect their* $\mathcal{O}_r$*-module structure (see Subsection 2.9 and Figure 5).*

Theorem 2.3. *Let* $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ *be a perfect matching, and* $\mathbf{d} \in \mathbb{Z}^2$*. Let* $f = \mathfrak{s}W$ *(which is therefore a Riemann function) and let* C *be the offset of* f*. Then for any* $\mathbf{K} \in \mathbb{Z}^2$ *we have*

$$b^0\big(\mathcal{M}_{W,\mathbf{d}}\big) = (\mathfrak{s}W)(\mathbf{d}) = f(\mathbf{d}), \tag{7}$$

$$b^1\big(\mathcal{M}_{W,\mathbf{d}}\big) = f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}), \tag{8}$$

$$\chi\big(\mathcal{M}_{W,\mathbf{d}}\big) = \deg(\mathbf{d}) + C.$$

Proof. See Theorems 4.2 (Subsection 4.6) of [12]. □

2.6.3 Duality Theorems

There is a large class of theorems that can be called "duality theorems." Here we formalize the minimal requirements of such a theorem in our context.

Let W be a perfect matching and $\mathbf{K}, \mathbf{L} \in \mathbb{Z}^2$ with $\mathbf{L} = \mathbf{K} + \mathbf{1}$. If we apply (7) with W replaced by $W^*_{\mathbf{L}}$ and $\mathbf{d}$ replaced with $\mathbf{K} - \mathbf{d}$, and we (8), we get that

$$f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = b^0(\mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}});$$

combining this with (8), we get that

$$b^1\big(\mathcal{M}_{W,\mathbf{d}}\big) = b^0\big(\mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}}\big). \tag{9}$$

By a *duality theorem* we mean any theorem that provides a perfect pairing of k-vector spaces:

$$H^1(\mathcal{M}_{W,\mathbf{d}}) \times H^0\big(\mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}}\big) \to k \tag{10}$$

for all perfect matchings W and $\mathbf{K}, \mathbf{L}, \mathbf{d} \in \mathbb{Z}^2$ with $\mathbf{L} = \mathbf{K} + \mathbf{1}$; since the vector spaces in (10) are finite dimensional, it is equivalent to give an isomorphism

$$H^1(\mathcal{M}_{W,\mathbf{d}})' \xrightarrow{\simeq} H^0\big(\mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}}\big) \tag{11}$$

where $'$ denotes the dual of the k-vector space; such an isomorphism immediately implies (9).

2.6.4 Mimicking the classical Riemann-Roch theorem

[These next two subsubsections can be skipped; they describe the relation of the above material from [12] with classical sheaf theory and the Riemann-Roch theorem.]

One interesting part of the proof of Theorem 4.2 of [12] is that it mimics the modern proof of the Riemann-Roch theorem. Namely for any $W, \mathbf{d}$ as above for $i = 1, 2$ it is easy to see that there is a *short exact sequence* of k-diagrams (we define this in the next section)

$$0 \to \mathcal{M}_{W,\mathbf{d}} \to \mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i} \to \mathcal{S}_i \to 0$$

where $\mathcal{S}_i$ is a *skyscraper k-diagram* (Definition 5.2 below); this immediately implies that

$$\chi\big(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}\big) = \chi\big(\mathcal{M}_{W,\mathbf{d}}\big) + 1.$$

This mimics the equation in the middle of page 296 of [18], where there is an exact sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + P) \to k(P) \to 0,$$

from which one concludes that

$$\chi\big(\mathcal{L}(D + P)\big) = \chi\big(\mathcal{L}(D)\big) + 1.$$

2.6.5 Connection to Classical Sheaf Theory and Grothendieck's Sheaf Theory

The reader familiar with sheaf theory can see that k-diagrams are just sheaf theory on a certain five-point topological space. Let us give some details; the reader can refer to Subsubsection 10.8.3 of [12] or Section 4 of [10] for more details.

It may help to understand some general underlying principle: if X is any finite set, and $\mathcal{O}$ is a topology on X, then each point $P \in X$ lies in a unique smallest open subset U_P containing P. The sets $\{U_P\}_{P \in X}$ form a basis for the topology, and the $\{U_P\}_{P \in X}$ also become the objects of category $\mathcal{C}_X$, with an arrow $U_P \to U_Q$ iff $U_P \subset U_Q$. Now any sheaf, $\mathcal{F}$, of k-vector spaces on $(X, \mathcal{O})$ restricts to a contravariant functor $\mathcal{F}|_{\mathcal{C}_X}$ from $\mathcal{C}_X$ to the category of k-vector spaces (which is Grothendieck's notion of a presheaf of k-vector spaces on the category $\mathcal{C}_X$); it is not hard to see that the map $\mathcal{F} \mapsto \mathcal{F}|_{\mathcal{C}_X}$ is an equivalence of categories (for a proof see [13], Theorem 2.1, which explains this as a special case of the Comparison Lemma [27] Exp. III, Theorem 4.1).

As a special case of the last paragraph, let $X = A_1, A_2, B_1, B_2, B_3$; for $i = 1, 2$ let

$$U_{A_i} = \{A_i\}, \quad U_{B_i} = \{A_i, B_i\},$$

and let

$$U_{B_3} = \{A_1, A_2, B_3\}.$$

Then if $\mathcal{O}$ is the topology on X with basis U_P with $P = A_1, A_2, B_1, B_2, B_3$, then $\{A_1\}$ and $\{A_2\}$ are open subsets, and $\{B_1\}, \{B_2\}, \{B_3\}$ are closed subsets. Let $\mathcal{C}$ be the category whose objects are U_P with $P \in X$. Then for any sheaf, $\mathcal{F}$, of k-vector spaces on $(X, \mathcal{O})$, for all $P \in X$, $\mathcal{F}(U_P)$ is a vector space, and for each inclusion $U_P \to U_Q$ (i.e., $U_{A_i} \to U_{B_j}$ for $i = j$ or $j = 3$) we get a map $\mathcal{F}(U_Q) \to \mathcal{F}(U_P)$. This gives a diagram of k-vector spaces, $\mathcal{F}|_{\mathcal{C}}$. The previous paragraph implies that the map $\mathcal{F} \to \mathcal{F}|_{\mathcal{C}}$ is an equivalence of vector spaces (i.e., knowing $\mathcal{F}|_{\mathcal{C}}$, one can determine $\mathcal{F}$ up to isomorphism).

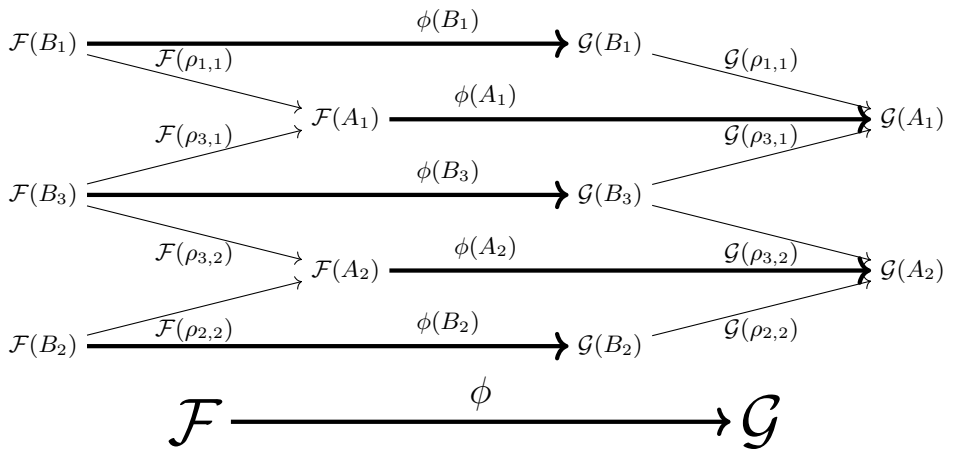


Figure 3: A morphism of diagrams $\phi\colon \mathcal{F} \to \mathcal{G}$, depicted in thick lines

2.7 Morphisms and Exact Sequences

To explain the duality theorem of [12], we need to define *morphisms* of k-diagrams; we refer the reader to Section 5 of [12] for more details and examples.

Definition 2.8. *Let $\mathcal{F}, \mathcal{G}$ be two k-diagrams. By a* morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ *we mean the data, ϕ, of linear maps from each value of $\mathcal{F}$ to the corresponding value on $\mathcal{G}$ in a way that commutes with the restriction maps: i.e., ϕ consists of k-linear maps $\phi(B_i)\colon \mathcal{F}(B_i) \to \mathcal{G}(B_i)$ for $i = 1, 2, 3$ and $\phi(A_j)\colon \mathcal{F}(A_j) \to \mathcal{G}(A_j)$ for $j = 1, 2$ such that $\mathcal{G}(\rho_{ij})\phi(B_i) = \phi(A_j)\mathcal{F}(\rho_{ij})$ whenever $\mathcal{F}(\rho_{ij}), \mathcal{G}(\rho_{ij})$ exist (i.e., $i = j$ or $i = 3$ and any j). For k-diagrams $\mathcal{F}, \mathcal{G}$ we use* $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ *to denote the set of morphism $\mathcal{F} \to \mathcal{G}$. We similarly define a morphism $\mathcal{F} \to \mathcal{G}$ when $\mathcal{F}, \mathcal{G}$ are two diagrams of rings, two diagrams of abelian groups, etc.*

If $\mathcal{F}, \mathcal{G}$ are two k-diagrams, then $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ becomes a k-vector space in an evident fashion, by the k-vector space structure on each linear map $\mathcal{F}(P) \to \mathcal{G}(P)$ for $P \in \{A_1, A_2, B_1, B_2, B_3\}$.

We illustrate a morphism of k-diagrams in Figure 3.

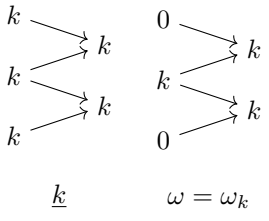
[The following remark can be skipped until Section 6.] If $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism, then we define the *kernel of ϕ*, the *cokernel of ϕ*, and the *image of ϕ* as their value-by-value k-diagrams; e.g., $\ker(\phi)$ is the sub-k-diagram of $\mathcal{F}$ whose value at $P = \{A_1, A_2, B_1, B_2, B_3\}$ is $\ker(\phi(P))$.[4] In particular, a sequence of morphisms $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence iff it is value-by-value, i.e., iff for all $P = \{A_1, A_2, B_1, B_2, B_3\}$, the sequence

$$0 \to \mathcal{F}_1(P) \to \mathcal{F}_2(P) \to \mathcal{F}_3(P) \to 0$$

is a short exact sequence. The "strong duality" theorem of Section 6 will make extensive use of short exact sequences and the resulting long exact sequences of Ext groups.

2.8 Duality for k-Diagrams

In [12], the following duality theory was described for k-diagrams. Let $\underline{k}$ and $\omega = \omega_k$ be the following diagrams:



(The k-diagram ω is also called $\underline{k}_{/B_1,B_2}$ in [12].) Theorem 9.1 of [12] showed that for any k-diagram, $\mathcal{F}$, we have

$$H^1(\mathcal{F})' \simeq \mathrm{Hom}(\mathcal{F}, \omega). \tag{12}$$

[4] In Section 6 we will use some more technical points of homological algebra. At that point we want to see that the category of k-diagrams (whose objects are k-diagrams and whose morphisms are as described above) is an Abelian category in which kernels, cokernels, and images are computed in value-by-value. The reader can either check this by hand, or refer to [27], the sentence following Proposition I.3.1. See also [14].

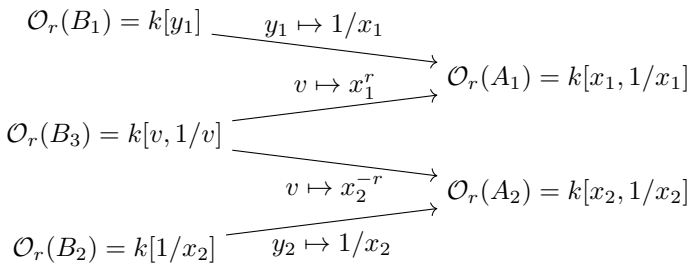


Figure 4: The Diagram of Rings, $\mathcal{O}_r = \mathcal{O}_{r,k}$: this k-diagram has more structure: its values are rings, and restriction maps are also morphisms of rings. Hence $\mathcal{O}_r = \mathcal{O}_{r,k}$ is much larger and more structured than $\underline{k}$ (the constant diagram whose values are k); hence for any $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is much smaller than $\mathrm{Hom}_{\underline{k}}(\mathcal{F}, \mathcal{G})$. This smallness is crucial if we want to get a stronger form of Serre duality.

Hence ω "represents" the functor $\mathcal{F} \mapsto H^1(\mathcal{F})'$ (and plays the role of the canonical sheaf in curve theory, e.g., [18], middle of page 295).

Theorem 9.2 of [12] then shows that if W is a perfect matching, and if $\mathbf{K}, \mathbf{L} \in \mathbb{Z}^2$ satisfy $\mathbf{L} = \mathbf{K} + \mathbf{1}$, then a natural isomorphism

$$\mathrm{Hom}(\omega, \omega) \to \mathrm{Hom}(\underline{k}, \underline{k})$$

yields an isomorphism

$$\mathrm{Hom}(\mathcal{M}_{W,\mathbf{d}}, \omega) \to \mathrm{Hom}(\underline{k}, \mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}}) \simeq H^0(\mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}}).$$

Combining this with (12) gives an isomorphism

$$H^1(\mathcal{M}_{W,\mathbf{d}})' \to H^0(\mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}}).$$

This is a "duality theorem," for then this shows that

$$b^1(\mathcal{M}_{W,\mathbf{d}}) = b^0(\mathcal{M}_{W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{d}}),$$

which proves the formula $b^1(\mathcal{M}_{W,\mathbf{d}}) = f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d})$ in Theorem 2.3 as a consequence of (12) and the fact that $b^0(\mathcal{M}_{W,\mathbf{d}}) = f(\mathbf{d})$ in Theorem 2.3.

(12) represents a "weaker" duality theory in that the canonical sheaf ω is independent of W and does not reflect any special structure of W. We expect that a "stronger" duality theory would have a canonical diagram that would involve the special structure of W. This is what we do next, when W is a periodic perfect matching.

2.9 $\mathcal{O}$-Modules

The notion of an *$\mathcal{O}_X$-module* is a standard notion in the context of a *ringed space*, $(X, \mathcal{O}_X)$ (see, e.g., [18] Section II.5, or [15] Section 0.4); in this subsection we explain what this amounts to in our context. The reason we do this is that if $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is an r-periodic perfect matching, then the $\mathcal{M}_{W,\mathbf{d}} = \mathcal{M}_{W,\mathbf{d},k}$ become *$\mathcal{O}_r$-modules* (of finite type) where $\mathcal{O}_r = \mathcal{O}_{r,k}$ is an interesting diagram of rings; this is crucial to the rest of this paper, i.e., to our newer duality theory. The rough idea is explained in Subsection 10.10 of [12], but is not further pursued there.

Definition 2.9. *Let k be a field and $r \in \mathbb{N}$. We define the k-diagram $\mathcal{O} = \mathcal{O}_{r,k}$ (or $\mathcal{O}_r$ with k understood), as follows:*

$$\forall i = 1, 2, \quad \mathcal{O}(A_i) = k[x_i, 1/x_i],\ \mathcal{O}(B_i) = k[y_i],\ \mathcal{O}(B_3) = k[v, 1/v],$$

and whose restriction maps take v to x_1^r, x_2^{-r} and take y_i to $1/x_i$, i.e.,

$$\mathcal{O}(\rho_{i,i})(y_i) = 1/x_i \text{ (for } i = 1, 2), \quad \mathcal{O}(\rho_{3,1})(v) = x_1^r, \quad \mathcal{O}(\rho_{3,2})(v) = x_2^{-r}$$

(and all restriction maps are the identity on k).

Since the values of $\mathcal{O}_{r,k}$ are rings, and its restriction maps are morphisms of rings, we view $\mathcal{O}_{r,k}$ as a diagram of rings or, more precisely, a diagram of k-algebras. We depict $\mathcal{O}_{r,k}$ in Figure 4.

Definition 2.10. *Let $\mathcal{O}$ be a diagram of rings (recall Definition 2.6). By an $\mathcal{O}$-module we mean any diagram of abelian groups, $\mathcal{F}$, such that for each $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$, $\mathcal{F}(P)$ is endowed with the structure of an $\mathcal{O}(P)$-module such that the restriction maps respect the ring structure, i.e., for each restriction map $\rho\colon \mathcal{F}(B) \to \mathcal{F}(A)$, for each $r \in \mathcal{O}(B)$ and $m \in \mathcal{F}(B)$ we have*

$$\mathcal{F}(\rho)(rm) = \big(\mathcal{O}(\rho)(r)\big)\big(\mathcal{F}(\rho)(m)\big).$$

For two $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$ we define $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ to be the set of morphisms of diagrams ϕ such that for for each $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$, $\phi(P)$ is a morphism of $\mathcal{O}(P)$-modules.

Note that $\mathcal{O}_r = \mathcal{O}_{r,k}$ are k-algebras, and hence any $\mathcal{O}_{r,k}$ is automatically a k-diagram of vector spaces.

It will be helpful to keep a few examples of $\mathcal{O}$-modules. We start with the main example, which justifies the definition of $\mathcal{O}_{r,k}$.

Example 2.1. *Let $\mathcal{O} = \mathcal{O}_{r,k}$. Then for any r-periodic perfect matching, W, $\mathcal{M}_{W,\mathbf{d}} = \mathcal{M}_{k,W,\mathbf{d}}$ is an $\mathcal{O}_{r,k}$-module as follows (here we are slightly renaming the values of $\mathcal{M}_{W,\mathbf{d}}$):*

1. *We identify $\mathcal{M}_{W,\mathbf{d}}(B_1) = k^{\oplus \mathbb{Z}_{\le d_1}}$ with the $k[y_1]$-module $k[y_1]y_1^{-d_1}$ in the evident fashion, i.e., if $f\colon \mathbb{Z}_{\le d_1} \to k$ is an element of $k^{\oplus \mathbb{Z}_{\le d_1}}$, then we identify f with $\sum_i f(i) y_1^{-i}$. Hence $\mathcal{M}_{W,\mathbf{d}}(B_1)$ is a $k[y_1]$-module, which makes it a $\mathcal{O}_r(B_1)$-module.*
2. *Similarly for $\mathcal{M}_{W,\mathbf{d}}(B_2)$.*
3. *We identify $\mathcal{M}_{W,\mathbf{d}}(A_1) = k^{\oplus \mathbb{Z}}$ with $k[x_1]$ by the map taking f to $\sum_i f(i) x_1^i$. The restriction map $\mathcal{M}_{W,\mathbf{d}}(\rho_{1,1})$ takes y_1 to $1/x_1$.*
4. *Similarly for $\mathcal{M}_{W,\mathbf{d}}(A_2)$ and $\mathcal{M}_{W,\mathbf{d}}(\rho_{2,2})$.*
5. *We identify $\mathcal{M}_{W,\mathbf{d}}(B_3) = k^{\oplus \mathbb{Z}}$ with $\mathcal{M}_{W,\mathbf{d}}(A_1) = k^{\oplus \mathbb{Z}}$ via the identity map, and $\rho_{3,1}$ is the identity map. Since $\mathcal{M}_{W,\mathbf{d}}(B_3)$ is a $k[x_1, 1/x_1]$-module, it is also a $k[x_1^r, 1/x_1^r]$-module, and identifying v with x_1^r, $\mathcal{M}_{W,\mathbf{d}}(B_3)$ is a $k[v, 1/v]$-module. This makes $\mathcal{M}_{W,\mathbf{d}}(B_3)$ isomorphic (non-canonically) to with $k[v, 1/v]^{\oplus r}$. To define the restriction $\mathcal{M}_{W,\mathbf{d}}(\rho_{32})$ we take $x_1^a \in \mathcal{M}_{W,\mathbf{d}}(B_3)$ to x_2^b where b is the unique integer with $W(a,b) = 1$. (Of course, we may equally well identify $\mathcal{M}_{W,\mathbf{d}}(B_3)$ with $\mathcal{M}_{W,\mathbf{d}}(A_2)$.)*

Another way to define $\mathcal{M}_{W,\mathbf{d}}(B_3)$ is to (non-canonically) choose integers $i_1, \dots, i_r$, one in each class mod r (for example, choose some $i_s \equiv s \pmod r$), and then for $s \in [r]$ let j_s be the unique integer with $W(i_s, j_s)$. We can then identify $\mathcal{M}_{W,\mathbf{d}}(B_3)$ with $k[v, 1/v]^{\oplus r}$, and define the $\rho_{3,1}$ restriction by mapping

$$\big(f_1(v), \dots, f_r(v)\big) \mapsto \sum_{s=1}^{r} f_s(v) x_1^{i_s}$$

and the $\rho_{3,2}$ restriction by mapping

$$\big(f_1(v), \dots, f_r(v)\big) \mapsto \sum_{s=1}^{r} f_s(1/v) x_2^{j_s}.$$

We depict this in Figure 5.

Example 2.2. *Let $\mathcal{O}$ be an arbitrary diagram of rings, and let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}$-modules. Then the* tensor product *of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{O}$, denoted $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is the $\mathcal{O}$-module whose value at $P \in \{A_1, A_2, B_1, B_3, B_2\}$ is $\mathcal{F}(P) \otimes_{\mathcal{O}(P)} \mathcal{G}(P)$, and whose restriction maps are the evident tensor product of restriction maps. We similarly define $\mathcal{F} \oplus_{\mathcal{O}} \mathcal{G}$.*

Example 2.3. *Let V be a k-vector space. The* constant diagram *V, denoted $\underline{V}$, refers to the k-diagram all of whose values are V and all of whose restrictions are the identity maps. Similarly for a ring R and the constant diagram of rings, $\underline{R}$. If k is a field, then it also is a ring. Then $\mathcal{O} = \underline{k}$ is a diagram of rings. We easily see that a $\underline{k}$-module is the same thing as a diagram of k-vector spaces.*

For the geometers we remark that for $r = 1$, $\mathcal{O}_r$ resembles the Riemann sphere with two closed points; then the $\mathcal{M}_{W,\mathbf{d}}$ are $\mathcal{O}_r$ line bundles. However, $r \ge 2$ this is a much more mysterious space.[5] [The $r \ge 2$ case of $\mathcal{O}_r$ is not an orbifold; perhaps it is a cover of an orbifold reflecting a Čech cohomology computation or something related.]

2.10 Our First Duality Theorem

We now wish to describe our first duality theory, that is the subject of Section 5.

Fix an r-periodic perfect matching W, and fix a $\mathbf{K} \in \mathbb{Z}^2$ and set $\mathbf{L} = \mathbf{K} + \mathbf{1}$. Section 5 begins by describing a pairing for any $\mathcal{O}_r$-module $\mathcal{F}$ of the form:

$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to k.$$

Later in Section 5 we show that this is a perfect pairing when $\mathcal{F}$ is equal to certain *line bundles* $\mathcal{L}_{\mathbf{d}} = \mathcal{L}_{\mathbf{d},r,k}$ (which we define in Definition 3.1).

The above pairing is built from the following two general construction steps, valid for any diagram of rings $\mathcal{O}$:

[5] We thank Ehud de Shalit for a discussion of $\mathcal{O}_{r,k}$, and Luc Illusie for questions regarding $\mathcal{M}_{W,\mathbf{d}}$ and $\mathcal{O}_{r,k}$-modules.

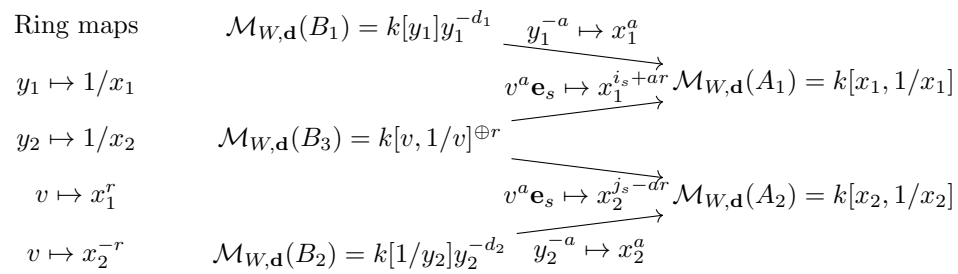


Figure 5: If W is an r-periodic perfect matching, then $\mathcal{M}_{W,\mathbf{d}}$ has a natural structure as an $\mathcal{O}_r$-module: for $k \in [r]$ we set

1. For any $\mathcal{O}$-modules $\mathcal{A}, \mathcal{B}, \mathcal{C}$ there is a natural map
$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{B}) \xrightarrow{\mathrm{id}_{\mathcal{C}} \otimes \cdot} \mathrm{Hom}_{\mathcal{O}}(\mathcal{C} \otimes \mathcal{A}, \mathcal{C} \otimes \mathcal{B})$$
where $\mathrm{id}_{\mathcal{C}}$ is the identity map on $\mathcal{C}$.
2. For any $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$, there is a natural "Yoneda pairing"
$$H^1(\mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to H^1(\mathcal{G}).$$

Hence for any $\mathcal{O}_r$-module $\mathcal{F}$ we have a natural map
$$\mathrm{Hom}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to \mathrm{Hom}\big(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}\big).$$
So setting
$$\omega_W = \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}},$$
we get a natural map
$$\mathrm{Hom}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to \mathrm{Hom}\big(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega\big).$$
Next, we have a Yoneda pairing
$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}\big(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega\big) \to H^1(\omega).$$
It turns out that
$$H^1(\omega) \simeq k$$
(this is an involved computation). Hence, fixing an isomorphism to k, we get a natural pairing
$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega) \to H^1(\omega) \to k.$$
Next we verify that this pairing is a perfect pairing when $\mathcal{F}$ is a certain type of *line bundle*, $\mathcal{L}_{\mathbf{d}} = \mathcal{L}_{r,k,\mathbf{d}}$ (this is a straightforward but tedious computation). While the $\mathcal{L}_{\mathbf{d}}$ are defined in Definition 3.1, let us mention some of their basic properties:
$$\mathcal{L}_{\mathbf{d}} \otimes \mathcal{L}_{\mathbf{d}'} \simeq \mathcal{L}_{\mathbf{d}+\mathbf{d}'}, \quad \mathcal{L}_{\mathbf{0}} \simeq \mathcal{O}_r, \quad \mathcal{M}_{W,\mathbf{d}'} \otimes \mathcal{L}_{\mathbf{d}} \simeq \mathcal{M}_{W,\mathbf{d}'+\mathbf{d}},$$
and for any $\mathcal{O}_r$-modules $\mathcal{F}, \mathcal{G}$,
$$\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F} \otimes \mathcal{L}_{\mathbf{d}}, \mathcal{G} \otimes \mathcal{L}_{\mathbf{d}}).$$
Since
$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to k$$
is a perfect pairing for $\mathcal{F} = \mathcal{L}_{\mathbf{d}}$, this easily implies that there is an isomorphism
$$H^1(\mathcal{M}_{W,\mathbf{d}})' \to \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{O}_r, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \simeq H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}).$$
Hence we get a duality theorem, implying that
$$b^1(\mathcal{M}_{W,\mathbf{d}}) = b^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}).$$

Remark 2.6. *If W, W' are r-periodic perfect matchings, then we have that for any $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$*
$$b^1\big(\mathcal{M}_{W,\mathbf{d}} \otimes \mathcal{M}_{W',\mathbf{d}'}\big) = 0$$
unless $W' = W^*_{\mathbf{L}}$ for some $\mathbf{L} \in \mathbb{Z}^2$. Hence the definition of ω is a rather exceptional choice for a "canonical" $\mathcal{O}$-module. We also remark that
$$b^0(\omega) = \infty,$$
so there are properties of ω that are not mirrored in the classical Riemann-Roch theorem (classically the canonical sheaf is a line bundle, and therefore has finite b^0 and b^1).

2.11 How Canonical is the Canonical $\mathcal{O}_r$-Module ω_W?

It turns out that $\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}$, ranging over all $\mathbf{K},\mathbf{L}$ with $\mathbf{L}=\mathbf{K}+\mathbf{1}$, is independent — up to isomorphism — of the particular pair $(\mathbf{K},\mathbf{L})$. Hence

$$\omega_{W,\mathbf{K}} = \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}$$

depends only on W. It is an interesting question to see how ω_W depends on W. We will briefly address this in Subsection 4.6.

2.12 A Stronger Duality Theorem

In Section 6 we prove a stronger duality theorem; this requires more of a background in homological algebra as it applies to $\mathcal{O}_r$-modules, part of which is given in Appendix A. Let us describe the main result.

If $\mathcal{O}$ is any diagram of rings, and $\mathcal{A},\mathcal{B},\mathcal{C}$ are any $\mathcal{O}$-modules, then the natural map (of Subsection 2.10)

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{A},\mathcal{B}) \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{C}\otimes_{\mathcal{O}}\mathcal{A},\mathcal{C}\otimes_{\mathcal{O}}\mathcal{B})$$

extends to a map of δ-functors

$$\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{A},\mathcal{B}) \to \mathrm{Ext}^i_{\mathcal{O}}(\mathcal{C}\otimes_{\mathcal{O}}\mathcal{A},\mathcal{C}\otimes_{\mathcal{O}}\mathcal{B})$$

provided that $\mathcal{C}$ is a *flat $\mathcal{O}$-module*. It is easy to see that if W is an r-periodic perfect matching, then $\mathcal{M}_{W,0}$ is a flat $\mathcal{O}_r$-module, and hence we get a map

$$\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{F},\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to \mathrm{Ext}^i_{\mathcal{O}}(\mathcal{M}_{W,0}\otimes\mathcal{F},\omega_{W,\mathbf{K}}).$$

Now there is a Yoneda pairing (generalizing that of Subsection 2.10)

$$\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{A},\mathcal{B})\times\mathrm{Ext}^j_{\mathcal{O}}(\mathcal{B},\mathcal{C}) \to \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{A},\mathcal{C})$$

which therefore gives a pairing

$$H^i(\mathcal{M}_{W,\mathbf{0}}\otimes\mathcal{F})\times\mathrm{Ext}^{1-i}_{\mathcal{O}_r}(\mathcal{F},\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to H^1(\omega_{W,\mathbf{K}}) \simeq k \tag{13}$$

for any $\mathcal{O}_r$-module, $\mathcal{F}$. We say that $\mathcal{F}$ satisfies *strong duality* if (13) is a perfect pairing for $i=0,1$.

It follows from general principles and the five-lemma that if $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$ is a short exact sequence, and if two of $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3$ satisfy strong duality, then so does the third.

In Section 6 we establish that a number of $\mathcal{F}$ satisfy strong duality. For example, in Section 5 we've seen that (13) is a perfect pairing for $\mathcal{F}=\mathcal{L}_{\mathbf{d}}$ and $i=1$. [This is true modulo some technicalities regarding the Yoneda pairing that we are hiding for now.] But for $\deg(\mathbf{d})$ sufficiently small we can see that for $\mathcal{F}=\mathcal{L}_{\mathbf{d}}$, both $H^i(\mathcal{M}_{W,\mathbf{0}}\otimes\mathcal{F})$ and $\mathrm{Ext}^{1-i}_{\mathcal{O}_r}(\mathcal{F},\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}})$ vanish for $i=0$. Hence $\mathcal{L}_{\mathbf{d}}$ satisfy strong duality for $\deg(\mathbf{d})\ll 0$. Now we use an exact sequence

$$0\to\mathcal{L}_{\mathbf{d}}\to\mathcal{L}_{\mathbf{d}+\mathbf{e}_1}\to\mathcal{S}_1\to 0$$

where $\mathcal{S}_1$ is a certain *skyscraper sheaf* to show that (1) $\mathcal{S}_1$ satisfies strong duality, and then (2) $\mathcal{L}_{\mathbf{d}}$ satisfies strong duality for all $\mathbf{d}$.

The "two out of three principle" shows that, for example, if $\mathcal{F}$ is the cokernel of a map of finite direct sums of $\mathcal{O}_r$-modules of the form $\mathcal{L}_{\mathbf{d}}$, then $\mathcal{F}$ also satisfies strong duality. Such $\mathcal{F}$ are examples of $\mathcal{O}_r$-modules that are *coherent* in the usual sense of $\mathcal{O}$-modules on a ringed topological $(X,\mathcal{O})$.[6]

2.13 The Interest in Perfect Matchings

It seems quite restrictive to study the set of $f\colon\mathbb{Z}^2\to\mathbb{Z}$ whose weight W is a perfect matching. As mentioned before, the interest in such f and W is that to any Riemann function $f\colon\mathbb{Z}^n\to\mathbb{Z}$ one can associate a *virtual sheaf* that is formally the sum and difference of sheaves $\mathcal{M}_{W,\mathbf{d}}$ where W is a perfect matching. Let us roughly describe the procedure, referring to [12] Subsections 1.3 and 1.4 for details.

First, any Riemann function $f\colon\mathbb{Z}^2\to\mathbb{Z}$ has a weight W that can be written as

$$W = W_1+\cdots+W_\ell-\tilde{W}_1-\cdots-\tilde{W}_{\ell-1} \tag{14}$$

for some ℓ and perfect matchings $W_1,\dots,W_\ell,\tilde{W}_1,\dots,\tilde{W}_{\ell-1}$. Assuming we work with *virtual* sheaves — meaning formal sums and differences — of the sheaves $\mathcal{M}_{W_i,\mathbf{d}}$ and $\mathcal{M}_{\tilde{W}_i,\mathbf{d}}$, one gets $f(\mathbf{d})$ and $f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d})$ as the zeroth and first Betti numbers of a virtual sheaf built from perfect matchings. One then proves the virtual sheaf obtained is independent of how one writes W as the RHS of (14).

[6] Hartshorne's textbook [18] defines a coherent $\mathcal{O}$-module only in the context of schemes; the more general definition can be found in, for example, EGA I, Section 0.5.3.

Second, if $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is any Riemann function, for any $\mathbf{d} \in \mathbb{Z}^n$ one gets a Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$ obtained by varying two of the n variables in $\mathbf{d}$; i.e., for distinct $i, j \in [n]$, we consider $g(a_1, a_2) \stackrel{\text{def}}{=} f(\mathbf{d} + a_1\mathbf{e}_i + a_2\mathbf{e}_j)$, where $\mathbf{e}_i, \mathbf{e}_j$ are standard basis vectors. Since $g\colon \mathbb{Z}^2 \to \mathbb{Z}$ is also a Riemann function, one gets a virtual sheaf for $g(0,0) = f(\mathbf{d})$. One then proves that the virtual sheaf is independent of the choice of i, j.

As alluded to in the introduction, the above was shown to hold in [12] in the context of k-diagrams, not $\mathcal{O}_r$-modules. If W is an r-periodic weight function, then Lemma 3.1 of [12] shows that (14) holds where the W_i and $\tilde{W}_i$ are also r-periodic. Hence we can define $\mathcal{M}_{W,\mathbf{d}}$ as a formal difference of sums of $\mathcal{O}_r$-modules $\mathcal{M}_{W_i,\mathbf{d}}$ and $\mathcal{M}_{W_i',\mathbf{d}}$. But [12] does not prove that the resulting virtual (or formal difference of) $\mathcal{O}_r$-module is independent of these choices; it only proves that for two different decompositions of (14), the two formal differences are isomorphic as virtual k-diagrams (not as virtual $\mathcal{O}_r$-modules). We hope to address this point in a future work.

2.14 Additional Remarks and Future Work

A good challenge for future work is to develop models that explain generalized Riemann-Roch formulas as a type of sheaf or diagram of k-vector spaces that does not have all the ad hoc choices we make, and that does not need to pass to virtual diagrams or virtual sheaves.

Another—perhaps independent challenge—is to use the theory of diagrams or sheaves to give proofs of self-duality, such as in the Baker-Norine formula [4] and some more general situations, such as those studied by Amini and Manjunath [2].

3. Some Basic $\mathcal{O}$-Modules

In this section we describe some basic $\mathcal{O}_r$-modules and some operations on them that we will need. In this section, k is a fixed field.

3.1 Line Bundles and Tensor Products

Let $\mathcal{O}$ be an arbitrary k-diagram of rings. Then if $\mathcal{M}_1, \mathcal{M}_2$ are two $\mathcal{O}$-modules, we define $\mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2$, respectively, to be the $\mathcal{O}$-modules whose values at $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$ are $\mathcal{M}_1(P) \oplus \mathcal{M}_2(P)$ and $\mathcal{M}_1(P) \otimes_{\mathcal{O}(P)} \mathcal{M}_2(P)$, respectively, and whose restriction maps are those induced by those of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Definition 3.1. *Let $r \in \mathbb{N}$ and $\mathbf{d} \in \mathbb{Z}^2$. The* line bundle of $\mathcal{O}_{r,k}$ with $\mathbf{d}$ shift*, denoted $\mathcal{L}_{\mathbf{d}} = \mathcal{L}_{r,\mathbf{d},k}$ is the $\mathcal{O}_{r,k}$-module $\mathcal{L}$ whose values are those of $\mathcal{O}_{r,k}$, and with the following restriction maps:*

1. *$\rho_{1,1}$ takes $1 \in \mathcal{L}(B_1)$ to $x_1^{d_1} \in \mathcal{L}(A_1)$ [and therefore, by the map $\mathcal{O}_{r,k}(B_1)$ to $\mathcal{O}_{r,k}(A_1)$, takes $p(y_1) \in \mathcal{L}(B_1)$ to $x_1^{d_1} p(1/x_1) \in \mathcal{L}(A_1)$];*
2. *$\rho_{2,2}$ takes $1 \in \mathcal{L}(B_2)$ to $x_2^{d_2} \in \mathcal{L}(A_2)$ [and therefore takes $p(y_2) \in \mathcal{L}(B_2)$ to $x_2^{d_2} p(1/x_2) \in \mathcal{L}(A_2)$].*
3. *$\rho_{3,1}$ takes $1 \in \mathcal{L}(B_3)$ to $1 \in \mathcal{L}(A_1)$ [and therefore takes $p(v) \in k[v^{\pm}] = \mathcal{L}(B_3)$ to $p(x_1^r) \in k[x^{\pm}] = \mathcal{L}(A_1)$; and*
4. *$\rho_{3,2}$ takes $1 \in \mathcal{L}(B_3)$ to $1 \in \mathcal{L}(A_2)$.*

Since k is fixed in this section, we will generally omit it as a subscript and simply write $\mathcal{O}_r$ and $\mathcal{L}_{\mathbf{d},r}$.

We depict the $\mathcal{L}_{r,\mathbf{d}}$ in Figure 6. Note that $\mathcal{L}_{r,\mathbf{0}} = \mathcal{O}_r$ (we do mean equality, not merely being isomorphic).

Remark 3.1. *Notice that since $\rho_{1,1}$ takes 1 to $x_1^{d_1}$, by the $\mathcal{O}_r$-module structure on $\mathcal{L}_{\mathbf{d}}$, 1 times $p(y_1)$ must be taken to 1 times $p(1/x_1)$. More generally, to specify a restriction map of an $\mathcal{O}$-module, $\mathcal{M}$, to specify $\rho_{i,j}$ it suffices to choose a generating set of $\mathcal{M}(B_i)$ and to say where each set element is mapped to in $\mathcal{M}(A_j)$. It is usually far simpler to specify restriction maps this way, and we generally do so in what follows; at times we write the entire map.*

The following proposition is easy.

Proposition 3.1. *For any $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$*

$$\mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'} \simeq \mathcal{L}_{r,\mathbf{d}+\mathbf{d}'}, \tag{15}$$

and in particular

$$\mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,-\mathbf{d}} \simeq \mathcal{O}_r. \tag{16}$$

If W is any r-periodic perfect matching, then

$$\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'} = \mathcal{M}_{W,\mathbf{d}+\mathbf{d}'}. \tag{17}$$

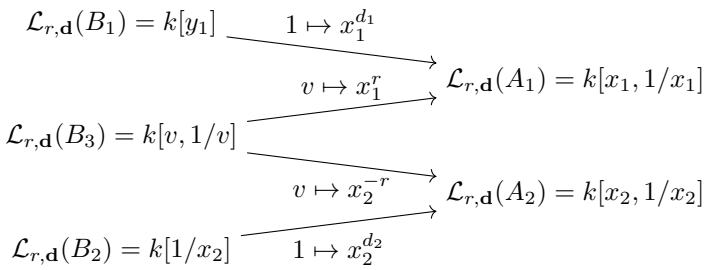


Figure 6: The $\mathcal{O}_r$-Modules, $\mathcal{L}_{r,\mathbf{d}}$. These are invertible $\mathcal{O}_r$-modules: they have the same values as $\mathcal{O}_r$, and we have $\mathcal{L}_{r,\mathbf{0}} = \mathcal{O}_r$. Since $y_1 \mapsto x_1^{-1}$ in $\mathcal{O}_r$, the fact that $\mathcal{L}_{r,\mathbf{d}}(\rho_{1,1})$ takes 1 to $x_1^{d_1}$ implies that $p(y_1) \in \mathcal{L}_{r,\mathbf{d}}(B_1)$ is mapped to $x_1^{d_1} p(x_1^{-1})$ under $\mathcal{L}_{r,\mathbf{d}}(\rho_{1,1})$. Hence writing $1 \mapsto x_1^{d_1}$ is more concise than describing the entire map $\mathcal{L}_{r,\mathbf{d}}(\rho_{1,1})\colon k[y_1] \to k[x_1^{\pm}]$. This concise notation is very helpful for the $\mathcal{O}_r$-modules in this article.

If $\mathcal{F}, \mathcal{G}$ are any $\mathcal{O}_r$-modules, then

$$\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{F}, \mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{G}), \tag{18}$$

and

$$\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{L}_{r,-\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{G}). \tag{19}$$

Proof. For $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$ we have

$$\mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'}(P) = \mathcal{O}(P) \otimes_{\mathcal{O}_r(P)} \mathcal{O}(P)$$

and hence there is an isomorphism for each P:

$$\phi(P)\colon \mathcal{O}(P) \otimes_{\mathcal{O}_r(P)} \mathcal{O}(P) \to \mathcal{O}(P) \quad \text{given by} \quad \phi(P)(1 \otimes 1) = 1.$$

Let us prove that the maps $\{\phi(P)\}_{P \in X}$ form an isomorphism $\phi\colon \mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'} \to \mathcal{L}_{r,\mathbf{d}+\mathbf{d}'}$ of k-diagrams. Glancing at Figure 3, we see that it suffices to show that for each i, j such that $\rho_{i,j}$ exists, we have a commuting diagram

$$\begin{array}{ccc} \mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'}(B_i) & \xrightarrow{\phi(B_i)} & \mathcal{L}_{r,\mathbf{d}+\mathbf{d}'}(B_i) \\ \Big\downarrow{\scriptstyle \mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'}(\rho_{i,j})} & & \Big\downarrow{\scriptstyle \mathcal{L}_{r,\mathbf{d}+\mathbf{d}'}(\rho_{i,j})} \\ \mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'}(A_j) & \xrightarrow[\phi(A_j)]{} & \mathcal{L}_{r,\mathbf{d}+\mathbf{d}'}(A_j) \end{array}$$

So consider $\rho_{1,1}$, i.e., $i = j = 1$: the restriction map $\mathcal{L}_{r,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}'}(\rho_{1,1})$ takes $1 \otimes 1 \in k[y_1]$ to

$$x_1^{d_1} \otimes x_1^{d_1'} = x_1^{d_1+d_1'}(1 \otimes 1)$$

which therefore agrees with $\mathcal{L}_{r,\mathbf{d}+\mathbf{d}'}$; similarly for $\rho_{2,2}$. The case of $\rho_{3,1}$ is similar, since $\mathcal{L}_{r,\mathbf{d}}(\rho_{3,1})$ and $\mathcal{L}_{r,\mathbf{d}}$ each take 1 to 1; and similarly for $\rho_{3,2}$. This shows that $\{\phi(P)\}_{P \in X}$ gives an isomorphism as in (15).

Since $\mathcal{L}_{r,\mathbf{0}} = \mathcal{O}_r$, (15) implies (16).

Similarly for (17), whose details we leave to the reader.

[The rest, (18) and (19), are general properties of invertible sheaves (e.g., [18], Exercise II.5.1); however, for self-containment, we now give the details.]

To check (18), let $\phi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and, for brevity, let $\mathcal{L} = \mathcal{L}_{r,\mathbf{d}}$; we therefore get maps $\phi(P)\colon \mathcal{F}(P) \to \mathcal{G}(P)$ for all $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$ that make Figure 3 commutative. Since $\mathcal{L}(P) = \mathcal{O}(P)$ for all P, for each $P \in X$ there is a natural bijection between maps

$$\phi(P)\colon \mathcal{F}(P) \to \mathcal{G}(P)$$

to maps

$$\mathrm{id}_{\mathcal{O}_r(P)} \otimes_{\mathcal{O}_r(P)} \phi(P)\colon \mathcal{L}(P) \otimes_{\mathcal{O}_r(P)} \mathcal{F}(P) \to \mathcal{L}(P) \otimes_{\mathcal{O}_r(P)} \mathcal{G}(P),$$

namely the map taking

$$1 \otimes f \mapsto 1 \otimes \big(\phi(P)\big)(f).$$

We claim that the family of maps $\{\mathrm{id}\otimes\phi(P)\}_{P\in X}$ form a morphism of $\mathcal{O}$-modules: indeed, looking at Figure 3, we see that for each ρ_{ij} there is a commutativity that we need to verify. So consider, for example, the $\rho_{1,1}$ commutativity: so consider the two squares:

$$\begin{array}{ccc} \mathcal{F}(B_1) & \xrightarrow{\phi(B_1)} & \mathcal{G}(B_1) \\ \downarrow{\scriptstyle \mathcal{F}(\rho_{1,1})} & & \downarrow{\scriptstyle \mathcal{G}(\rho_{1,1})} \\ \mathcal{F}(A_1) & \xrightarrow[\phi(A_1)]{} & \mathcal{G}(A_1) \end{array} \qquad \begin{array}{ccc} \mathcal{L}\otimes\mathcal{F}(B_1) & \xrightarrow{\mathrm{id}\otimes\phi(B_1)} & \mathcal{L}\otimes\mathcal{G}(B_1) \\ \downarrow{\scriptstyle \mathcal{L}\otimes\mathcal{F}(\rho_{1,1})} & & \downarrow{\scriptstyle \mathcal{L}\otimes\mathcal{G}(\rho_{1,1})} \\ \mathcal{L}\otimes\mathcal{F}(A_1) & \xrightarrow[\mathrm{id}\otimes\phi(A_1)]{} & \mathcal{L}\otimes\mathcal{G}(A_1) \end{array}$$

we claim that the left square commutes iff the right square does: indeed, say that $f\in\mathcal{F}(B_1)$ gives rise to f', g, g' as follows:

$$\begin{array}{ccc} f & \xrightarrow{\phi(B_1)} & g=\phi(B_1)f \\ \downarrow{\scriptstyle \mathcal{F}(\rho_{1,1})} & & \downarrow{\scriptstyle \mathcal{G}(\rho_{1,1})} \\ f'=\mathcal{F}(\rho_{1,1})f & \overset{???}{\underset{\phi(A_1)}{\cdots\cdots\cdots

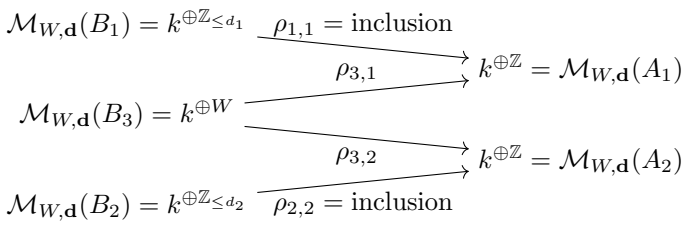


Figure 7: The k-Diagram $\mathcal{M}_{W,\mathbf{d}}$.

Definition 3.2. *Let k be a field. If S is a set, we use $k^{\oplus S}$ to denote the k-vector space that is direct sum of one copy of k for each element of S, i.e., whose elements are collections $\{v_s\}_{s\in S}$ with $v_s \neq 0$ for at most finitely many values of s; for $s \in S$, we use $\mathbf{e}_s$ to denote the vector that is 1 in component s and 0 elsewhere. If T is another set and $\alpha\colon S \to T$ a map of sets, then α gives rise to a unique k-linear transformation, denoted $k^{\oplus\alpha}$, from $k^{\oplus S} \to k^{\oplus T}$ taking $\mathbf{e}_s$ to $\mathbf{e}_{\alpha(s)}$. If $S \subset T$, then the inclusion map $\iota\colon S \to T$ gives an injection $k^{\oplus\iota}$ which we call the* inclusion map (of $k^{\oplus S}$ to $k^{\oplus T}$).

In the above, one easily checks that if α is an injection, surjection, or bijection, then the same is true of $k^{\oplus\alpha}$. Next we fix a convention for multisets (any reasonable convention would suffice).

Definition 3.3. *Let S_1, S_2 be sets, and $W\colon S_1 \times S_2 \to \mathbb{Z}_{\ge 0} \cup \{\infty\}$. The* multiset on $S_1 \times S_2$ with multiplicities W *refers to the set*

$$\mathrm{Multi}(W) = \{(s_1, s_2, i) \in S_1 \times S_2 \times \mathbb{N} \mid i \le W(s_1, s_2)\}, \tag{20}$$

where if $W(s_1, s_2) = \infty$, then we view all i as satisfying $i \le W(s_1, s_2)$. We refer to the maps $\mathrm{Multi}(W) \to S_1$ *and* $\mathrm{Multi}(W) \to S_2$ *taking (s_1, s_2, i) to, respectively, s_1 and s_2, as, respectively, the* first and second projections. *We use the notation $k^{\oplus W}$ to denote $k^{\oplus \mathrm{Multi}(W)}$, which comes with maps*

$$\mathrm{proj}_i\colon k^{\oplus W} \to k^{\oplus S_i} \tag{21}$$

induced by the first and second projections. The support of W *is the set of $(s_1, s_2) \in S_1 \times S_2$ such that $W(s_1, s_2) \ge 1$. When W takes on only the values $\{0, 1\}$, then with mild abuse of notation we may identify* $\mathrm{Multi}(W)$ *with its support, which is a subset of $S_1 \times S_2$, since in this case W is determined by its support. For $r \ge 1$ we say that W is* r-periodic *if for all $i, j \in \mathbb{Z}$ we have*

$$W(i, j) = W(i + r, j - r).$$

Example 3.1. *If $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is a perfect matching, and $\pi\colon \mathbb{Z} \to \mathbb{Z}$ is its associated bijection, then $k^{\oplus W}$ has one copy of k for each pair $(a_1, \pi(a_1)) \in \mathbb{Z}^2$ varying over all $a_1 \in \mathbb{Z}$. In this case we may identify $k^{\oplus W}$ with $k^{\oplus\mathbb{Z}}$, where the first projection is the identity map on $k^{\oplus\mathbb{Z}}$, and the second projection is the map $k^{\oplus\mathbb{Z}} \to k^{\oplus\mathbb{Z}}$, takes $\mathbf{e}_{a_1}$ to $\mathbf{e}_{\pi(a_1)}$. Hence both maps* (21) *are isomorphisms.*

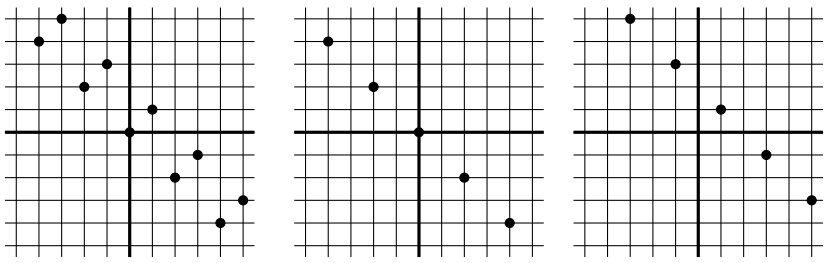
Definition 3.4. *Let k be a field, $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\ge 0} \cup \{\infty\}$, and $\mathbf{d} \in \mathbb{Z}^2$. We use $\mathcal{M}_{W,\mathbf{d}}$ to denote the following k-diagram (Definition 2.6):*

1. *for $i = 1, 2$, $\mathcal{M}_{W,\mathbf{d}}(B_i) = k^{\oplus\mathbb{Z}_{\le d_i}}$, $\mathcal{M}_{W,\mathbf{d}}(A_i) = k^{\oplus\mathbb{Z}}$, $\rho_{i,i}$ is the inclusion,*
2. *$B_3 = k^{\oplus W}$, and for $j = 1, 2$, $\rho_{3,j}$ are the projection maps (as in* (21)*).*

We depict these k-diagrams in Figure 7.

Note that if W is r-periodic, then $\mathcal{M}_{W,\mathbf{d}}$ can be viewed as a ring of $\mathcal{O}_r$-modules: in other words, for $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$, we can view $\mathcal{M}_{W,\mathbf{d}}(P)$ as an $\mathcal{O}_r(P)$ module as follows:

1. For $i = 1, 2$ we can identity $\mathcal{M}_{W,\mathbf{d}}(A_i) = k^{\oplus\mathbb{Z}}$ with $k[x_i^{\pm}]$, i.e., where the element $\alpha\colon \mathbb{Z} \to k$ in $k^{\oplus\mathbb{Z}}$ is taken to
$$\sum_{j\in\mathbb{Z}} \alpha(j) x_i^j.$$
2. Similarly, for $i = 1, 2$ we can identity $\mathcal{M}_{W,\mathbf{d}}(B_i) = k^{\oplus\mathbb{Z}_{\le d_i}}$ with the $k[y_i]$ module $k[y_i]y^{d_1} \subset k[x_i^{\pm}]$ where y_i is identified with $1/x_i$; hence $\beta\colon \mathbb{Z}_{\le d_i} \to k$ is taken to
$$\sum_{j\in\mathbb{Z}} \beta(j) y_i^j.$$



$W = \text{Simple}_{2;(0,0),(1,1)}$ $\quad$ $\text{Simple}_{2;(0,0)}$ $\quad$ $W = \text{Simple}_{2;(1,1)}$

Figure 8: W is 2-periodic that satisfies $W(2k,-2k) = W(2k+1,-2k+1) = 1$ for $k \in \mathbb{Z}$ and otherwise $W = 0$. We depict W by showing in bold points where $W = 1$. W can be written as the sum of two 2-periodic functions, $\text{Simple}_{2;(0,0)}$ plus $\text{Simple}_{2;(1,1)}$, each of which we depict above. This is the unique way of writing W as the sum of two simple 2-periodic functions. Of course, $\text{Simple}_{2;(0,0)} = \text{Simple}_{2;(2k,-2k)}$ for all $k \in \mathbb{Z}$, so simple functions aren't uniquely described as $\text{Simple}_{2;(i,j)}$.

3. If W is r-periodic, then for all i, j we have

$$W(i,j) = W(i+r, j-r).$$

This gives us a map on $\text{Multi}(W)$ (20) to itself taking (s_1, s_2, i) to $(s_1 + r, s_2 - r, i)$, and therefore a map $k^{\oplus W}$ to itself. We view $v \in k[v^{\pm}]$ as acting on $k^{\oplus W}$ as taking $W(i,j)$ to $W(i+r, j-r)$; for any $q \in \mathbb{Z}$, this gives a map v^q, taking $W(i,j)$ to $W(i+qr, j-qr)$, from which we get a $k[v^{\pm}]$ action on $k^{\oplus W}$.

Since our ring maps of $\mathcal{O}_r$ take y_i to $1/x_i$ and v to x_1^r and x_2^{-r}, $\mathcal{M}_{W,\mathbf{d}}$ becomes an $\mathcal{O}_r$ module.

Example 3.2. *If $W\colon \mathbb{Z}^2 \to \{0,1\}$ is a perfect matching, then the definition of $\mathcal{M}_{W,\mathbf{d}}$ in Definition 3.4 agrees with its definition in Section 2 (i.e., Definition 2.7, and its $\mathcal{O}_r$-module structure explained in Example 2.1).*

3.3 Tensor Products of Certain $\mathcal{O}_r$-modules

The point of this subsection is to explicitly describe $\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'}$. We will do this in greater generality.

Definition 3.5. *We say that a function $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ has* bounded support *if for some $C_1, C_2 \in \mathbb{Z}$ we have*

$$\forall \mathbf{d} \in \mathbb{Z}^2, \qquad W(\mathbf{d}) \neq 0 \quad \Rightarrow \quad C_1 \le \deg(\mathbf{d}) \le C_2.$$

Definition 3.6. *If $(i,j) \in \mathbb{Z}^2$, we use $\delta_{i,j}$ to denote the function $\mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$ that is 1 on (i,j) and 0 elsewhere. By the* simple r-periodic weight at (i,j) *we mean the function $\mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$ given by*

$$\text{Simple}_{r;i,j} = \sum_{q \in \mathbb{Z}} \delta_{i+rq, j-rq}$$

(and we often just write $\text{Simple}_{i,j}$ when r is understood).

Hence $\text{Simple}_{r;i,j}$ is r-periodic, and for any $q \in \mathbb{Z}$

$$\text{Simple}_{r;i,j} = \text{Simple}_{r;i+qr,j-qr}.$$

Moreover, the only function $f\colon \mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$ that satisfies $f \le \text{Simple}_{r;i,j}$ is either $f = \text{Simple}_{r;i,j}$ or $f = 0$. For these reasons, functions of the form $\text{Simple}_{r;i,j}$ are the "simplest" nonzero r-periodic weight functions.

Example 3.3. *Let W be the 2-periodic matching with $W(0,0) = W(1,1) = 1$. (Hence $W(2k,-2k) = W(2k+1,-2k+1) = 1$ for all $k \in \mathbb{Z}$.) Then*

$$W = \text{Simple}_{2;0,0} + \text{Simple}_{2;1,1} = \text{Simple}_{2;(0,1),(0,1)}.$$

We depict these functions in Figure 8.

The following is easy.

Proposition 3.2. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$ be an r-periodic degree bounded function. Then W is a finite sum of r-periodic simple functions*

$$W = \sum_{i=0}^{r-1} \sum_{j \in \mathbb{Z}} W(i,j) \text{Simple}_{r;i,j}.$$

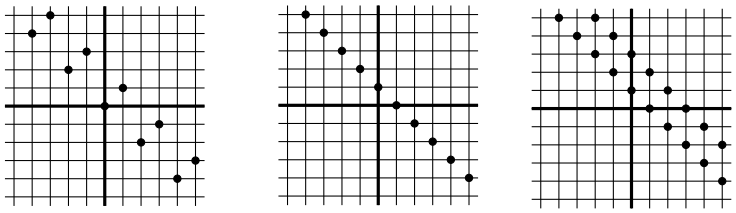


Figure 9: Here is an example of W and W' that are both 2-periodic weights; since W' is also 1-periodic, we have $W \star_2 W'$ is also 1-periodic. W, W' are both the sum of two simple 2-periodic functions, and $W \star_2 W'$ a sum of four. Later we will see that since W' is not of the form $W^*_{\mathbf{L}}$ for any $\mathbf{L} \in \mathbb{Z}^2$, it follows that $\mathcal{F} = \mathcal{M}_{W,\mathbf{0}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{0}}$ cannot serve as a *canonical $\mathcal{O}_r$-module* since for any $\mathbf{d}$ (no matter how small), $\mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_r} \mathcal{L}_{r,\mathbf{d}}$ has $H^1(\mathcal{G}) = 0$.

Proof. The sum on the left is finite, since W is a weight function, and hence for each $0 \le i \le r-1$, there are only finitely many j at which $W(i,j)$ is nonzero. The sum on the left clearly agrees with W at all (i',j') with $0 \le i' \le r-1$ and j' arbitrary. Therefore the two sides agree on all (i',j') by the r-periodicity of both sides. □

Now we define a $\mathcal{O}_r$-module for any W that is the sum of simple, r-periodic functions.

Definition 3.7. *Let $r \in \mathbb{N}$, $\ell \in \mathbb{Z}_{\ge 0}$, k be a field, $\mathbf{i} = (i_1, \dots, i_\ell), \mathbf{j} = (j_1, \dots, j_\ell) \in \mathbb{Z}^\ell$, and $\mathbf{d} \in \mathbb{Z}^2$. We define the $\mathcal{O}_r$-module*

$$\mathcal{N} = \mathcal{N}_{k,r;\mathbf{i},\mathbf{j};\mathbf{d}}$$

to be the module such that:

1. *for $P \in \{A_1, A_2, B_1, B_2\}$, $\mathcal{N}(P) = \mathcal{O}_r(P)$;*
2. *$\mathcal{N}(B_3) = k[v^{\pm}]^\ell$;*
3. *the restriction map $\rho_{1,1}$ takes 1 to $x_1^{d_1}$;*
4. *the restriction map $\rho_{2,2}$ takes 1 to $x_2^{d_1}$;*
5. *for $1 \le q \le \ell$, the restriction map $\rho_{3,1}$ takes $\mathbf{e}_q$, the vector $(0, \dots, 0, 1, 0, \dots, 0) \in k[v^{\pm}]^\ell$ to $x_1^{i_q}$; and*
6. *similarly $\rho_{3,2}$ takes $\mathbf{e}_q$ to $x_2^{j_q}$.*

Example 3.4.

$$\mathcal{L}_{r;\mathbf{d}} = \mathcal{N}_{(0,0);\mathbf{d}},$$

since for $\mathcal{N} = \mathcal{N}_{(0,0);\mathbf{d}}$ we have $\ell = 1$, $\mathbf{i} = (0)$, $\mathbf{j} = (0)$, and $\mathcal{N}(B_3) = k[v^{\pm}]$ and takes 1 to $x_1^0 = 1$ in $\mathcal{N}(A_1)$ and to $x_2^0 = 1$ in $\mathcal{N}(A_2)$.

Example 3.5. *Let W be an r-periodic perfect matching, with associated permutation π. Then*

$$\mathcal{M}_{W,\mathbf{d}} = \mathcal{N}_{(0,1,\dots,r-1),(\pi(0),\dots,\pi(r-1));\mathbf{d}}.$$

If $\mathbf{i} \in \mathbb{Z}^\ell$ and $\mathbf{i}' \in \mathbb{Z}^{\ell'}$ we use the notation $\mathbf{i}+\mathbf{i}' \in \mathbb{Z}^{\ell\ell'}$ to denote the vector whose components are the sum of those of $\mathbf{i}$ and $\mathbf{i}'$:

$$\mathbf{i}+\mathbf{i}' = (i_1 + i'_1, i_2 + i'_1, \dots, i_\ell + i'_{\ell'}) \in \mathbb{Z}^{\ell\ell'}$$

The following proposition follows directly from the definitions.

Proposition 3.3. *Let $r \in \mathbb{N}$, $\ell, \ell' \in \mathbb{Z}_{\ge 0}$, $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^\ell$, $\mathbf{i}', \mathbf{j}' \in \mathbb{Z}^{\ell'}$, and $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$. Then*

$$\mathcal{N}_{r;\mathbf{i},\mathbf{j};\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{N}_{r;\mathbf{i}',\mathbf{j}';\mathbf{d}'} \simeq \mathcal{N}_{r;\mathbf{i}+\mathbf{i}',\mathbf{j}+\mathbf{j}';\mathbf{d}+\mathbf{d}'}.$$

We illustrate this proposition in a case of interest in Figure 9.

Corollary 3.1. *Say that W, W' from $\mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$ have bounded support and are r-periodic. Say that*

$$W = \sum_{q=1}^{\ell} \text{Simple}_{r;i_q,j_q}, \quad W' = \sum_{q=1}^{\ell'} \text{Simple}_{r;i'_q,j'_q}.$$

Then for any $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$ *we have*

$$\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'} \simeq \mathcal{M}_{W'',\mathbf{d}+\mathbf{d}'},$$

where

$$W'' = \sum_{q=1}^{\ell} \sum_{q'=1}^{\ell'} \mathrm{Simple}_{r;i_q+i'_{q'},j_q+j'_{q'}}.$$

Corollary 3.2. *Say that* W, W' *from* $\mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ *have bounded support and are* r*-periodic. Define the* r-periodic convolution of W and W'*, denoted* $W \star_r W'$*, to be the function* $\mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ *give by*

$$W \star_r W'(\mathbf{y}) = \sum_{a_1=0}^{r-1} \sum_{a_2 \in \mathbb{Z}} W(\mathbf{a}) W'(\mathbf{y} - \mathbf{a}).$$

Then for any $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$ *we have*

$$\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'} \simeq \mathcal{M}_{W \star_r W',\mathbf{d}+\mathbf{d}'}.$$

Corollary 3.3. *Say that* W, W' *are two* r*-periodic perfect matchings, with associated bijections* π, π'*. Then*

$$\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'} \simeq \mathcal{M}_{W'',\mathbf{d}+\mathbf{d}'},$$

where

$$W'' = \sum_{q=0}^{r-1} \sum_{q'=0}^{r-1} \mathrm{Simple}_{r;q+q',\pi(q)+\pi'(q')}.$$

Hence $\mathcal{M}_{W'',\mathbf{d}+\mathbf{d}'}(B_3)$ is isomorphic to $k[v^{\pm}]^{r^2}$.

Remark 3.3. *If* W, W' *are two perfect matchings that are not necessarily periodic, we can define the tensor product of* k*-diagrams*

$$\mathcal{M}_{W,\mathbf{d}} \otimes \mathcal{M}_{W',\mathbf{d}'} = \mathcal{M}_{W,\mathbf{d}} \otimes_{\underline{k}} \mathcal{M}_{W',\mathbf{d}'}.$$

However, the B_3 *value of this diagram is*

$$k^{\oplus W} \otimes k^{\oplus W'},$$

and we easily see that this is isomorphic to $k^{\oplus W''}$ *where* $W'' = W * W'$ *where*

$$(W * W')(\mathbf{s}) \stackrel{def}{=} \sum_{\mathbf{a} \in \mathbb{Z}} W(\mathbf{a}) W'(\mathbf{s} - \mathbf{a}),$$

and the sum above can be infinite. (In fact, since W, W' *is of bounded support,* $W * W'$ *must be infinite.) However, if* W, W' *are* r*-periodic, then we can see that*

$$\forall \mathbf{s} \in \mathbb{Z}^2, \qquad W \star_r W'(\mathbf{s}) > 0 \iff W * W'(\mathbf{s}) > 0.$$

In the next section we will see that $H^1(\mathcal{M}_{W,\mathbf{d}})$ *depends only on the* $\mathbf{s}$ *where* $W(\mathbf{s}) > 0$*, not on the particular value of* $W(\mathbf{s})$ *(even if* $W(\mathbf{s}) = \infty$*). Hence it turns out that*

$$H^1(\mathcal{M}_{W,\mathbf{d}} \otimes_{\underline{k}} \mathcal{M}_{W',\mathbf{d}'}) = H^1(\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'}).$$

Similarly if W, W' *are both* r*-periodic and* r'*-periodic, then*

$$H^1(\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'}) = H^1(\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_{r'}} \mathcal{M}_{W',\mathbf{d}'}).$$

4. A Canonical $\mathcal{O}$-Module

Again, in this section k will be a fixed field, and we will often suppress k, writing, e.g., $\mathcal{O}_r$ instead of $\mathcal{O}_{r,k}$.

The basis for our duality theory will be an $\mathcal{O}_r$-module, ω, such that

1. $H^1(\omega) \simeq k$;
2. $\omega = \mathcal{M} \otimes \mathcal{M}'$ where $\mathcal{M} = \mathcal{M}_W$ and $\mathcal{M}_{W'}$ where W is an arbitrary r-periodic perfect matching and W' is an appropriately chosen r-periodic perfect matching;
3. ω allows up to set up a satisfactory "duality theory."

Although we leave (3) above vague for now, there is a rather striking result for ω satisfying (1) and (2).

Theorem 4.1. *Let W, W' be r-periodic perfect matchings. If for all $\mathbf{L} \in \mathbb{Z}^2$ we have $W' \neq W^*_{\mathbf{L}}$, then for all $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$ we have*

$$H^1(\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'}) = 0,$$

This theorem will be proved in Subsection 4.4.

This theorem tells us that the only possible ω as above comes from W' of the form $W^*_{\mathbf{L}}$. Moreover, we have the following result.

Theorem 4.2. *Let W be a periodic perfect matchings, and let r be the period of W (i.e., the minimum $r \geq 1$ such that W is r-periodic). Let $\mathbf{K} \in \mathbb{Z}^2$, $\mathbf{L} = \mathbf{K} + \mathbf{1}$, $W' = W^*_{\mathbf{L}}$, and let $\omega_{W,\mathbf{K}} = \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W',\mathbf{K}}$. Then*

$$b^1(\omega_{W,\mathbf{K}}) = 1,$$

and a basis for (the one-dimensional) $H^1(\omega)$ is given by

$$(x^{L_1}, 0) = (0, -x_2^{L_2}) \in H^1(\omega) \simeq \omega(A)/\omega(\partial).$$

More generally, for any $\mathbf{d}, \mathbf{d}' \in \mathbb{Z}^2$ let $\mathbf{K}' = \mathbf{d} + \mathbf{d}'$; we have that

$$b^1(\mathcal{M}_{W,\mathbf{d}} \otimes \mathcal{M}_{W',\mathbf{d}'})$$

equals the size of the set

$$Q = \{q \in \mathbb{Z} \mid L_1 + qr > K'_1 \text{ and } L_2 - qr > K'_2\}.$$

Moreover:

1. *A basis for $H^1(\mathcal{M}_{W,\mathbf{d}} \otimes \mathcal{M}_{W',\mathbf{d}'})$ is the set of $(x_1^{L_1+qr}, 0)$ ranging over all $q \in Q$; and similarly with $(0, x_2^{L_2-qr})$ over $q \in Q$.*
2. *If $(x_1^a, 0) \in \omega_W(A)$ and $a \neq L_1 + qr$ with $q \in Q$, then the image of $(x_1^a, 0)$ in $H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W',\mathbf{K}'})$ is zero; similarly for $(0, x_2^a)$ with $a \neq L_2 - qr$ with $q \in Q$.*

This theorem will be proven in Subsection 4.5.

Remark 4.1. *Theorem 4.2 also shows that*

$$b^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W',\mathbf{K}-\mathbf{d}}) = 1$$

provided that $\mathbf{d} \in [r' - 1, r' - 1]$, where r' is the period of W; hence for $r' \geq 2$, there would be other candidates for our canonical diagram (if our only criteria was that b^1 should equal 1). However, it turns out that $\mathbf{d} = \mathbf{0}$ is the right choice of canonical diagram for our duality theorem. [Of course, we also have that

$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W',\mathbf{K}-\mathbf{d}'}) = 1$$

whenever $\mathbf{d}'$ equals one of the $\mathbf{d}$ above plus a multiple of $(r, -r)$.]

Remark 4.2. *The Folinsbee-Friedman algorithm in Subsection 4.1 below, along with (3.3), show that if W, W' are r-periodic perfect matchings, then $H^1(\mathcal{M}_{W,\mathbf{d}} \otimes \mathcal{M}_{W',\mathbf{d}})$ whether one takes the tensor product as $\mathcal{O}_r$-modules or as $\underline{k}$-modules (i.e., k-diagrams).*

Definition 4.1. *Let W be an r-periodic perfect matchings. Let $\mathbf{K} \in \mathbb{Z}^2$ and $\mathbf{L} = \mathbf{K} + \mathbf{1}$. Then the* canonical $\mathcal{O}$-module *on W and $\mathbf{K}$ refers to*

$$\omega_{W,\mathbf{K}} = \omega_{W,\mathbf{K},r} \stackrel{def}{=} \mathcal{M}_{W,\mathbf{0}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}.$$

We alert the reader that the only result we need in Sections 5 and 6 is that

$$H^1(\omega_{W,\mathbf{K}}) \simeq k.$$

Hence Theorem 4.1 will not be needed there. (The purpose of this theorem is to explain the special nature of the canonical $\mathcal{O}_r$-module.) (However, the proofs of Theorem 4.1 and 4.2 are both based on the same fundamental Lemma 4.1.)

Remark 4.3. *Since $H^1(\omega_{W,\mathbf{K}})$ is generated $(x^{L_1}, 0)$, we see that $\omega_{W,\mathbf{K}}$ generally depends on $\mathbf{K}$. However, we easily see that for any $\mathbf{K}' \in \mathbb{Z}$ and $\mathbf{L}' = \mathbf{K} + \mathbf{1}$ there is an isomorphism*

$$\phi\colon \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}} \xrightarrow{\simeq} \mathcal{M}_{W^*_{\mathbf{L}'},\mathbf{K}'};$$

namely for $i = 1, 2$, $\phi(B_i)$ takes y_i^a to $y_i^{a+K_i-K_i'}$, $\phi(A_i)$ takes x_i^a to $x_i^{a-K_i'+K_i}$, and $\phi(B_3)$ takes $\mathbf{d}$ to $\mathbf{d}+\mathbf{L}'-\mathbf{L}$; to check that ϕ is an isomorphism we use the facts that

$$W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L}-\mathbf{d}) = W(\mathbf{L}'+\mathbf{L}-\mathbf{L}'-\mathbf{d}) = W^*_{\mathbf{L}'}(\mathbf{d}+\mathbf{L}'-\mathbf{L})$$

and that $\mathbf{L}'-\mathbf{L} = \mathbf{K}'-\mathbf{K}$. It follows that $\omega_{W,\mathbf{K}}$ and $\omega_{W,\mathbf{K}'}$ are isomorphic. In Subsection 4.6 we give some remarks about how the isomorphism class of $\omega_{W,\mathbf{K}}$ is independent of $\mathbf{K}$ but might depend on W.

In the next two subsections we prove the above theorems. Recall [12] that if W, W' are any r-periodic perfect matchings, then

$$\mathcal{M}_W \otimes \mathcal{M}_{W'} = \mathcal{M}_{W''}$$

where W'' is an r-periodic function $\mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ that can be written as the sum

$$W'' \simeq W_1 + \cdots + W_r$$

where $W_1, \ldots, W_r$ are r-periodic perfect matchings. As such it is easy to see that for $r \geq 2$ we have

$$H^0(\mathcal{M}_{W''}) = \infty.$$

Hence these $\mathcal{O}$-modules are more chaotic than their counterparts in algebraic geometry.

Remark 4.4. *We also remark that for any W, W', $\gamma = \mathcal{M}_W \otimes \mathcal{M}_{W'}$ has the same values as $\mathcal{M}_{W,\mathbf{d}}$ (and as $\mathcal{O}$) except at B_3; and $\gamma(B_3) = k[v^{\pm}]^{\oplus r^2}$ and $\mathcal{M}(B_3) = k[v^{\pm}]^{\oplus r}$. A crude dimension count suggests that the image of $\mathcal{M}(B)$ in $\mathcal{M}(A)$ is roughly "of the same size" as $\mathcal{M}(A)$: indeed, the image of $\mathcal{M}(B_3)$ in $\mathcal{M}(A)$ gives a bijection between $\mathcal{M}(A_1)$ and $\mathcal{M}(A_2)$, both isomorphic to $k^{\oplus\mathbb{Z}}$; for $i = 1, 2$, the image of $\mathcal{M}(B_i)$ in $\mathcal{M}(A_i)$ is $k^{\oplus\mathbb{Z}_{\leq d_i}}$; hence the image of $\mathcal{M}(B)$ in $\mathcal{M}(A)$ is roughly one copy of $k^{\oplus\mathbb{Z}}$ from $\mathcal{M}(B_3)$ and roughly half a copy of $k^{\oplus\mathbb{Z}}$ for each $\mathcal{M}(B_i)$. However, for $r \geq 2$, since γ has the same values as $\mathcal{M}_{W,\mathbf{d}}$ except that it has "$(r-1)$ more copies of $k^{\oplus\mathbb{Z}}$ at B_3," it seems clear that $H^0(\gamma) = \infty$ and likely that $H^1(\gamma)$ should equal 0. Hence one might expect $H^1(\gamma)$ to be 0. For this reason Theorem 4.2 seems quite remarkable to us.*

4.1 The Algorithm of Folinsbee-Friedman [12] To Compute $H^i(\mathcal{M}_{W,\mathbf{d}})$

To prove the main theorems in this section, we review the method of Folinsbee-Friedman [12] to compute $H^i(\mathcal{M}_{W,\mathbf{d}})$ where $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is an arbitrary function. This is stated as Theorem 4.3 there, although the basis for $H^1(\mathcal{M}_{W,\mathbf{d}})$, which we need in this article, is only apparent from the proof of Theorem 4.3, not its statement.

Recall that by definition, $H^1(\mathcal{M}_{W,\mathbf{d}})$ is the cokernel of the map

$$\mathcal{M}_{W,\mathbf{d}}(\partial)\colon \mathcal{M}_{W,\mathbf{d}}(B_1) \oplus \mathcal{M}_{W,\mathbf{d}}(B_2) \oplus \mathcal{M}_{W,\mathbf{d}}(B_3) \to \mathcal{M}_{W,\mathbf{d}}(A_1) \oplus \mathcal{M}_{W,\mathbf{d}}(A_2),$$

which, setting $\tau = \mathcal{M}_{W,\mathbf{d}}(\partial)$, is the map

$$\tau\colon k^{\oplus\mathbb{Z}_{\leq d_1}} \oplus k^{\oplus\mathbb{Z}_{\leq d_2}} \oplus k^{\mathrm{Multi}(W)} \to k^{\oplus\mathbb{Z}} \oplus k^{\oplus\mathbb{Z}} \simeq k^{\oplus\mathbb{Z}\times\{1,2\}}.$$

Theorem 4.3. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be an arbitrary function. Let*

1. *G' be the (bipartite) graph whose vertex set is $\mathbb{Z} \times \{1, 2\}$ such that the vertices $(s_1, 1)$ and $(s_2, 2)$ are joined by $W(s_1, s_2)$ edges (this is therefore a multigraph);*
2. *let G be the graph obtained from G' by collasping the vertices of $\mathbb{Z}_{\leq d_1} \times \{1\}$ and $\mathbb{Z}_{\leq d_2} \times \{2\}$ into a singe vertex v_0 (keeping as a self-loop any edge from $\mathbb{Z}_{\leq d_1} \times \{1\}$ to $\mathbb{Z}_{\leq d_2} \times \{2\}$).*

(Hence G may have self-loops and is not generally bipartite.) Also set $V_{\mathrm{first}}, V_{\mathrm{second}}$ to be the subsets $\mathbb{Z}_{\geq d_1+1} \times \{1\}$ and $\mathbb{Z}_{\geq d_2+1} \times \{2\}$, so that V is partitioned as $\{v_0\} \amalg V_{\mathrm{first}} \amalg V_{\mathrm{second}}$. Then

$$b^1(\mathcal{M}_{W,\mathbf{d}}) = b^0(G) - 1 \tag{22}$$

$$b^0(\mathcal{M}_{W,\mathbf{d}}) = b^1(G). \tag{23}$$

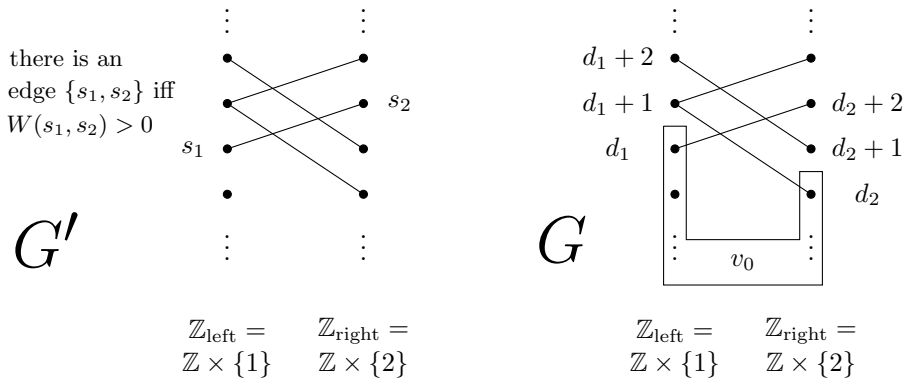


Figure 10: The graph $G' = (V', E')$: V' consists of two copies of $\mathbb{Z}$, one on the left, one on the right, plus $W(s_1, s_2)$ edges between $s_1 \in \mathbb{Z}_{\text{left}}$ and $s_2 \in \mathbb{Z}_{\text{right}}$. G is obtained from G' by collasping the left vertices $\le d_1$ and the right vertices $\le d_2$ into a single vertex, v_0. Therefore G generally has self-loops about v_0, and G is not generally bipartite. To compute $H^1(\mathcal{M}_{W,\mathbf{d}})$ with G' and G, multiple edges have no effect; so it is enough to put an edge from s_1 on the left to s_2 on the right when $W(s_1, s_2) > 0$.

Moreover, let $\{G_i\}_{i \in I}$ be the connected components of G, where $0 \in I$ and G_0 is the connected component of G containing v_0; set $I' = I \setminus \{0\}$. (Hence G_i with $i \in I'$ are precisely the connected components of G' whose vertex set is disjoint from $\mathbb{Z}_{\le d_1} \times \{1\} \cup \mathbb{Z}_{\le d_2} \times \{2\}$.) For each $i \in I'$, and $G_i = (V_i, E_i)$, choose some $v_i \in V_i$. Then

$$H^1(\mathcal{M}_{W,\mathbf{d}}) \simeq \bigoplus_{i \in I'} H^0(G_i) \simeq k^{\oplus I'},$$

and a basis for $H^1(\mathcal{M}_{W,\mathbf{d}})$ is $\{\mathbf{e}_{v_i}\}_{i \in I'}$, the standard basis vectors for $k^{\oplus \mathbb{Z} \times \{1,2\}}$.

We depict G' and G in Figure 10.

Proof. Theorem 4.3 of [12] explicitly describes the formulation of G and G', and states (22) and (23). The rest can be inferred from the proof of Theorem 4.3: let us review the main point. Because for $i = 1, 2$ we have

$$\operatorname{coker}\big(\mathcal{M}_{W,\mathbf{d}}(\rho_{ii})\big) \simeq \mathcal{M}_{W,\mathbf{d}})(A_i)/\mathcal{M}_{W,\mathbf{d}})(B_i) \simeq k^{\oplus \mathbb{Z}_{\ge d_i + 1}},$$

and because $\mathcal{M}_{W,\mathbf{d}}(B_3) = k^{\text{Multi}(W)}$, it follows that the cokernel of τ above is the same as the cokernel of the map

$$\tilde{\tau} \colon k^{\text{Multi}(W)} \to k^{\oplus \mathbb{Z}_{\ge d_1 + 1}} \oplus k^{\oplus \mathbb{Z}_{\ge d_2 + 1}}$$

each copy of k coming from a pair (s_1, s_2) with $W(s_1, s_2) \neq 0$ is sent to $(-\mathbf{e}_{s_1}, -\mathbf{e}_{s_2})$, if $s_1 \ge d_1 + 1$ and $s_2 \ge d_2 + 1$, but is sent to 0 otherwise. But this is the same as the map

$$\bigoplus_{i \in I'} k^{\oplus E_i} \to \bigoplus_{i \in I'} k^{\oplus V_i}$$

where each $e \in E_i$ corresponding to an (s_1, s_2) with $W(s_1, s_2) > 0$ is taken to $(-\mathbf{e}_{s_1}, -\mathbf{e}_{s_2})$. But each $G_i = (V_i, E_i)$ with $i \in I'$ is also bipartite, so by scaling this map by negating the $V_i \cap \mathbb{Z}_{\ge d_1 + 1}$ vertices, this is equivalent to the map taking $e \in E_i$ as above to $(\mathbf{e}_{s_1}, -\mathbf{e}_{s_2})$; hence we get a direct sum of indicence matrices for G_i. □

Notation 4.4. *In the formulation of G' and G above from W, we will use $\mathbb{Z}_{\text{left}}, \mathbb{Z}_{\text{right}}$ to denote $\mathbb{Z} \times \{1\}, \mathbb{Z} \times \{2\}$, respectively, and refer to vertices in $\mathbb{Z}_{\text{left}}$ as* left vertices *and similarly for $\mathbb{Z}_{\text{right}}$.*

Remark 4.5. *In computing $H^1(\mathcal{M}_{W,\mathbf{d}})$ from G, we can replace any multiple edge of G with a single edge, since this doesn't change the connected components of G.*

Remark 4.6. *In Theorem 4.3, may be simpler to think of I' and the G_i with $i \in I'$ as representing the connected components of G' that have no vertices in $\mathbb{Z}_{\le d_1}$ on the left and none in $\mathbb{Z}_{\le d_2}$ on the right. This is because G' is a simpler graph than G. In fact, we will state our fundamental lemma in terms of G'.*

The following proposition is straightforward; it is involved in how we state our fundamental lemma.

Proposition 4.1. *Say that W, W' are r-periodic perfect matchings whose associated permutations are π, π'. Let $W'' = W \star_r W'$.*

1. *For all* $\mathbf{s} \in \mathbb{Z}^2$*, the following are equivalent:*
   (a) $W''(\mathbf{s}) > 0$*;*
   (b) *there exists* $\mathbf{a} \in \mathbb{Z}^2$ *such that* $W(\mathbf{a}) > 0$ *and* $W'(\mathbf{s}-\mathbf{a}) > 0$*; and*
   (c) *there exist* $a, a' \in \mathbb{Z}$ *such that*
   $$s_1 = a + a', \quad s_2 = \pi(a) + \pi'(a').$$
2. *For any* $L_1, L_2 \in \mathbb{Z}$*, the following are equivalent:*
   (a) $\forall a \in \mathbb{Z}$*,* $\pi(a) + \pi'(L_1 - a) = L_2$*;*
   (b) $W''(L_1, s_2) > 0$ *implies that* $s_2 = L_2$*;*
   (c) $W''(s_1, L_2) > 0$ *implies that* $s_1 = L_1$*; and*
   (d) G' *in Theorem 4.3 with* $W = W''$ *has a connected component whose vertex set consists entirely of vertex* L_1 *on the left and vertex* L_2 *on the right.*

Proof. (1): Corollary 3.2 shows that $W''(\mathbf{s}) > 0$ iff for some $a_1 = 0, \dots, r-1$ and $a_2 \in \mathbb{Z}$ we have

$$W(\mathbf{a}) > 0 \quad \text{and} \quad W'(\mathbf{s}-\mathbf{a}) > 0. \tag{24}$$

The latter implies the (1b). Now assume (1b), i.e., that $\tilde{a}_1 \in \mathbb{Z}$ and $\tilde{a}_2 \in \mathbb{Z}$ satisfy

$$W(\tilde{a}_1, \tilde{a}_2) > 0 \quad \text{and} \quad W(s_1 - \tilde{a}_1, s_2 - \tilde{a}_2) > 0$$

then for some $q \in \mathbb{Z}$ we have $0 \le \tilde{a}_1 - qr \le r-1$. Since W, W' are r-periodic, the latter implies that

$$W(\tilde{a}_1 - qk, \tilde{a}_2 + qk) > 0, \quad \text{and} \quad W'(s_1 - \tilde{a}_1 + qk, s_2 - \tilde{a}_2 - qk) > 0$$

which implies (24) for $a_1 = \tilde{a}_1 - qr$ and $a_2 = \tilde{a}_2 + qk$, which makes $0 \le a_1 \le r-1$. Hence (1a) and (1b) are equivalent.

Now we show that (1b) and (1c) are equivalenet: (1b) is equivalent to $a_2 = \pi(a_1)$ and $s_2 - a_2 = \pi'(s_1 - a_1)$; and setting $a_1' = s_1 - a_1$, this equivalent the existence of a_1, a_1' such that

$$s_1 = a_1 + a_1' \quad \text{and} \quad s_2 = a_2 + \pi'(s_1 - a_1);$$

but in these last equations

$$s_2 = a_2 + \pi'(s_1 - a_1) = \pi(a_1) + \pi'(a_1').$$

Hence (1b) and (1c) are equivalent.

Hence (1a)–(1c) are equivalent.

(2): Let us show that (2b) and (2a) are equivalent, by showing that their negations are equivalent: the negation of (2b) is that there exists $s_2 \ne L_2$ such that $W''(L_1, s_2) > 0$, which by (1) is equivalent to there exist $a, a' \in \mathbb{Z}$ such that $L_1 = a + a'$ and $\pi(a) + \pi(a') = s_2 \ne L_2$; since a' is determined as $a' = L_1 - a$, this is equivalent to the existence of a such that $\pi(a) + \pi'(L_1 - a) \ne L_2$. But this is the negation of (2a); hence (2a) and (2b) are equivalent.

Similarly, the negation of (2c) is equivalent to the existence of $a, a' \in \mathbb{Z}$ such that $a + a' \ne L_1$ and $\pi(a) + \pi'(a') = L_2$. But this implies $\pi(a) + \pi'(L_1 - a) \ne L_2$, since π' is a bijection. Hence, by (1c), $W(L_1, s_2) > 0$ where

$$s_2 = \pi(a) + \pi'(L_1 - a) \ne L_2.$$

Hence the negation of (2c) implies the negation of (2b). By the symmetric argument, the negation of (2b) implies the negation of (2c). Hence (2b) and (2c) are equivalent, which are therefore also equivalent to (2a).

(2b) and (2c) are equivalent to (2d), by the definition of G'. Hence (2a)–(2d) are equivalent. □

4.2 The Fundamental Lemma

The discussion in the last subsection motivates the fundamental lemma we now give.

Lemma 4.1. *Let* π, π' *be permutations of bounded support (i.e., for some constants* C, C' *we have* $|\pi(a)+a| \le C$ *and* $|\pi'(a)+a| \le C'$ *for all* $a \in \mathbb{Z}$*). Let* G' *be the following bipartite graph:*

1. *its vertex set is* $V = \mathbb{Z}_{\text{left}} \amalg \mathbb{Z}_{\text{right}}$*, where* $\mathbb{Z}_{\text{left}}, \mathbb{Z}_{\text{right}}$ *are each a copy of* $\mathbb{Z}$*;*
2. *each* $a, a' \in \mathbb{Z}$ *there is an edge joining* $a + a'$ *on the left (i.e.,* $a + a' \in \mathbb{Z}_{\text{left}}$*) to* $\pi(a) + \pi'(a')$ *on the right (i.e., in* $\mathbb{Z}_{\text{right}}$*)*

(multiple edges are unimportant here). Then for any $L_1 \in \mathbb{Z}$, either

1. *all edges from L_1 on the left are incident upon a single vertx L_2 on the right (in which case $\pi(x)+\pi'(L_1 - x) = L_2$, i.e., if W, W' are the weights associated to π, π', then $W' = W^*_{\mathbf{L}}$), or*
2. *for any $k \in \mathbb{Z}$, there is a path from L_1 on the left to some left or right vertex whose value is at most k.*

Proof. Say that the hypotheses (1) and (2) hold, but for some L_1 neither (1) nor (2) holds; we will derive a contradiction. So let S_1 be all the left vertices in the connected component of L_1 in G', and S_2 all the right vertices. Then S_1 must have a minimum element; but since π, π' are of bounded support, then S_1 must have a maximum element, for otherwise L_1 has a path to an arbitrarily large left vertex and therefore a path to an arbitrarily small right vertex. Let $L_1^{\min}$ and $L_1^{\max}$ be the minimum and maximum values of S_1; similarly S_2 has a minimum and a maximum value which we respectively denote $L_2^{\min}$ and $L_2^{\max}$.

Let

$$\Delta_1 = L_1^{\max} - L_1^{\min}, \quad \Delta_2 = L_2^{\max} - L_2^{\min};$$

let us prove that $\Delta_1 = \Delta_2$: we have that for any $a \in \mathbb{Z}$

$$\pi(a) + \pi'(L_1^{\min} - a) \le L_2^{\max},$$

and

$$\pi(a) + \pi'(L_1^{\max} - a) \ge L_2^{\min}.$$

Subtracting these two equations gives

$$\forall a \in \mathbb{Z}, \quad \pi'(L_1^{\max} - a) - \pi'(L_1^{\min} - a) \ge -\Delta_2;$$

substituting $b = L_1^{\min} - a$, we have that

$$\forall b \in \mathbb{Z}, \quad \pi'(b + \Delta_1) - \pi'(b) \ge -\Delta_2.$$

Taking b to be $0, \Delta_1, \dots, (\ell-1)\Delta_1$ for some $\ell \in \mathbb{N}$ and adding, we have

$$\pi'(\ell\Delta_1) - \pi'(0) \ge -\ell\Delta_2.$$

Since π' is of bounded support, we have

$$-\ell\Delta_1 - \pi'(\ell\Delta_1) \ge C'$$

for some constant C' (depending only on π'). Adding the last two inequalities we get

$$-\ell\Delta_1 - \pi'(0) \ge -\ell\Delta_2 + C'.$$

Dividing by ℓ and taking $\ell \to \infty$ we get $-\Delta_1 \ge -\Delta_2$ or $\Delta_1 \le \Delta_2$. The symmetric argument shows that $\Delta_2 \le \Delta_1$, and hence $\Delta_1 = \Delta_2$.

Now let $\Delta = \Delta_1 = \Delta_2$; since conclusion (1) of the lemma does not hold, we have $\Delta > 0$. Let us show that this is impossible.

By assumption, $L_1^{\max}$ on the left is connected only to ℓ on the right with $\ell \ge L_2^{\min}$, i.e.,

$$\forall y \in \mathbb{Z}, \quad \pi(y) + \pi'(L_1^{\max} - y) \ge L_2^{\min}.$$

Letting $x = y - \Delta$, we have

$$\forall x \in \mathbb{Z}, \quad \pi(x + \Delta) + \pi'(L_1^{\min} - x) \ge L_2^{\min}. \tag{25}$$

Since $L_1^{\min}$ only has right vertices somewhere between $L_2^{\min}$ and $L_2^{\max}$, if

$$X_0 = \{x \mid \pi(x) + \pi'(L_1^{\min} - x) \le L_2^{\max} - 1\}$$

then

1. $x \in X_0$ implies that

   $$\pi(x) + \pi'(L_1^{\min} - x) \le L_2^{\max} - 1$$

   and hence, in view of (25),

   $$\pi(x) - \pi(x + \Delta) \le -\Delta - 1;$$

   and

2. otherwise,
$$\forall x \notin X_0, \quad \pi(x) + \pi'(L_1^{\min} - x) = L_2^{\max}. \tag{26}$$
and hence
$$\forall x \notin X_0, \quad \pi(x) - \pi(x + \Delta) = -\Delta.$$

We aim to show that $X_0 = \emptyset$.

First, let us show that $|X_0|$ is finite: for any $a, B \in \mathbb{N}$, if
$$s = s_{a,B} = \left|\{a - B\Delta,\ a - (B-1)\Delta, \ldots, a + B(-1)\Delta\} \cap X_0\right|$$
then
$$\pi(a - B\Delta) - \pi(a + B\Delta) \le -(2B)\Delta - s.$$
Hence
$$s \le -(2B)\Delta + \pi(a + B\Delta) - \pi(a - B\Delta).$$
Since π is of bounded support, the right-hand side above is bounded for any fixed a and $B \to \infty$; but
$$s_{a,+\infty} \stackrel{\text{def}}{=} \lim_{B\to\infty} s_{a,B} = \{x \in X_0 \mid x \equiv a \pmod{\Delta}\},$$
and each is bounded for fixed a. Taking $a = 0, 1, \ldots, B-1$, we have
$$|X_0| = s_{0,+\infty} + \cdots s_{B-1,+\infty}$$
which is bounded. Hence X_0 is finite.

Now we prove that X_0 is empty: indeed, let
$$\tilde{\pi}(y) = \pi(y + \Delta) + \Delta.$$
Since π is a permutation, so is $\tilde{\pi}$. But $\tilde{\pi}(x) = \pi(x)$ for $x \notin X_0$, and $\tilde{\pi}(x) \le \pi(x) - 1$ for $x \in X_0$. Hence, since X_0 is finite,
$$\sum_{x\in X_0} \tilde{\pi}(x) \le -|X_0| + \sum_{x\in X_0} \pi(x);$$
and so
$$|X_0| \le \sum_{x\in X_0} \pi(x) - \sum_{x\in X_0} \tilde{\pi}(x). \tag{27}$$
But since $\tilde{\pi}$ and π agree precisely on the complement of X_0, we have
$$\tilde{\pi}(\mathbb{Z} \setminus X_0) = \pi(\mathbb{Z} \setminus X_0)$$
(an equality of sets), and therefore we have the equality of sets
$$\tilde{\pi}(X_0) = \pi(X_0).$$
Hence
$$\sum_{x\in X_0} \tilde{\pi}(x) = \sum_{x\in X_0} \pi(x),$$
and hence (27) implies that $|X_0| = 0$.

Hence X_0 is empty. Then (26) implies that
$$\forall x \in \mathbb{Z}, \qquad \pi(x) + \pi'(L_1^{\min} - x) = L_2^{\max}.$$
Hence it follows that all edges from $L_1^{\min}$ on the left join $L_2^{\max}$ on the right. Hence L_1 can only be connected to $L_1^{\min}$ and $L_2^{\max}$ if $L_1 = L_1^{\min}$. Hence $\Delta = 0$, which is precisely conclusion (1) of the lemma. (Which is a contradiction, since we assumed both conclusions (1) and (2) do not hold.) □

4.3 An Easy Consequence of the Fundamental Lemma

To use Lemma 4.1 to prove Theorem 4.2, we will need some easy facts; we will combine them with Lemma 4.1 in the statement of Lemma 4.4 below.

Definition 4.2. *Let π be a bijection $\mathbb{Z} \to \mathbb{Z}$ of bounded support. For any $r \in \mathbb{Z}$, we say that π is r-periodic if*

$$\forall x \in \mathbb{Z}, \quad \pi(x+r) = \pi(x) - r.$$

The period *of π is the smallest $r \geq 1$ such that π is r-periodic; if no such r exists, then we say that π is* unperiodic.

Lemma 4.2. *Let π, π' be perfect matchings of bounded support. Say that for some $L_1, L_2 \in \mathbb{Z}$ and all $x \in \mathbb{Z}$ we have*

$$\pi(x) + \pi'(L_1 - x) = L_2.$$

Then either π, π' are unperiodic, or they have the same period.

Proof. Say that π is r-periodic, i.e.,

$$\forall x \in \mathbb{Z}, \quad \pi(x+r) = \pi(x) - r. \tag{28}$$

Then the assumption that $\pi(x) = L_2 - \pi'(L_1 - x)$ for all x implies

$$\forall x \in \mathbb{Z}, \quad \pi'(L_1 - x + r) = L_2 - \pi(x-r) = L_2 - \pi(x) - r;$$

since $\pi(x) = L_2 - \pi'(L_1 - x)$, the RHS above becomes

$$L_2 - \big(L_2 - \pi'(L_1 - x)\big) - r = \pi'(L_1 - x) - r;$$

setting $y = L_1 - x$, it follows that

$$\forall y \in \mathbb{Z}, \quad \pi'(y+r) = \pi'(y) - r, \tag{29}$$

and therefore π' is r-periodic. Arguing backwards, we see that (29) implies (28); hence if π' is r-periodic, then so is π. Hence π and π' are r-periodic for the exact same set of $r \in \mathbb{Z}$. It follows that if either π or π' is periodic, then they have the same period, and if one of them is unperiodic then so is the other. □

Lemma 4.3. *Let π, π' be perfect matchings of bounded support. Say that π has period $r \geq 1$, and that for some $L_1, L_2 \in \mathbb{Z}$ we have*

$$\forall x, \quad \pi(x) + \pi'(L_1 - x) = L_2$$

Then we have that for all $L_1', L_2' \in \mathbb{Z}$ the following are equivalent:

1. $$\forall x, \quad \pi(x) + \pi'(L_1' - x) = L_2' \tag{30}$$

2. *For some $q \in \mathbb{Z}$ we have $L_1' = L_1 + qr$ and $L_2' = L_2 - qr$.*

Proof. (2) easily implies (1), since if π is r-periodic, then so is π', and then for any $q \in \mathbb{Z}$ we have

$$\pi(x) + \pi'(L_1' - x) = \pi(x) + \pi'(L_1 + qr - x) = \pi(x) + \pi'(L_1 - x) - qr = L_2 - qr = L_2'.$$

Hence it suffices to show (1) implies (2).

If (30) holds and $L_1' \neq L_1 + kr$ for some $k \in \mathbb{Z}$, then we have $L_1 + kr < L_1' < L_1 + kr + r$ for some $k \in \mathbb{Z}$; set $r' = L_1' - L_1 - kr$, so $1 \leq r' < r$. Then, using the r-periodicity of π', we have

$$\forall x, \quad \pi(L_1 - x) - L_2 = \pi'(L_1' - x) - L_2' = \big(\pi'(L_1' - x - kr) + kr\big) - L_2' = \pi'(L_1 + r' - x) - L_2' + kr.$$

Setting $y = L_1' - x$, we have for all y,

$$\pi'(y) - \pi'(r' + y) = C$$

for some constant C. Now we claim that $C = r'$, for applying the equation above to $y, y + r', \ldots, y + r'(a-1)$ for some $a \in \mathbb{N}$ and adding we get

$$\pi'(y) - \pi'(ar' + y) = Ca;$$

taking $a \to \infty$, and using the fact that π' is bounded, we get that $C = -r'$. But then

$$\forall y \in \mathbb{Z}, \quad \pi'(y) - \pi'(r' + y) = -r',$$

and so π' is r'-periodic for some $1 \leq r' < r$, and hence so is π, which is impossible. □

Combining Lemmas 4.2 and 4.3 we immediately get an important conclusion.

Lemma 4.4. *Say that in Lemma 4.1, π has period $r \geq 1$ and there exists $L_1, L_2 \in \mathbb{Z}$ such that*

$$\forall x \in \mathbb{Z}, \quad \pi(x) + \pi'(L_1 - x) = L_2.$$

Then for the graph G' of Lemma 4.1 (and of Theorem 4.3), for all $m \in \mathbb{Z}$:

1. *if $m = L_1 + qr$ for some $q \in \mathbb{Z}$, then all edges from m on the left are adjacent to and only to the single vertex $L_2 - qr$; and*
2. *otherwise, m on the left is connected to some left or right vertex of G' of arbitrarily small value (and hence lies in the v_0 component of G).*

Note that in the above lemma (which assumes the hypotheses of Lemma 4.1), if y is a vertex of G' on the right, then is connected to some vertex on the left (namely, if $b + b' = y$, then y on the right is connected to $\pi^{-1}(b) + (\pi')^{-1}(b')$ on the left, by (1c) of Proposition 4.1 applied symmetrically). Hence Lemma 4.4 identifies all connected components of G' and therefore of G in Theorem 4.3.

4.4 Proof of Theorem 4.1

Proof of Theorem 4.1. Say that $b^1(\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'}) > 0$. Then $\mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'} \simeq \mathcal{M}_{W'',\mathbf{d}+\mathbf{d}'}$ where $W'' = W \star_r W'$. Then the graph G' of the Folinsbee-Friedman algorithm has a connected component whose vertices are disjoint from $\mathbb{Z}_{\leq d_1+d_1'}$ on the left and $\mathbb{Z}_{\leq d_2+d_2'}$ on the right. Then this connected component has a left vertex of G', say $L_1 \in \mathbb{Z}_{\text{left}}$. Applying Lemma 4.1, we see that conclusion (1) must hold. Hence $\pi(x) + \pi'(L_1 - x) = L_2$ for all $x \in \mathbb{Z}$. It follows that for all $x, y \in \mathbb{Z}$,

$$W(x,y) = 1 \iff y = \pi(x) \iff y = L_2 - \pi'(L_1 - x) \iff \pi'(L_1 - x) = L_2 - y$$

$$\iff W'(L_1 - x, L_2 - y) = 1.$$

Hence $W(\mathbf{d}) = W'(\mathbf{L} - \mathbf{d})$ for all $\mathbf{d} \in \mathbb{Z}^2$ and hence $W' = W^*_{\mathbf{L}}$. □

4.5 Proof of Theorem 4.2

Proof of Theorem 4.2. Let us prove the statement for a general $\mathbf{K}' = \mathbf{d} + \mathbf{d}'$. Let $W'' = W \star_r W'$. Since $W'(\mathbf{L} - \mathbf{d}) = W(\mathbf{d})$ for all $\mathbf{d} \in \mathbb{Z}^2$, the proof in the last subsection shows that $\pi(x) + \pi'(L_1 - x) = L_2$ for all x. It follows from Lemma 4.4 that the connected components of G' in Theorem 4.3 (with W'' replacing W) are either (1) connected to some left $\mathbb{Z}_{\leq K_1'}$ vertex or some right $\mathbb{Z}_{\leq K_2'}$ vertex, or (2) are a single connected component consisting of a left vertex $L_1 + qr > K_1'$ and a right vertex $L_2 - qr > K_2'$ (for some $q \in \mathbb{Z}$). Combining this with Theorem 4.3 proves the claim regarding the basis of H^1 for

$$\mathcal{M}_{W'',\mathbf{d}+\mathbf{d}'} \simeq \mathcal{M}_{W,\mathbf{d}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{d}'}$$

For the special case of $\omega_{W,\mathbf{K}} = \mathcal{M}_{W,\mathbf{0}} \oplus_{\mathcal{O}_r} \mathcal{M}_{W',\mathbf{K}}$, where $\mathbf{K}' = \mathbf{K}$, we know that the only $q \in \mathbb{Z}$ with $L_1 + qr > K_1 = L_1 - 1$ and $L_2 - qr > K_2 = L_2 - 1$ is $q = 0$. □

4.6 How Does ω_W Depend on W?

We have noted that if W is an r-periodic perfect matching, then $\omega_{W,\mathbf{K}}$ depends — up to isomorphism — only on W. It is interesting to ask how ω_W depends on W. The remarks below show that when $\tilde{W}$ is a translation of W or of $W^*_{\mathbf{0}}$, then $\omega_W \simeq \omega_{\tilde{W}}$; otherwise we don't know if $\omega_W \simeq \omega_{\tilde{W}}$, although below we argue that if $\omega_W \simeq \omega_{\tilde{W}}$ then this isomorphism would have to be somewhat "exotic."

Since

$$\omega_{W,\mathbf{K}} = \mathcal{M}_{W,\mathbf{0}} \otimes_{\mathcal{O}_r} \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}},$$

let us restrict to $\mathbf{L} = \mathbf{0}$ and $\mathbf{K} = -1$. Then

$$\omega_{W,-\mathbf{1}} \simeq \mathcal{M}_{W \star_r W^*_{\mathbf{0}},-1}.$$

But if $W'' = W \star_r W^*_{\mathbf{0}}$, then

$$W''(\mathbf{s}) = \sum_{a_1=0}^{r-1} \sum_{a_2 \in \mathbb{Z}} W(\mathbf{a}) W^*_{\mathbf{0}}(\mathbf{s} - \mathbf{a}) = \sum_{a_1=0}^{r-1} \sum_{a_2 \in \mathbb{Z}} W(\mathbf{a}) W(\mathbf{a} - \mathbf{s})$$

So if A is the support of W in $\mathbb{Z}^2$, then W'' is supported on the set of $\mathbf{s}$ such that some $\mathbf{a} \in A$ has an $\mathbf{a}' \in A$ such that $\mathbf{a}' = \mathbf{a} - \mathbf{s}$, i.e., $\mathbf{s} = \mathbf{a} - \mathbf{a}'$. Hence W'' is determined by the multiset of differences $A - A$, where we consider $A - A$ as a multiset based on the values of W'' there (they may be greater than 1). For example, if $\tilde{W}$ is a translation of W, or a translation of $W_{\mathbf{0}}^*$, then

$$W'' = W \star_r W_{\mathbf{0}}^* \quad \text{and} \quad \tilde{W}'' = \tilde{W} \star_r \tilde{W}_{\mathbf{0}}^*$$

are equal, since the respective supports $A, \tilde{A}$ of $W, \tilde{W}$ satisfy $A - A = \tilde{A} - \tilde{A}$ as multisets.

Note that in the last paragraph, $\omega_W \simeq \omega_{\tilde{W}}$ if the supports of $A, \tilde{A}$ satisfy $A - A = \tilde{A} - \tilde{A}$ as multisets. We remark that it is well known that there are subsets $I, \tilde{I} \in \mathbb{Z}$ such that $I - I = \tilde{I} - \tilde{I}$ as multisets and $\tilde{I}$ is not a translate of I or of $-I$ (e.g., [5, 26]). However, we don't know if this is true, modulo $(r, -r)$, of sets $A, \tilde{A} \subset \mathbb{Z}^2$ where $W, \tilde{W}$ corresponding to $A, \tilde{A}$ are perfect matchings.

Note that if we can find such $A, \tilde{A}$ such that $A - A = \tilde{A} - \tilde{A}$ as sets, not multisets (the above multiplicity convention), then we still get $\omega_W \simeq \omega_{\tilde{W}}$, although via a somewhat "exotic" isomorphism: to see this, first note that ω_W decomposes as the sum of two $\mathcal{O}_r$ modules, one that is nonzero only on B_3, whose value there is the kernel, $K = K(\omega_W)$ of

$$\omega_W(B_3) \to \omega_W(A_1) \oplus \omega_W(A_2).$$

(this consists of $r(r-1)$ copies of $\mathcal{O}_r(B_3) = k[v, 1/v]$ as an $k[v, 1/v]$-module). Hence if $A - A = \tilde{A} - \tilde{A}$ simply as sets, then one could take an identity morphism from $\omega_W \to \omega_{\tilde{W}}$ on A_1, A_2, B_1, B_2, and take the identity on the B_3 values corresponding to the "first" nonzero value of $W'' = W \star_r W_{\mathbf{0}}^*$ and $\tilde{W} \star_r \tilde{W}_{\mathbf{0}}^*$. One can then take an arbitrary $\mathcal{O}_r$ module map from $K(\omega_W)$ to $K(\omega_{\tilde{W}})$. However, this is a rather "exotic" isomorphism. Moreover, we don't know any such examples of r-periodic perfect matchings $W, \tilde{W}$ whose supports, $A, \tilde{A}$ satisfy $A - A$ and $\tilde{A} - \tilde{A}$ agree as sets and not as multisets.

We don't know if more "exotic" isomorphisms $\omega_W \to \omega_{\tilde{W}}$ can exist if $A - A \neq \tilde{A} - \tilde{A}$ as sets.

5. Duality

Again, in this section k will be a fixed field, and we will often suppress k, writing, e.g., $\mathcal{O}_r$ instead of $\mathcal{O}_{r,k}$.

Let W be an r-periodic weight, and let $\mathcal{F}$ be an $\mathcal{O}_r$-module. In this section we will describe a natural (in $\mathcal{F}$) pairing

$$H^1_{\mathcal{O}}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to k \tag{31}$$

whenever $\mathbf{K} = \mathbf{L} + \mathbf{1}$ are elements of $\mathbb{Z}^2$. We will show that when $\mathcal{F}$ is a line bundle $\mathcal{L}_{r,\mathbf{d}}$, then this gives a perfect pairing of finite dimensional vector spaces. Hence this gives an isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \xrightarrow{\simeq} H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F})' \tag{32}$$

(where $'$ denotes the dual space as a k-vector space). It will easily follow that we get an isomorphism

$$H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \xrightarrow{\simeq} H^1(\mathcal{M}_{W,\mathbf{d}})',$$

which is therefore a "duality theorem," according to the discussion around (11).

In Section 6 we will strengthen this result in two ways: (1) we will give a second "duality" formula that $\mathcal{F} = \mathcal{L}_{r,\mathbf{d}}$ satisfies, and (2) we will show that there are more examples of $\mathcal{F}$ for which both duality statements hold, which include certain "skyscraper diagrams" and any *coherent* $\mathcal{O}$-module. Section 6 requires more background in homological algebra; this section will proceed more "naively."

We begin by describing the ingredients we need to produce the pairing (34)

Remark 5.1. *The k-vector spaces*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}}}) \quad \textit{and} \quad \mathrm{Hom}_{\underline{k}}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}}})$$

are very different; in particular, we will later explain that the second space is infinite-dimensional for $\mathcal{F} = \mathcal{L}_{r,\mathbf{d}}$. Hence it is essential for us to work with Hom sets of $\mathcal{O}$-modules in (31) *in our duality theorem.*

5.1 A Yoneda Pairing

In this subsection $\mathcal{O}$ is an arbitrary diagram of k-algebras; the only reason we insist on this is that we have only defined $H^i(\mathcal{F})$ when $\mathcal{F}$ is a k-diagram (of vector spaces). However, had we defined $H^i(\mathcal{F})$ for any diagram of abelian groups, $\mathcal{F}$ (in the evident fashion), then the discussion below would be valid for any diagram of rings, $\mathcal{O}$.

There is a very simple way to define a map:

$$H^i(\mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G}) \to H^1(\mathcal{G});$$

we will call this map a *Yoneda pairing*. In Section 6 we will explain that the map we describe is really an example of the well-known *Yoneda pairing*. For now we just describe this map from scratch.

So let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of diagrams of k-vector spaces. Then ϕ induces a map

$$\mathcal{F}(A_1) \oplus \mathcal{F}(A_2) \xrightarrow{\phi(A_1)\oplus\phi(A_2)} \mathcal{G}(A_1) \oplus \mathcal{G}(A_2).$$

Since ϕ is a morphism, one easily checks that this map yields a map

$$H^1(\mathcal{F}) = \big(\mathcal{F}(A_1) \oplus \mathcal{F}(A_2)\big)/\big(\mathrm{Image}(\mathcal{F}(\rho_{\mathrm{tot}}))\big)$$

to

$$H^1(\mathcal{G}) = \big(\mathcal{G}(A_1) \oplus \mathcal{G}(A_2)\big)/\big(\mathrm{Image}(\mathcal{G}(\rho_{\mathrm{tot}}))\big).$$

This gives a pairing

$$H^1(\mathcal{F}) \times \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to H^1(\mathcal{G}).$$

To determine this pairing it suffices to determine how each $\alpha \in H^1(\mathcal{F})$ and $\beta \in \mathrm{Hom}(\mathcal{F},\mathcal{G})$ are mapped to in $H^1(G)$, which is determined by the two morphisms

$$\beta(A_1)\colon \mathcal{F}(A_1) \to \mathcal{G}(A_1), \quad \beta(A_2)\colon \mathcal{F}(A_2) \to \mathcal{G}(A_2),$$

plus the equivalence classes of $\mathcal{G}(A_1) \oplus \mathcal{G}(A_2)$ modulo the image of $\mathcal{G}(\partial)$.

Of course, in case $\mathcal{F},\mathcal{G}$ are $\mathcal{O}$-modules where each value of $\mathcal{O}$ is a k-algebra, then

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G}) \subset \mathrm{Hom}_k(\mathcal{F},\mathcal{G}),$$

and the above Yoneda pairing is also a pairing

$$H^1(\mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G}) \to H^1(\mathcal{G}).$$

5.2 Tensoring Hom

Let $\mathcal{O}$ be any diagram of rings, and let $X = \{A_1, A_2, B_1, B_2, B_3\}$. For any $\mathcal{O}$-modules $\mathcal{A},\mathcal{B},\mathcal{C}$ there is a natural map

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{A},\mathcal{B}) \xrightarrow{\mathrm{id}_{\mathcal{C}}\otimes\cdot} \mathrm{Hom}_{\mathcal{O}}(\mathcal{C}\otimes\mathcal{A}, \mathcal{C}\otimes\mathcal{B}) \tag{33}$$

where $\mathrm{id}_{\mathcal{C}}$ is the identity map on $\mathcal{C}$: indeed, an element $\phi \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{A},\mathcal{B})$ is just a family of maps $\phi = \{\phi_P\}_{P\in X}$, where

$$\phi_P\colon \mathcal{A}(P) \to \mathcal{B}(P)$$

is a map of $\mathcal{O}(P)$-modules, and the $\{\phi_P\}$ are compatible with the restriction maps of $\mathcal{A}$ and $\mathcal{B}$. In this case there is a map called $\mathrm{id}_{\mathcal{C}} \otimes \phi$ which is a morphism $\mathcal{C}\otimes\mathcal{A} \to \mathcal{C}\otimes\mathcal{B}$ whose map at $P \in X$ is the unique morphism of $\mathcal{O}(P)$-modules taking (m,a) to $(m,\phi(a))$ (we easily check that there is a unique such morphism).

5.3 The Duality Map

In this subsection we describe the duality map (32) based on the pairing (31).

The map (33) gives a a natural map

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}}}) \to \mathrm{Hom}\big(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}\big) = \mathrm{Hom}\big(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega_W\big).$$

So fix an isomorphism

$$H^1(\omega) \simeq k.$$

We therefore get a map

$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega) \to H^1(\omega) \to k. \tag{34}$$

This is our desired map (31).

Definition 5.1. *We say that an $\mathcal{O}_r$-module $\mathcal{F}$* satisfies duality *if the map*

$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to k$$

induced from (34) *is a perfect pairing.*

5.4 The Main Computation

In this subsection, we show that (34) is a perfect pairing when $\mathcal{F} = \mathcal{L}_{r,\mathbf{d}}$. This is a straightforward calculation, but one has to carefully keep track of things.

Let us first make some preliminary remarks and lemmas, which hold for any r-periodic perfect matching, W, and choice of $\mathbf{K} \in \mathbb{Z}^2$ and $\mathbf{L} = \mathbf{K} + \mathbf{1}$.

First, recall from Theorem 4.2 that for $\omega = \omega_{W,\mathbf{K}}$, $H^1(\omega)$ is by definition

$$\omega(A)/\mathrm{Image}(\omega(\partial))$$

and is the one-dimensional k-vector space generated by $(x_1^{L_1}, 0)$, which in $H^1(\omega)$ is equivalent to the vector $-(0, x_2^{L_2})$.

Second, we may identify

$$\mathrm{Hom}(\mathcal{L}_{r;\mathbf{d}}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \simeq \mathrm{Hom}(\mathcal{O}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}).$$

Third,

$$\mathrm{Hom}(\mathcal{O}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})$$

may be described as a direct sum by the following general proposition.

Proposition 5.1. *Let W' be any r-periodic perfect matching, and π' is the associated permutation of W'. For any $\mathbf{d} \in \mathbb{Z}^2$, let*

$$I_1 = \{i_i \mid i_1 \geq d_1,\ \pi'(i_1) \geq d_2\}$$

(which is finite since π' is bounded). Then for each $\phi \in \mathrm{Hom}(\mathcal{O}, \mathcal{M}_{W',\mathbf{d}})$, the map

$$\phi(A_1)\colon \mathcal{O}(A_1) \to \mathcal{M}_{W',\mathbf{d}}(A_1) \tag{35}$$

— seeing as $\mathcal{O}(A_1) = \mathcal{M}_{W',\mathbf{d}}(A_1) = k[x_1^{\pm}]$ — must take $1 \in \mathcal{O}(A_1)$ to $p(x_1) \in \mathcal{M}_{W',\mathbf{d}}(A_1)$ where p is the Laurent polynomial; then p is necessarily of the form

$$p(x_1) = \sum_{i_1 \in I_1} c_{i_1} x_1^{i_1}. \tag{36}$$

Conversely, for any p as in (36)*, the map from $k[x^{\pm}]$ to itself (as a module over itself) taking 1 to $p(x_1)$ determines a (unique) map $\phi(A_1)$ in* (35)*, which extends to a unique morphism ϕ as in $\phi \in \mathrm{Hom}(\mathcal{O}, \mathcal{M}_{W',\mathbf{d}})$. This sets up a bijection*

$$\mathrm{Hom}(\mathcal{O}, \mathcal{M}_{W',\mathbf{d}}) \to \left\{ p \;\middle|\; p(x_1) = \sum_{i_1 \in I_1} c_{i_1} x_1^{i_1},\ c_{i_i} \in k \right\} \simeq k^{I_1}.$$

Proof. Let $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$. First note that if $m \in \mathcal{M}_{W',\mathbf{d}}(P)$, then there is a unique morphism of $\mathcal{O}(P)$-modules $\mathcal{O}(P) \to \mathcal{M}_{W',\mathbf{d}}(P)$ that maps 1 to m. Hence a $\phi \in \mathrm{Hom}(\mathcal{O}, \mathcal{M}_{W',\mathbf{d}})$ gives for each $P \in X$ an element $m_P \in \mathcal{M}_{W',\mathbf{d}}(P)$ such that $\phi(P)(1) = m_P$. Hence such a ϕ is equivalent to a family $\{m_P\}_{P \in X}$ such that the maps $1 \mapsto m_P$ are compatible with the restriction maps; i.e., give a commutative diagram as in Figure 3 with $\mathcal{F} = \mathcal{O}$ and $\mathcal{G} = \mathcal{M}_{W',\mathbf{d}}(P)$.

So consider a $\phi \in \mathrm{Hom}(\mathcal{O}, \mathcal{M}_{W',\mathbf{d}})$; then $\phi(A_1)$ is a map of $k[x_1^{\pm}]$ to itself as a $k[x_1^{\pm}]$; let $p(x) = \phi(A_1)1 \in k[x_1^{\pm}]$, and therefore

$$p(x) = \sum_{i \in \mathbb{Z}} c_i x_1^i.$$

Similarly $\phi(B_1)$ must take 1 to $q(y_1) \in k[y_1]$. Since ϕ is compatible with the restriction map $\rho_{1,1}$, we have $\phi(B_1)$ must take 1 to $q(y_1)$ and the commutativity of:

$$\begin{array}{ccc}
\mathcal{O}(B_1) = k[y_1] & \xrightarrow{\quad 1 \mapsto q(y_1) \quad} & \mathcal{M}_{W',\mathbf{d}}(B_1) = k[y_1] \\
{\scriptstyle 1 \mapsto 1}\searrow & & \searrow{\scriptstyle 1 \mapsto x_1^{d_1}} \\
\mathcal{O}(A_1) = k[x_1^{\pm}] & \xrightarrow{\quad 1 \mapsto p(x_1) \quad} & \mathcal{M}_{W',\mathbf{d}}(A_1) = k[x_1^{\pm}]
\end{array}$$

(see the top of Figure 3) we see that the left arrow followed by the bottom arrow takes 1 to $p(x_1)$, and the top takes 1 to $q(y_1)$, which the right arrow takes to $x_1^{d_1} q(1/x_1)$. Hence

$$x_1^{d_1} q(1/x_1) = p(x_1).$$

Hence

$$q(y_1) = y_1^{d_1} p(1/y_1) = y_1^{d_1} \sum_i c_i y_1^{-i}. \tag{37}$$

Since $q(y_1) \in k[y_1]$, we must have $c_i \neq 0$ implies that $d_1 - i \geq 0$, or, equivalently, $i \leq d_1$. Similarly chasing through the restrictions $\rho_{3,1}$ with the diagram:

$$\begin{array}{ccc} \mathcal{O}(A_1) = k[x_1^{\pm}] & \xrightarrow{1 \mapsto p(x_1) = \sum_i c_i x_1^i} & \mathcal{M}_{W',\mathbf{d}}(A_1) = k[x_1^{\pm}] \\ 1 \mapsto 1 \nearrow & & \nearrow (x_1^i, x_2^{\pi'(i)}) \mapsto x_1^i \\ \mathcal{O}(B_3) = k[v^{\pm}] & \xrightarrow{1 \mapsto \sum_i c_i (x_1^i, x_2^{\pi'(i)})} & \mathcal{M}_{W',\mathbf{d}}(B_3) = k^{\oplus W} \simeq k[v^{\pm}]^{\oplus r} \end{array}$$

we see that $\phi(B_3)$ must be the map

$$1 \mapsto \sum_i c_i \big(x_1^i, x_2^{\pi'(i)}\big),$$

and chasing through $\rho_{3,2}$ we see that $\phi(A_2)$ must be the map

$$1 \mapsto \sum_i c_i x_2^{\pi'(i)},$$

and then chasing through $\rho_{2,2}$ as we did $\rho_{1,1}$, we see that $\phi(B_2)$ must be the map $1 \mapsto \tilde{r}(y_2)$ where (compare with (37))

$$r(y_2) = y_2^{d_2} \sum_i c_i y_2^{-\pi'(i)}.$$

Since $r(y_2)$ must lie in $k[y_2]$ we have $c_i \neq 0$ implies $\pi'(i) \leq d_2$.

Moreover, as long as $c_i \neq 0$ implies $i \leq d_1$ and $\pi'(i) \leq d_2$, then ϕ, which is determined by $\phi(A_1)$, is a valid morphism $\mathcal{O} \to \mathcal{M}_{W',\mathbf{d}}$, seeing as each of the candidates for $\phi(P)$ for $P \in X$ is indeed a valid morphism $\mathcal{O}(P) \to \mathcal{M}_{W',\mathbf{d}}(P)$ that comprise ϕ. □

We use Proposition 5.1 as follows.

Corollary 5.1. *Let W be an r-periodic perfect matching, and $\mathbf{L}, \mathbf{K}, \mathbf{d} \in \mathbb{Z}^2$ satisfy $\mathbf{L} = \mathbf{K} + \mathbf{1}$. Then a basis for $\mathrm{Hom}\big(\mathcal{O}_r, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}\big)$ is given by the unique $\phi \in \mathrm{Hom}\big(\mathcal{O}_r, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}\big)$ such that $\phi(A_1)$ is multiplication by $x_1^{i_1}$ where i_1 varies in the set*

$$I_1 = \{i_1 \mid \text{for the unique } \mathbf{i} = (i_1, i_2) \text{ with } W(\mathbf{L} - \mathbf{i}) = 1 \text{ we have } \mathbf{i} \leq \mathbf{K} - \mathbf{d}\}.$$

Hence, (substituting $\mathbf{b} = \mathbf{L} - \mathbf{i}$ and) setting

$$b_1 = \{b_1 \mid \text{for the unique } \mathbf{b} = (b_1, b_2) \text{ with } W(\mathbf{b}) = 1 \text{ we have } \mathbf{d} + \mathbf{1} \leq \mathbf{b}\}, \tag{38}$$

$\mathrm{Hom}\big(\mathcal{O}_r, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}\big)$ has a basis consisting of

$$\{\phi \in \mathrm{Hom}\big(\mathcal{O}_r, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}\big) \mid \phi(A_1) \text{ is multiplication by } x_1^{L_1 - b_1}\}.$$

Remark 5.2. *In Corollary 5.1, if $\phi \in \mathrm{Hom}(\mathcal{O}_r, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})$ is given by $\phi(A_1)$ maps $1 \mapsto p(x_1)$, then the corresponding $\phi' \in \mathrm{Hom}(\mathcal{L}_{r,\mathbf{d}}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}})$ also has $\phi'(A_1)$ maps $1 \mapsto p(x_1)$, since the isomorphism*

$$\mathrm{Hom}(\mathcal{O}_r, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \to \mathrm{Hom}(\mathcal{L}_{r,\mathbf{d}}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}})$$

is just tensoring with $\mathcal{L}_{r,\mathbf{d}}$ on both sides, which doesn't affect A_1 since $\mathcal{L}_{r,\mathbf{d}}(A_1) = \mathcal{O}(A_1)$.

Theorem 5.1. *Let k be a field and $W \colon \mathbb{Z}^2 \to \{0,1\}$ be a perfect matching with associated bijection π. Let $\mathbf{L} \in \mathbb{Z}^2$ set $\mathbf{K} = \mathbf{L} - \mathbf{1}$. For an arbitrary $\mathcal{O}_r$-module, $\mathcal{F}$, consider the pairing of k-diagrams*

$$H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to H^1(\omega_{W,\mathbf{K}}) \xrightarrow{\simeq} k \tag{39}$$

given by composing the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) &\to \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \\ &= \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}, \omega_W) \end{aligned}$$

and the Yoneda map

$$H^1(\mathcal{F}\otimes\mathcal{M}_{W,\mathbf{0}})\times \mathrm{Hom}_{\mathcal{O}_r}(\mathcal{M}_{W,\mathbf{0}}\otimes\mathcal{F},\omega_W)\to H^1(\omega_W)$$

and the isomorphism

$$H^1(\omega_W)\to k$$

which takes

$$(x_1^{L_1},0)\in\omega_W(A)$$

to $1\in k$ *(and therefore takes* $(0,x_2^{L_2})$ *to* -1*). Consider the particular case where* $\mathcal{F}$ *is the line bundle* $\mathcal{L}_{\mathbf{d}}=\mathcal{L}_{r,\mathbf{d}}$ *for some* $\mathbf{d}\in\mathbb{Z}^2$*. Then*

1. *If* $(x_1^a,0)\in\mathcal{M}_{W,\mathbf{d}}(A)$*, and if* $\phi\in\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{O}_r,\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})$ *has* $\phi(A_1)=x_1^b$ *for some* $b\in\mathbb{Z}$*, then under the pairing* (39) *and the natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{O}_r,\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})\simeq\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{L}_{r,\mathbf{d}},\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}),$$

   $((x_1^a,0),\phi)$ *is taken to* $(x_1^{a+b},0)\in\omega_{W,\mathbf{K}}(A)$*.*

2. *Hence* (x_1^a,ϕ) *above is taken to a non-zero element of* $H^1(\omega_{W,\mathbf{K}})$ *iff* $a+b=L_1$*.*

3. *The pairing* (39) *for* $\mathcal{F}=\mathcal{L}_{k,\mathbf{d}}$ *is a perfect pairing. Hence* $\mathcal{L}_{k,\mathbf{d}}$ *satisfies duality (Definition 5.1.)*

Proof. To prove (1), consider how a ϕ fits into the pairing (39): according to Remark 5.2, we identify ϕ with the element

$$\phi'\in\mathrm{Hom}(\mathcal{L}_{r,\mathbf{d}},\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}})\simeq\mathrm{Hom}(\mathcal{O},\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}),$$

where $\phi'(A_1)$ is multiplication by x_1^b in

$$\mathrm{Hom}_{k[x_1^{\pm}]}(k[x_1^{\pm}],k[x_1^{\pm}]).$$

Hence

$$(\mathrm{Id}_{\mathcal{M}_{W,\mathbf{0}}}\otimes\phi')(A_1)\colon\ \mathrm{Hom}_{k[x_1^{\pm}]}(k[x_1^{\pm}]\otimes k[x_1^{\pm}],k[x_1^{\pm}]\otimes k[x_1^{\pm}])\simeq\mathrm{Hom}_{k[x_1^{\pm}]}(k[x_1^{\pm}],k[x_1^{\pm}])$$

is again multiplication by x_1^b, and therefore takes $x_1^a\in k[x_1^{\pm}]=\mathcal{M}_{W,d}(A_1)$ to $x_1^{a+b}\in\omega_{W,\mathbf{K}}(A_1)$. Hence $(x_1^a,0)\in\mathcal{M}_{W,\mathbf{d}}(A)$ is taken to $(x_1^{a+b},0)\in\omega_W(A)$.

(2) follows from Theorem 4.2, since the only $(x_1^c,0)$ that is nonzero in $H^1(\omega_W)$ is $(x_1^{L_1},0)$.

To prove (3) let us first show that $H^1(\mathcal{M}_{W,\mathbf{d}})$ the following basis:

$$\{(x_1^{a_1},0)\ |\text{for the unique }\mathbf{a}=(a_1,a_2)\text{ with }W(\mathbf{a})=1\text{ we have }\mathbf{d}+\mathbf{1}\le\mathbf{a}\};\tag{40}$$

the reader can either prove this from scratch, or use the following material from [12]: first, we have

$$\mathcal{M}_{W,\mathbf{d}}\simeq\mathcal{I}_{\mathbf{d}}^{\oplus W}=\bigoplus_{W(\mathbf{a})=1}\mathcal{I}_{\mathbf{d}\ge\mathbf{a}}\tag{41}$$

(see Definitions 6.1 and 6.2 for this notation, and Proposition 6.1 of [12] for the isomorphism). We then have

$$H^1(\mathcal{M}_{W,\mathbf{d}})\simeq\bigoplus_{W(\mathbf{a})=1}H^1(\mathcal{I}_{\mathbf{d}\ge\mathbf{a}})$$

(see end Section 4.1 of [12]). Now according to Example 5.2 of [12], $H^1(\mathcal{I}_{\mathbf{d}\ge\mathbf{a}})$ is nonzero iff $\mathcal{I}_{\mathbf{d}\ge\mathbf{a}}$ is a copy of $\underline{k}_{/B_1,B_2}$, which holds iff $\mathbf{d}+\mathbf{1}\le\mathbf{a}$; moreover, the definition of $\mathcal{I}_{\mathbf{d}\ge\mathbf{a}}$ shows that if $\mathbf{d}+\mathbf{1}\le\mathbf{a}$, then $H^1(\mathcal{I}_{\mathbf{d}\ge\mathbf{a}})$ is generated by $(1,0)$ in $\mathcal{I}_{\mathbf{d}\ge\mathbf{a}}(A_1)$, which corresponds to $(x_1^{a_1},0)$ in $\mathcal{M}_{W,\mathbf{d}}$ under the isomorphism of (41). Hence (40) is a basis for $H^1(\mathcal{M}_{W,\mathbf{d}})$.

Now, compare (40) with (38): the set of $\mathbf{a}$ with $W(\mathbf{a})=1$ and $\mathbf{d}+\mathbf{1}\ge\mathbf{a}$ is a finite set (since $W(\mathbf{a})=0$ if $\deg(\mathbf{a})$ is sufficiently large), and is the same condition that $\mathbf{b}$ in (38) satisfies. By (2) above, $(x^{a_1+L_1-b_1},0)\in H^1(\omega_{W,\mathbf{K}})$ is nonzero iff $a_1=b_1$; hence (39) with $\mathcal{F}=\mathcal{L}_{r,\mathbf{d}}$ is a perfect pairing of finite dimensional k-vector spaces. □

5.5 More Diagrams $\mathcal{F}$ Satisfying Duality

At this point we can conclude that other diagrams satisfy (39). We will only outline the ideas, because in the next section we will prove much stronger results.

Say that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_r$-modules, and both $\mathcal{F}_1, \mathcal{F}_2$ satisfy (39). Then we claim the same is true of $\mathcal{F}_3$. To see this, one has to verify that our definition of cohomology groups and the Yoneda pairing guarantees that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}_3, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) & \longrightarrow & \mathrm{Hom}(\mathcal{F}_2, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) & \longrightarrow & \mathrm{Hom}(\mathcal{F}_1, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}_3)^* & \longrightarrow & H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}_2)^* & \longrightarrow & H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}_1)^* \end{array} \tag{42}$$

If so, then the five-lemma shows that $\mathcal{F}_3$ also satisfies (39). Let us give an important example.

Definition 5.2. *For $i = 1, 2$, the* small skyscraper at B_i*, denoted $\mathcal{S}_i$, is the $\mathcal{O}_r$-module whose values are* 0 *everywhere except at B_i, where its value is $k[y_1]/y_1k[y_1]$.*

Hence $\mathcal{S}_1, \mathcal{S}_2$ are $\mathcal{O}_r$-modules (whose definition is independent of r), and as k-diagrams they are the diagrams whose values are 0 everywhere except at B_i, where its value is k. Note that $\mathcal{S}_i$ is an example of a "skyscraper diagram" in the sense of Subsubsection A.2.2.

Example 5.1. *For each $r \in \mathbb{N}$ and $\mathbf{d} \in \mathbb{Z}^2$, there is an exact sequence of $\mathcal{O}_r$-modules*

$$0 \to \mathcal{L}_{r,\mathbf{d}} \to \mathcal{L}_{r,\mathbf{d}+\mathbf{e}_1} \to \mathcal{S}_1 \to 0.$$

where the map $\mathcal{L}_{r,\mathbf{d}} \to \mathcal{L}_{r,\mathbf{d}+\mathbf{e}_1}$ is the inclusion, and $\mathcal{S}_1$ is therefore the quotient. By the five-lemma, $\mathcal{S}_1$ also satisfies duality. There is an analogous short exact sequence with $\mathcal{S}_2$, and similarly $\mathcal{S}_2$ satisfies duality.

We remark that one can equally well see that the $\mathcal{S}_j$ satisfy duality by verifying that for $j \in [2]$, both

$$\mathrm{Hom}(\mathcal{S}_j, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}), \quad H^1(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{S}_j)$$

vanish.

It follows that any diagram that can be written as the cokernel of a morphism $\mathcal{F}_1 \to \mathcal{F}_2$ where $\mathcal{F}_1$ are sums of $\mathcal{L}_{r,\mathbf{d}}$ and $\mathcal{S}_j$ above also satisfies (39).

We remark that the commutativity of the diagram (42) follows from results in the next section, which appeal to results about the Yoneda pairing using the powerful framework of derived categories.

6. Strong Duality

Again, in this section k will be a fixed field, and we will often suppress k, writing, e.g., $\mathcal{O}_r$ instead of $\mathcal{O}_{r,k}$.

In Theorem 5.1 we showed that pairing (31) is a perfect pairing for $\mathcal{F} = \mathcal{L}_{r,\mathbf{d}}$, and we therefore get an isomorphism (32) for this value of $\mathcal{F}$. The point of this section is to give a stronger duality theorem: namely, we will build a pairing

$$H^i(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F}) \times \mathrm{Ext}^{1-i}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to k$$

for any $\mathcal{F}$ and $i = 0, 1$, which therefore gives a morphism

$$\mathrm{Ext}^{1-i}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}) \to H^i(\mathcal{M}_{W,\mathbf{0}} \otimes \mathcal{F})' \tag{43}$$

for $i = 0, 1$. We will say that $\mathcal{F}$ satisfies *strong duality* if this map is an isomorphism for $i = 0, 1$; for $i = 1$, (43) turns out to be just the map (32), and so strong duality is, indeed, a stronger property than duality (Definition 5.1).

In this section we give examples of $\mathcal{O}_r$-modules that satisfy strong duality, namely (1) the skyscraper diagrams $\mathcal{S}_j$ for $j = 1, 2$ (Definition 5.2), and (2) $\mathcal{L}_{r,\mathbf{d}}$ for any $\mathbf{d} \in \mathbb{Z}^2$. This is proven in Theorem 6.3, which is based on Theorem 6.2.

It is also a standard type of result that if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence, and any two of the $\mathcal{F}_i$ satisfy strong duality, then so does the third. This is proven in Theorem 6.1.

To prove these theorems we wll need to first exploit the *flatness* of $\mathcal{M}_{W,\mathbf{d}}$; we explain flatness in Subsection 6.1, and its application to Ext groups in Subsection 6.2. We also need a more general Yoneda pairing than in Section 5; this we develop in Subsection 6.3. The remaining subsections of this section prove Theorems 6.1, 6.2, and 6.3.

Parts of this section will also refer to facts regarding homological algebra and injective and projective resolutions of k-diagrams developed in Appendix A.

Throughout this section we will work either with $\mathcal{O}_r$-modules or $\mathcal{O}$-modules where $\mathcal{O}$ is a more general structure (either a diagram of rings or a diagram of k-algebras). For brevity we omit $\mathcal{O}_r$ or $\mathcal{O}$ in subscripts when confusion is unlikely; for example, we often write $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ instead of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ or $\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F},\mathcal{G})$, and similarly for $\mathrm{Ext}^i(\mathcal{F},\mathcal{G})$.

6.1 Flatness

Flatness is a standard concept in homological algebra: if R is a ring and M an R-module, we say M is *flat* if for every exact sequence of R-modules $0 \to A \to B \to C \to 0$ we have that

$$0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

(obtained by tensoring with id_M) is also exact. It is a standard result that a direct sum of copies of any ring, R, is (a projective R-module and) a flat R-module; see [20] Proposition III.7.4 (and its proof), [25] Proposition 3.46, or [28] beginning of Section 3.2.

Definition 6.1. *Let $\mathcal{O}$ be a diagram of k-algebras. We say that an $\mathcal{O}$-module, $\mathcal{M}$, is* flat *if for every exact sequence of $\mathcal{O}$-modules*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

we have that

$$0 \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}_1 \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}_2 \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}_3 \to 0$$

is exact.

Here, of course, the maps $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}_i \to \mathcal{M} \otimes_{\mathcal{O}} \mathcal{F}_{i+1}$ are the maps taking (m, f) to (m, f') where f maps to f' in the arrow $\mathcal{F}_i \to \mathcal{F}_{i+1}$.

Lemma 6.1. *Let $\mathcal{O}$ be a diagram of k-algebras. Let $\mathcal{M}$ be an $\mathcal{O}$-module such that for any $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$, $\mathcal{M}(P)$ is isomorphic to a direct sum of copies of $\mathcal{O}(P)$. Then $\mathcal{M}(P)$ is flat.*

Proof. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence. Then for each $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$ we have an exact sequence

$$0 \to \mathcal{F}_1(P) \to \mathcal{F}_2(P) \to \mathcal{F}_3(P) \to 0.$$

Since $\mathcal{M}(P)$ is isomorphic to a direct sum of copies of $\mathcal{O}(P)$, $\mathcal{M}(P)$ is a (projective and) flat $\mathcal{O}(P)$-module. Therefore

$$0 \to \mathcal{M}(P) \otimes_{\mathcal{O}} \mathcal{F}_1(P) \to \mathcal{M}(P) \otimes_{\mathcal{O}} \mathcal{F}_2(P) \to \mathcal{M}(P) \otimes_{\mathcal{O}} \mathcal{F}_3(P) \to 0.$$

Since $P \in X$ is arbitrary, we have that

$$0 \to \mathcal{M} \otimes \mathcal{F}_1 \to \mathcal{M} \otimes \mathcal{F}_2 \to \mathcal{M} \otimes \mathcal{F}_3 \to 0$$

is exact. □

Of course, more generally $\mathcal{M}$ is a flat $\mathcal{O}$-module if for all $P \in X$, $\mathcal{M}(P)$ is a flat $\mathcal{O}(P)$-module.

6.2 The Map $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{M}\otimes\mathcal{F}, \mathcal{M}\otimes\mathcal{G})$ and Ext Groups for Flat $\mathcal{M}$

In this subsection we will describe a map

$$\mathrm{Ext}^i(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^i(\mathcal{M}\otimes\mathcal{F}, \mathcal{M}\otimes\mathcal{G})$$

that results from standard homological algebra whenever $\mathcal{M}$ is flat.

At this point we will assume the homological algebra in [28], namely Theorem 2.7.6 on page 63, which states that $\mathrm{Ext}^n_R(A,B)$, defined to be the right derived functors of $B \mapsto \mathrm{Hom}(A,B)$ for A fixed, is isomorphic to the right derived functors of $A \mapsto \mathrm{Hom}(A,B)$ for B fixed. Note that by Subsection A.3, for any diagram of rings, $\mathcal{O}$, the category of $\mathcal{O}$-modules has enough injectives and projectives. Hence, by the Freyd-Mitchell embedding theorem (see, e.g., [28], Theorem 1.61, page 25), Theorem 2.7.6 (which holds for R-modules for a ring, R) also holds for $\mathcal{O}$-modules.

Let $\mathcal{O}$ be any diagram of rings. Then for $\mathcal{O}$-modules $\mathcal{F},\mathcal{G}$, one defines $\mathrm{Ext}^i(\mathcal{F},\mathcal{G})$ for $i \geq 0$ as the right derived functors of $\mathcal{G} \mapsto \mathrm{Hom}(\mathcal{F},\mathcal{G})$ (with $\mathcal{F}$ fixed); by Theorem 2.7.6 of [28] (page 63), these functors are isomorphic to the right derived functors of $\mathcal{F} \mapsto \mathrm{Hom}(\mathcal{F},\mathcal{G})$ with $\mathcal{G}$ fixed. The groups $\mathcal{F} \mapsto \mathrm{Ext}^i(\mathcal{F},\mathcal{G})$ is a universal set of δ-functors, since the category of $\mathcal{O}$-modules has enough projectives.

If $\mathcal{M}$ is any $\mathcal{O}$-modules, we get a natural map

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{M}\otimes\mathcal{F},\mathcal{M}\otimes\mathcal{G}) \tag{44}$$

by tensoring with id_M; i.e., for each $\phi\colon \mathcal{F}\to\mathcal{G}$ we give a map

$$\mathrm{id}_M\otimes\phi\colon \mathcal{M}\otimes\mathcal{F}\to\mathcal{M}\otimes\mathcal{G}$$

which for each $P\in X = A_1, A_2, B_1, B_2, B_3$ is the map

$$\mathcal{M}(P)\otimes\mathcal{F}(P) \xrightarrow{\mathrm{id}_M\otimes\phi(P)} \mathcal{M}(P)\otimes\mathcal{G}(P)$$

(taking $m\otimes f$ to $m\otimes(\phi(P)(f))$).

Now assume that $\mathcal{M}$ is flat. Then the functors

$$\mathcal{F}\mapsto \mathrm{Ext}^i(\mathcal{M}\otimes\mathcal{F},\mathcal{M}\otimes\mathcal{G})$$

for $i\ge 0$ give another δ-functor, since if $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$ is short exact, then $0\to\mathcal{M}\otimes\mathcal{F}_1\to\mathcal{M}\otimes\mathcal{F}_2\to\mathcal{M}\otimes\mathcal{F}_3\to 0$ is short exact, and we get a long exact sequence of groups $\mathrm{Ext}^i(\mathcal{M}\otimes\mathcal{F}_j,\mathcal{M}\otimes\mathcal{G})$.

Since the functors $\mathcal{F}\to\mathrm{Ext}^i(\mathcal{F},\mathcal{G})$ form a universal δ-functor, the map

$$\mathrm{Ext}^0(\mathcal{F},\mathcal{G})\to\mathrm{Ext}^0(\mathcal{M}\otimes\mathcal{F},\mathcal{M}\otimes\mathcal{G})$$

(given by (44)), gives rise to a unique set of maps (as i varies)

$$\mathrm{Ext}^i(\mathcal{F},\mathcal{G})\to\mathrm{Ext}^i(\mathcal{M}\otimes\mathcal{F},\mathcal{M}\otimes\mathcal{G})$$

of δ-functors. By definition this means that for every short exact sequence

$$0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$$

we have vertical arrows

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}_3,\mathcal{G}) & \longrightarrow & \mathrm{Hom}(\mathcal{F}_2,\mathcal{G}) & \longrightarrow & \mathrm{Hom}(\mathcal{F}_1,\mathcal{G}) & \longrightarrow & \mathrm{Ext}^1(\mathcal{F}_3,\mathcal{G}) & \longrightarrow & \mathrm{Ext}^1(\mathcal{F}_2,\mathcal{G}) & \longrightarrow & \mathrm{Ext}^1(\mathcal{F}_1,\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathcal{M}\otimes\mathcal{F}_3,\omega) & \longrightarrow & \mathrm{Hom}(\mathcal{M}\otimes\mathcal{F}_2,\omega) & \longrightarrow & \mathrm{Hom}(\mathcal{M}\otimes\mathcal{F}_1,\omega) & \longrightarrow & \mathrm{Ext}^1(\mathcal{M}\otimes\mathcal{F}_3,\omega) & \longrightarrow & \mathrm{Ext}^1(\mathcal{M}\otimes\mathcal{F}_2,\omega) & \longrightarrow & \mathrm{Ext}^1(\mathcal{M}\otimes\mathcal{F}_1,\omega) & \longrightarrow & 0
\end{array}$$

where $\omega=\mathcal{M}\otimes\mathcal{G}$.

6.3 The Yoneda Pairing

Let $\mathcal{O}$ be any diagram of rings. Then the *Yoneda pairing* is a map

$$\mathrm{Ext}^i_{\mathcal{O}}(A,B)\times\mathrm{Ext}^j_{\mathcal{O}}(B,C)\to\mathrm{Ext}^{i+j}_{\mathcal{O}}(A,C) \tag{45}$$

which is functorial in A,B,C in an appropriate sense.

We will only need the pairing in the cases $i=0$ and $j=1$ and $i=1$ and $j=0$. In these cases there are two standard ways to describe this pairing.

The first way is simpler to work with: it involves working in the derived category[7] of $\mathcal{O}$-modules, whose foundations do most of the work for us. This is spelled out in [17] Remark III.5.4(b), page 166: namely, we can identify $\mathrm{Ext}^i_{\mathcal{O}}(X,Y)$ with $\mathrm{Hom}_{\mathcal{D}(\mathcal{O})}(X[k],Y[i+k])$, where $\mathcal{D}(\mathcal{O})$ is the derived category of $\mathcal{O}$-modules. (For background on this and more references, see Subsection A.1.) Then for $\mathcal{O}$-modules A,B,C, the map (45) is the composition of Hom's in the derived category

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{O})}(X[k],Y[k+i])\times\mathrm{Hom}_{\mathcal{D}(\mathcal{O})}(Y[k+i],Z[k+i+j])\to\mathrm{Hom}_{\mathcal{D}(\mathcal{O})}(X[k],Z[k+i+j]).$$

The wonderful thing about this approach is that we automatically know that Hom is associative, which is the main computation we need to prove Proposition 6.1.[8]

Therefore this is the approach we take to the Yoneda pairing.

[7] Here we are working with the full derived category of $\mathcal{O}$-modules, $\mathcal{D}(\mathcal{O})$. However, since all the $\mathcal{O}$-modules of interest to us have finite projective and injective resolutions, we could equally well work in $\mathcal{D}^+(\mathcal{O})$, $\mathcal{D}^-(\mathcal{O})$, or $\mathcal{D}^b(\mathcal{O})$ of the derived category of $\mathcal{O}$-modules with boundedness conditions; see, e.g., the beginning of Subsection III.2.5 of [17] for definitions.

[8] Another advantage of the derived category approach is that the cases $i=0$ and $j=0$ do not have to be treated differently from the $i,j>0$ cases, which the classical Yoneda Ext pairing requires.

The more classical approach to the Yoneda pairing — which we won't use — is to define $E(X,Y)$ as the class of "extensions" of X by Y, i.e., exact sequences

$$0 \to Y \to F \to X \to 0, \tag{46}$$

and to then show that (1) $E(X,Y) \simeq \mathrm{Ext}^1(X,Y)$, and (2) $E(X,Y)$ is a bifunctor in X and Y; for the details to this approach, see, for example, [20], Sections III.1 and III.2. This approach requires a lot less foundations, but the computations that we need are longer computations. [To turn $E(X,Y)$ into a bifunctor, one needs to pullback (46) along a morphism $X' \to X$ and pushout (46) along a morphism $Y \to Y'$ (see [20], Section III.1); this makes some computations we need more involved.][9]

6.3.1 First Type of Result: Prior Computations Are Correct

In Definition 2.6 we defined $H^i(\mathcal{F})$, and in Subsection 5.1 we defined a pairing

$$H^1(\mathcal{F}) \times \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to H^1(\mathcal{G}).$$

To use the machinery of the Yoneda pairing, we have to check that both these definitions coincide with the definitions given by the Yoneda pairing above; i.e., we have to verify that:

1. $H^i(\mathcal{F}) \stackrel{\text{def}}{=} \mathrm{Ext}^i_{\mathcal{O}}(\mathcal{O},\mathcal{F})$ agrees with the definition of $H^i(\mathcal{F})$ in Definition 2.6; we prove this in Subsection A.5.

2. The pairing
$$\mathrm{Ext}^1(\mathcal{O},\mathcal{F}) \times \mathrm{Ext}^0(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^1(\mathcal{O},\mathcal{G}),$$
given by the Yoneda pairing (as computed in the derived category) agrees with the pairing used in Theorem 5.1 (i.e., in Subsection 5.1)
$$H^1(\mathcal{F}) \times \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to H^1(\mathcal{G}),$$
when defining $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \stackrel{\text{def}}{=} \mathrm{Ext}^0(\mathcal{F},\mathcal{G})$ (and $H^i(\mathcal{F}) \stackrel{\text{def}}{=} \mathrm{Ext}^i_{\mathcal{O}}(\mathcal{O},\mathcal{F})$); we prove this in Subsection A.6.

6.3.2 Second Needed Result: Two "Adjointness" Properties

The second result we need is that the pairing (45) satisfies two properties that might be called "adjointness" properties.

To describe these properties, for each A,B,C and i,j, for each $\alpha \in \mathrm{Ext}^i(A,B)$ and $\beta \in \mathrm{Ext}^j(B,C)$, let

$$\langle \alpha,\beta\rangle = \langle \alpha,\beta\rangle_{A,B,C,i,j}$$

denote the image of (α,β) under the map (45).

1. If $f\colon B_1 \to B_2$ is any map, and
$$f_*\colon \mathrm{Ext}^i(A,B_1) \to \mathrm{Ext}^i(A,B_2) \quad \text{and} \quad f^*\colon \mathrm{Ext}^j(B_2,C) \to \mathrm{Ext}^j(B_1,C)$$
are the natural morphisms (of Ext as a bifunctor), then for all $\alpha \in \mathrm{Ext}^i(A,B_1)$ and $\gamma \in \mathrm{Ext}^j(B_2,C)$ we need to prove that
$$\langle \alpha, f^*\gamma\rangle = \langle f_*\alpha,\gamma\rangle. \tag{47}$$
In fact, we will only need the case $i+j=1$ (but this holds in general).

2. If $0 \to B_1 \to B_2 \to B_3 \to 0$ is a short exact sequence, and
$$\delta_*\colon \mathrm{Ext}^i(A,B_3) \to \mathrm{Ext}^{i+1}(A,B_1) \quad \text{and} \quad \delta^*\colon \mathrm{Ext}^{j-1}(B_3,C) \to \mathrm{Ext}^j(B_1,C)$$
are the connecting δ-maps, then for all $\alpha \in \mathrm{Ext}^i(A,B_3)$ and $\gamma \in \mathrm{Ext}^j(B_1,C)$ we have
$$\langle \alpha, \delta^*\gamma\rangle = \langle \delta_*\alpha,\gamma\rangle. \tag{48}$$

For proofs of (47) and (48), see respectively Subsections A.8 and A.9.

[9] To construct the pairing (45), one can splice together sequences in the "Yoneda Ext," which generalizes $E(X,Y)$ above, in the cases $i,j>0$; however, since we only need the cases $i+j=1$, we are never splicing together Yoneda Ext sequences.

6.3.3 Our Interest in Adjointness

We use the above adjointness properties for one reason alone.

Proposition 6.1. *Let C be an $\mathcal{O}_r$-module, and say that $H^1(C) \simeq k$ and fix one such isomorphism. Then for any short exact sequence of $\mathcal{O}_r$-modules $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$, the Yoneda pairing gives vertical arrows in the diagram*

$$\begin{array}{ccccccccccccc}
0 & \to & \mathrm{Hom}(\mathcal{G}_3, C) & \to & \mathrm{Hom}(\mathcal{G}_2, C) & \to & \mathrm{Hom}(\mathcal{G}_1, C) & \to & \mathrm{Ext}^1(\mathcal{G}_3, C) & \to & \mathrm{Ext}^1(\mathcal{G}_2, C) & \to & \mathrm{Ext}^1(\mathcal{G}_1, C) \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(\mathcal{G}_3)' & \longrightarrow & H^1(\mathcal{G}_2)' & \longrightarrow & H^1(\mathcal{G}_1)' & \longrightarrow & H^0(\mathcal{G}_3)' & \longrightarrow & H^0(\mathcal{G}_2)' & \longrightarrow & H^0(\mathcal{G}_1)' \longrightarrow 0
\end{array}$$

which is a commutative diagram (where $'$ is the dual space), and where we write $\mathrm{Hom}(\mathcal{G}_j, C)$ in place of $\mathrm{Ext}^0(\mathcal{G}_j, C)$, and $H^i(\mathcal{G}_j)$ in place of $\mathrm{Ext}^i(\mathcal{O}_r \mathcal{G}_j)$.

In this proposition it is important that $\mathrm{Hom}(\mathcal{G}_j, C)$ and $H^i(\mathcal{G}_j)$ are really shorthand notation for the Ext groups $\mathrm{Ext}^0(\mathcal{G}_j, C)$ and $\mathrm{Ext}^i(\mathcal{G}_j, C)$; in the proof of Theorem 6.2 below, it is crucial to use the results of Subsubsection 6.3.1 to see that the Yoneda pairing computed in Theorem 5.1 agrees with the Yoneda pairing on Ext groups as defined in this section (in terms of the derived category).

To prove this proposition, we will introduce the following easy lemma.

Lemma 6.2. *Let U, V, W, X be k-vector spaces. Let $g\colon U \to W$ and $h\colon X \to V$ be linear maps, and let $h'\colon V' \to X'$ be the dual linear map. Let*

$$\mu = \langle \cdot, \cdot \rangle_{VU}\colon V \times U \to k, \quad \nu = \langle \cdot, \cdot \rangle_{XW}\colon X \times W \to k$$

be two bilinear forms of k-vector spaces, and let $\tilde{\mu}\colon U \to V'$ and $\tilde{\nu}\colon W \to X'$. Then the following are equivalent:

1. *for each $u \in U$ and $x \in X$ we have $\langle x, gu \rangle_{XW} = \langle hx, u \rangle_{VU}$; and*
2. *the following diagram commutes:*

$$\begin{array}{ccc}
U & \xrightarrow{\ g\ } & W \\
{\scriptstyle \tilde{\mu}}\downarrow & & \downarrow{\scriptstyle \tilde{\nu}} \\
V' & \xrightarrow{\ h'\ } & X'
\end{array}$$

Proof. Let us write down condition (2), i.e., what it means for the diagram above to commute: so let $u \in U$. Then $gu \in W$, and the image of gu under $\tilde{\nu}$ is the linear functional $x \mapsto \langle x, gu \rangle_{XW}$. On the other hand, the image of u under $\tilde{\mu}$ is the linear functional $v \mapsto \langle v, u \rangle_{VU}$; the image of this linear functional under h' is therefore the linear map $x \mapsto \langle hx, u \rangle_{VU}$. Hence the above diagram commutes iff for all $u \in U$ the two linear functionals

$$x \mapsto \langle x, gu \rangle_{XW} \quad \text{and} \quad x \mapsto \langle hx, u \rangle_{VU}$$

are the same linear functional on X. This holds iff condition (1) of the proposition holds. □

Proof of Proposition 6.1. The horizontal maps are given by long exact sequences of Ext groups associated to any short exact sequence. The vertical maps are given by the Yoneda pairing and the fact that — by Subsubsection 6.3.1 — we can define $H^i(G_j)$ to be $\mathrm{Ext}^i(\mathcal{O}, G_j)$, and define $\mathrm{Hom}(\mathcal{G}_j, C)$ to be $\mathrm{Ext}^0(\mathcal{G}_j, C)$, and then our computation of $H^i(G_j)$ as $\mathrm{Ext}^i(\mathcal{O}, G_j)$ agrees with that used in earlier sections, namely with Definition 2.6. We therefore have to check that the diagram commutes.

First, according to (47), if B_1, B_2, C are any elements of the derived category and $f\colon B_1 \to B_2$ a morphism, for any $\alpha \in \mathrm{Ext}^{1-i}(\mathcal{O}, B_1)$ and $\gamma \in \mathrm{Ext}^i(B_2, C)$ we have

$$\langle \alpha, f^* \gamma \rangle = \langle f_* \alpha, \gamma \rangle \in \mathrm{Ext}^1(\mathcal{O}, C) \simeq H^1(C),$$

where $\langle \cdot, \cdot \rangle$ in $\langle \alpha, f^* \gamma \rangle$ is the Yoneda pairing

$$\mathrm{Ext}^{1-i}(\mathcal{O}, B_1) \times \mathrm{Ext}^i(B_1, C) \to \mathrm{Ext}^1(\mathcal{O}, C),$$

and similarly the $\langle \cdot, \cdot \rangle$ in $\langle f_* \alpha, \gamma \rangle$ is the Yoneda pairing

$$\mathrm{Ext}^{1-i}(\mathcal{O}, B_2) \times \mathrm{Ext}^i(B_2, C) \to \mathrm{Ext}^1(\mathcal{O}, C).$$

Now apply Lemma 6.2 with U, V, W, X there being, respectively,

$$\mathrm{Ext}^i(B_2, C),\ H^{1-i}(B_2) \stackrel{\text{def}}{=} \mathrm{Ext}^{1-i}(\mathcal{O}, B_2),\ \mathrm{Ext}^i(B_1, C),\ H^{1-i}(B_1) \stackrel{\text{def}}{=} \mathrm{Ext}^{1-i}(\mathcal{O}, B_1)$$

(the two instances of $\stackrel{\text{def}}{=}$ above are using the results of Subsubsection 6.3.1); we get a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^i(B_2, C) & \stackrel{f^*}{\to} & \mathrm{Ext}^i(B_1, C) \\ \downarrow & & \downarrow \\ H^{1-i}(B_2)' & \underset{(f_*)'}{\longrightarrow} & H^{1-i}(B_1)' \end{array}$$

Taking $f\colon B_1 \to B_2$ to be $\mathcal{G}_2 \to \mathcal{G}_3$ (in the short exact sequence) gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^i(\mathcal{G}_3, C) & \longrightarrow & \mathrm{Ext}^i(\mathcal{G}_2, C) \\ \downarrow & & \downarrow \\ H^{1-i}(\mathcal{G}_3)' & \longrightarrow & H^{1-i}(\mathcal{G}_2)' \end{array}$$

and similarly with $f\colon B_1 \to B_2$ to be $\mathcal{G}_1 \to \mathcal{G}_2$ gives a commutative digram

$$\begin{array}{ccc} \mathrm{Ext}^i(\mathcal{G}_2, C) & \longrightarrow & \mathrm{Ext}^i(\mathcal{G}_1, C) \\ \downarrow & & \downarrow \\ H^{1-i}(\mathcal{G}_2)' & \longrightarrow & H^{1-i}(\mathcal{G}_1)' \end{array}$$

Taking $i = 1$ we see that the two leftmost squares of the diagram in the proposition commute, and taking $i = 0$ gives the two rightmost squares.

Hence it suffices to check that the middle square commutes. But this follows from the analogous argument as above, and with (48) replacing (47), and applying Lemma 6.2 with U, V, W, X being

$$\mathrm{Hom}(\mathcal{G}_1, C) \stackrel{\text{def}}{=} \mathrm{Ext}^0(\mathcal{G}_1, C),\ H^1(\mathcal{G}_1) \stackrel{\text{def}}{=} \mathrm{Ext}^1(\mathcal{O}, \mathcal{G}_1),\ \mathrm{Ext}^1(\mathcal{G}_3, C),\ H^0(\mathcal{G}_3) \stackrel{\text{def}}{=} \mathrm{Ext}^0(\mathcal{O}, \mathcal{G}_3).$$

□

6.4 The Duality Commutative Diagram

Theorem 6.1. *Let $\mathcal{O}$ be a diagram of k-algebras, let $\mathcal{M}$ be a flat $\mathcal{O}$-module, and let $\mathcal{M}'$ be any $\mathcal{O}$-module. Say that $\omega = \mathcal{M} \otimes \mathcal{M}'$ satisfies $H^1(\omega) \simeq k$, and fix such an isomorphism. Then for $i = 0, 1$ and any $\mathcal{O}$-module, $\mathcal{F}$, there is a natural map*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{M}') \to H^{1-i}(\mathcal{M} \otimes \mathcal{F})'$$

such that for any short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of $\mathcal{O}$-modules, the two resulting long exact sequences (given by horizontal arrows) fit into a commutative diagram:

$$\begin{array}{ccccccccccccc} 0 & \to & \mathrm{Hom}(\mathcal{F}_3, \mathcal{M}') & \to & \mathrm{Hom}(\mathcal{F}_2, \mathcal{M}') & \to & \mathrm{Hom}(\mathcal{F}_1, \mathcal{M}') & \to & \mathrm{Ext}^1(\mathcal{F}_3, \mathcal{M}') & \to & \mathrm{Ext}^1(\mathcal{F}_2, \mathcal{M}') & \to & \mathrm{Ext}^1(\mathcal{F}_1, \mathcal{M}') & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^1(\mathcal{M} \otimes \mathcal{F}_3)' & \to & H^1(\mathcal{M} \otimes \mathcal{F}_2)' & \to & H^1(\mathcal{M} \otimes \mathcal{F}_1)' & \to & H^0(\mathcal{M} \otimes \mathcal{F}_3)' & \to & H^0(\mathcal{M} \otimes \mathcal{F}_2)' & \to & H^0(\mathcal{M} \otimes \mathcal{F}_1)' & \to & 0 \end{array}$$

Moreover, one has the following "two out of three" principle: say that two of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ satisfy strong duality, i.e., for two of $j = 1, 2, 3$ we have

$$\mathrm{Ext}^i(\mathcal{F}_j, \mathcal{M}') \to H^{1-i}(\mathcal{M} \otimes \mathcal{F}_j)'$$

is an isomorphism for both $i = 0, 1$; then all the $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ satisfy strong duality.

Proof. We combine the commutative diagram at the end of Subsection 6.2, replacing $\mathcal{G}$ with $\mathcal{M}'$ (hence $\omega = \mathcal{M} \otimes \mathcal{M}'$) with the commutative diagram in Proposition 6.1, setting $\mathcal{G}_i = \mathcal{M} \otimes \mathcal{F}_i$ and $C = \omega$. This gives the desired commutative diagram.

The "two out of three" principle follows from the five-lemma. □

6.5 The Fundamental Lemma

Lemma 6.3. *Fix an r-periodic perfect matching W and an $\mathbf{M} \in \mathbb{Z}^2$. For any $\mathbf{d}$ with $\deg(\mathbf{d})$ sufficiently small, we have*

1. $H^0(\mathcal{M}_{W,\mathbf{M}} \otimes \mathcal{L}_{r,\mathbf{d}}) = 0$, *and*
2. $\mathrm{Ext}^1_{\mathcal{O}_r}(\mathcal{L}_{r,\mathbf{d}}, \mathcal{M}_{W,\mathbf{M}}) = 0$.

Proof. (1): we have

$$\beta^0(\mathcal{M}_{W,\mathbf{M}} \otimes \mathcal{L}_{r,\mathbf{d}}) = \beta^0(\mathcal{M}_{W,\mathbf{M}+\mathbf{d}}) = \big|\{\mathbf{d}' \mid \mathbf{d}' \le \mathbf{M}+\mathbf{d},\ W(\mathbf{d}') = 1\}\big|$$

which vanishes for $\deg(\mathbf{d})$ sufficiently small, since $\mathbf{M}$ is fixed and W has bounded support.

(2): fix any $\mathbf{L} = \mathbf{K} + \mathbf{1}$. By Proposition III.6.7 of [18] we have[10]

$$\mathrm{Ext}^1_{\mathcal{O}_r}(\mathcal{L}_{r,\mathbf{d}}, \mathcal{M}_{W,\mathbf{M}}) \simeq \mathrm{Ext}^1_{\mathcal{O}_r}(\mathcal{O}_r, \mathcal{L}_{r,-\mathbf{d}} \otimes \mathcal{M}_{W,\mathbf{M}}),$$

and by Proposition 3.1, the right-hand-side is

$$\simeq \mathrm{Ext}^1_{\mathcal{O}_r}(\mathcal{O}_r, \mathcal{M}_{W,\mathbf{M}-\mathbf{d}}),$$

and according to [18], Proposition III.6.3(c)

$$\mathrm{Ext}^1_{\mathcal{O}_r}(\mathcal{O}_r, \mathcal{M}_{W,\mathbf{M}-\mathbf{d}}) \simeq H^1(\mathcal{M}_{W,\mathbf{M}-\mathbf{d}})$$

which for any fixed $\mathbf{L} = \mathbf{K} + \mathbf{1}$ is isomorphic to

$$H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{M}+\mathbf{d}}) \simeq H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{M}} \otimes \mathcal{L}_{r,\mathbf{d}})$$

but by (1) (with $W, \mathbf{M}$ respectively replaced by $W^*_{\mathbf{L}}, \mathbf{K}-\mathbf{M}$), this vanishes for $\deg(\mathbf{d})$ sufficiently small (since $\mathbf{K}, \mathbf{M}$ are fixed). □

6.6 Strong Duality for $\mathcal{L}_{\mathbf{d}}$ with $\deg(\mathbf{d})$ Sufficiently Small

Theorem 6.2. *Let W be an r-periodic perfect matching, let $\mathcal{M} = \mathcal{M}_{W,\mathbf{0}}$, and for some $\mathbf{L} = \mathbf{K} + \mathbf{1}$ let $\mathcal{M}' = \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}}$. Then for all $\mathbf{d}$ with $\deg(\mathbf{d})$ sufficiently small we have that $\mathcal{F} = \mathbf{L}_{\mathbf{d}}$ satisfies strong duality, i.e., the map*

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{M}') \to H^{1-i}(\mathcal{M} \otimes \mathcal{F})' \tag{49}$$

is an isomorphism for $i = 0, 1$.

Proof. For $i = 0$ this follows from Theorem 5.1 and Subsubsection 6.3.1. According to Lemma 6.3, both sides of (49) are 0. □

6.7 Strong Duality for Skyscrapers and All $\mathcal{L}_{\mathbf{d}}$

Theorem 6.3. *The skyscraper diagrams $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy strong duality, as well as all diagrams $\mathcal{L}_{\mathbf{d}}$.*

Proof. Consider the short exact sequence:

$$0 \to \mathcal{L}_{\mathbf{d}} \to \mathcal{L}_{\mathbf{d}+\mathbf{e}_1} \to \mathcal{S}_1 \to 0,$$

By Theorem 6.2, both $\mathcal{L}_{\mathbf{d}}$ and $\mathcal{L}_{\mathbf{d}+\mathbf{e}_1}$ satisfy strong duality for $\deg(\mathbf{d})$ sufficiently small. Taking such a $\mathbf{d}$, the "two out of three principle" (Theorem 6.1) implies that $\mathcal{S}_1$ also satisfies strong duality.

Similarly $\mathcal{S}_2$ satisfies strong duality.

Now consider an arbitrary $\mathbf{d} \in \mathbb{Z}^2$. According to Theorem 6.2, for $a_0 \in \mathbb{Z}$ sufficiently large, we have $\mathcal{L}_{\mathbf{d}-a_0\mathbf{e}_1}$ satisfies strong duality. So fix such an a_0. Then, considering the exact sequence

$$0 \to \mathcal{L}_{\mathbf{d}-a\mathbf{e}_1} \to \mathcal{L}_{\mathbf{d}-(a-1)\mathbf{e}_1} \to \mathcal{S}_1 \to 0$$

with $a = a_0, a_0 - 1, a_0 - 2, \ldots$, and repeatedly applying the "two out of three" principle, we see that $\mathcal{L}_{\mathbf{d}-a'\mathbf{e}_1}$ for $a' = a_0 - 1, a_0 - 2, \ldots, 0$. □

[10] To apply Proposition III.6.7 of [18] as is, we have to verify that $\mathcal{L}_{r,-\mathbf{d}}$ equals the dual sheaf of $\mathcal{L}_{r,\mathbf{d}}$, i.e., to verify that $\mathcal{L}^\vee \stackrel{\text{def}}{=} \mathscr{H}om_{\mathcal{O}_r}(\mathcal{L}_{r,\mathbf{d}}, \mathcal{O}_r)$ equals $\mathcal{L}_{r,-\mathbf{d}}$. We leave this to the interested reader, or notice that we have essentially already verified this in Proposition 3.1, (19): indeed, this equation implies that for all $\mathcal{O}_r$-modules, $\mathcal{F}$, we have $\mathrm{Hom}(\mathcal{L}_{r,\mathbf{d}} \otimes \mathcal{F}, \mathcal{O}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{L}_{r,-\mathbf{d}})$: moreover, Exercise II.5.1 of [18], implies that for any ringed space $(X, \mathcal{O})$, and locally free $\mathcal{O}$-module $\mathcal{L}$ we have $\mathrm{Hom}(\mathcal{L} \otimes \mathcal{F}, \mathcal{O}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{L}^\vee)$ where $\mathcal{L}^\vee$ is the dual sheaf of $\mathcal{L}$; hence $\mathcal{L}^\vee_{r,\mathbf{d}}$ and $\mathcal{L}_{r,-\mathbf{d}}$ represent the same functor, and are therefore isomorphic by Yoneda's lemma. One can also entirely circumvent Proposition III.6.7 of [18], by using the proofs of Lemma III.6.6 and Proposition III.6.7, which results in a more direct argument: namely, $\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}_{r,\mathbf{d}}, \mathcal{G}) \simeq \mathrm{Ext}^i(\mathcal{F}, \mathcal{L}_{r,-\mathbf{d}} \otimes \mathcal{G})$ for all i since they agree for $i = 0$ by Proposition 3.1, (19), and both vanish for all $\mathcal{G}$ injective using (19) and the exactness of tensoring by $\mathcal{L}_{r,\mathbf{d}}$, which results from (18).

Remark 6.1. *Since the value of $\mathcal{S}_1$ vanishes at all points except at B_1, where its value is k, it follows that $H^0(\mathcal{S}_1) \simeq k$ and $H^1(\mathcal{S}_1) = 0$. Similarly for $\mathcal{S}_2$.*

Remark 6.2. *Any $\mathcal{O}_r = \mathcal{O}_{r,k}$-module can also be regarded as a sheaf of k-vector spaces, and hence a $\underline{k}$ module. A general theorem tells us that for any k-diagram, $\mathcal{F}$, $H^i_{\underline{k}}(\mathcal{F}) \simeq H^i_{\underline{\mathcal{O}}}$, and hence we can compute H^i as $\underline{k}$-modules or $\mathcal{O}_r$-modules. However,*

$$\mathrm{Hom}_{\mathcal{O}_r}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}}}) \quad \textit{and} \quad \mathrm{Hom}_{\underline{k}}(\mathcal{F}, \mathcal{M}_{W^*_{\mathbf{L}}})$$

are very different; in particular, we will later explain that the second space is infinite-dimensional for $\mathcal{F} = \mathcal{L}_{\mathbf{d}}$. Hence it is essential for us to work with $\mathcal{O}_r$-modules to prove our strong duality theorems.

A. The Yoneda Pairing, Skyscrapers, and CoSkyscrapers

In this section we address a number of points in homological algebra. The reader familiar with algebraic geometry will likely find all these points believable — if not already known to them — and can safely skip the details.

Again, in this section k will be a fixed field, and we will often suppress k, writing, e.g., $\mathcal{O}_r$ instead of $\mathcal{O}_{r,k}$. Also, we sometimes suppress the subscript r in $\mathcal{O}_r$, or suppress $\mathcal{O}$ altogether (e.g., writing $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ instead of $\mathrm{Ext}^i_{\mathcal{O}_r}(\mathcal{F}, \mathcal{G})$) if confusion is unlikely.

The results in this section actually hold where $\mathcal{O}$ is an arbitrary diagram of rings; however, in Subsections A.5 and onward, we insist that $\mathcal{O}$ is a diagram of k-algebras simply because our definitions of $H^i(\mathcal{F})$, of Ext groups, etc., insist on working with k-diagrams (of vector spaces).

A.1 Background on Ext groups and the Derived Category

For some of the facts we need about Ext groups, we refer to [18], Sections III.1 and III.6. For facts about the derived category, see [17] Chapter III and [28] Chapter 10.

Note that [17] defines $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ as $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}[0], \mathcal{G}[i])$, where $\mathcal{D} = \mathcal{D}(\mathcal{O})$ is the derived category of $\mathcal{O}$-modules and $\mathcal{G}[i]$ denotes the complex that is all zeros except for a $\mathcal{G}$ in position $-i$ (see Remark III.5.4(b), page 166 there); only later does [17] show that this definition coincides with the usual derived functor definition of $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ (see Subsection III.6.14, page 194 there). By contrast, [28] defines $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ in terms of derived functors of $\mathrm{Ext}^i(\mathcal{F}, \cdot)$ (Definition 2.5.2, page 50), and shows that this agrees with those of $\mathrm{Ext}^i(\cdot, \mathcal{G})$ (Theorem 2.7.6, page 63) (and it follows from the Freyd Full Embedding Theoerm[11] that the same holds in the category of $\mathcal{O}'$-modules for any ringed space $(X, \mathcal{O}')$, provided that this category has enough projectives and injectives. Since the category of $\mathcal{O}_{r,k}$-modules is equivalent to such a category, we conclude this holds for $\mathcal{O}_{r,k}$-modules. Then [28] shows that $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ and agrees with the i-th hyperext of $\mathcal{F}[0], \mathcal{G}[0]$ (Corollary 10.7.5, page 400 there; [28] writes $\mathcal{F}$ for both $\mathcal{F}$ and the complex $\mathcal{F}[0]$).

A.2 Skyscrapers and Coskyscrapers for $\mathcal{O}$-modules

We will need a projective resolution for $\mathcal{O}$, and to know that the category of $\mathcal{O}$-modules has enough injectives and enough projectives. This will involve what we call *coskyscraper* $\mathcal{O}$-modules which are a sort of dual diagram to the commonly used *skyscraper $\mathcal{O}$-modules* that one uses in sheaf theory.

[All statements above are well-known to experts, but it seems difficult to find this anywhere, at least in our simple context. Once one realizes that the category of diagrams of $\mathcal{O}$-modules is equivalent to sheaves of $\mathcal{O}'$-modules for a ringed space $(X, \mathcal{O}')$, then it is well known that there are enough injectives.]

What we do in this subsection is valid over a general diagram of rings $\mathcal{O}$; let us fix notation, setting

$$R_i = \mathcal{O}(B_i),\ S_j = \mathcal{O}(A_j),\ \sigma_{ij} = \mathcal{O}(\rho_{ij}),$$

and see Figure 11.

[11] See the comments in [18], Section III.1, page 203, or [22], Theorem 9.6.10, page 238. Moreover, below we will show that the category of $\mathcal{O}_{r,k}$-modules have enough injectives and projectives.

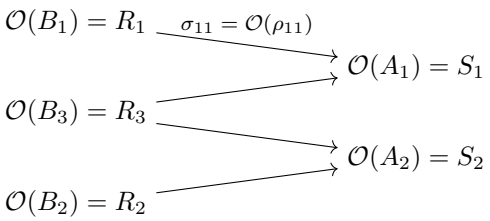
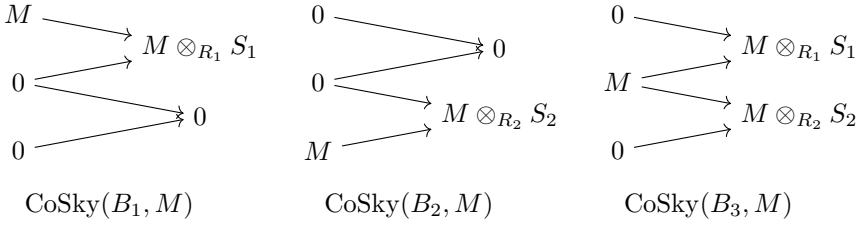


Figure 11: In this section we work with a general diagram of rings, $\mathcal{O}$. We introduce the notation $R_i = \mathcal{O}(B_i)$, $S_j = \mathcal{O}(A_j)$, and $\sigma_{ij} = \mathcal{O}(\rho_{ij})$.

A.2.1 Coskyscraper Diagrams

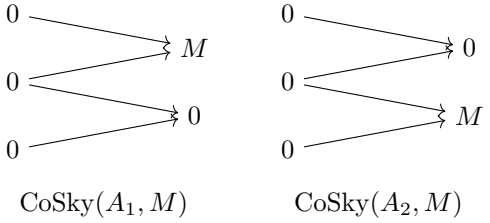
Proposition A.1. *Let $\mathcal{O}$ be a diagram of rings For $i \in [3]$ and M and R_i-module, define the* coskyscraper at B_i of value M, *denoted* $\mathrm{CoSky}(B_i, M)$, *to be the $\mathcal{O}$-module depicted below:*



Then for any $\mathcal{F}$ we have an isomorphism that is functorial in $\mathcal{F}$:

$$\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{CoSky}(B_i, M), \mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{O}(B_i)}\big(M, \mathcal{F}(B_i)\big) \tag{50}$$

which takes $\phi \in \mathrm{Hom}_{\mathcal{O}}\big(\mathrm{CoSky}(B_i, M), \mathcal{F}\big)$ to $\phi(B)$. Similarly for any $j \in [2]$, an S_j-module M, and the diagrams



and the isomorphism

$$\mathrm{Hom}_{\mathcal{O}}\big(\mathrm{CoSky}(A_j, M), \mathcal{F}\big) \simeq \mathrm{Hom}_{\mathcal{O}(A_j)}\big(M, \mathcal{F}(A_j)\big) \tag{51}$$

taking ϕ to $\phi(A_j)$.

(To understand the meaning of $M \otimes_{R_i} S_j$, notice that if M is an R_i-module, and ρ_{ij} is a restriction map for some $i \in [3]$ and $j \in [2]$, then S_j is an R_i-module via the map $\sigma_{ij} \colon R_i \to S_j$. Hence one can take the tensor product $M \otimes_{R_i} S_j$, which is an S_j-module with S_j acting on the second factor.)

Proof. We leave the details to the reader. For some details, see [10], Section 10.1. □

We remark that this proposition follows from the general construction of adjoints, [27], Exp. I, 5.1; see [13], Section 2.1, specifically the discussion around equation (6) (this is done for diagrams of k-vector spaces, and the above proposition is the generalization to $\mathcal{O}$-modules for a diagram of rings, $\mathcal{O}$). We also remark that the above proposition is very similar to what one sees in algebraic geometry.

Here is an immediate consequence of Proposition A.1.

Theorem A.1. *For some $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$, let M be a projective $\mathcal{O}(P)$-module. Then* $\mathrm{CoSky}(P, M)$ *is a projective $\mathcal{O}$-module.*

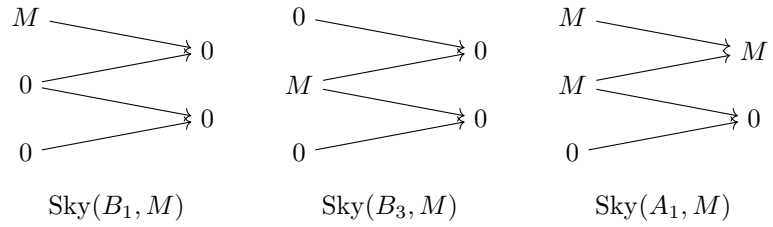
It is a standard result that M is a projective R-module iff it is a direct summand of a free R-module, i.e., of a direct sum of copies of R.

A.2.2 Skyscraper Diagrams

Similarly to coskyscraper diagrams, for each $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$, if M is an $\mathcal{O}(P)$, then for any $\mathcal{F}$ one has the isomorphism

$$\mathrm{Hom}_{\mathcal{O}}\big(\mathcal{F}, \mathrm{Sky}(P, M)\big) \simeq \mathrm{Hom}_{\mathcal{O}(P)}\big(\mathcal{F}(P), M\big) \tag{52}$$

where $\mathrm{Sky}(P, M)$ is the *skyscraper diagram at P with value M* which are defined by:



and similarly for $\mathrm{Sky}(B_2, M)$ and $\mathrm{Sky}(A_2, M)$. Here we note that if M is a module over $\mathcal{O}(A_j) = S_j$, and ρ_{ij} is a restriction, then M is automatically an R_i module via the map $\sigma_{ij} \colon R_i \to S_j$. Again we leave the proof of (52) to the reader, and refer the reader to [10], Section 10.1, for a diagram and helpful discussion.

We therefore get the following "dual theorem" to Theorem A.1.

Theorem A.2. *For some $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$, let M be an injective $\mathcal{O}(P)$-module. Then* $\mathrm{Sky}(P, M)$ *is an injective $\mathcal{O}$-module.*

A.3 $\mathcal{O}$-modules Have Enough Injectives and Projectives

For any ring, R, the category of R-modules has enough injectives and projectives (see [28], Sections 2.2 and 2.3).

If $\mathcal{F}$ is any $\mathcal{O}$-module, let us show that there is an injection $\mathcal{F} \to \mathcal{I}$ where $\mathcal{I}$ is an injective $\mathcal{O}$-module. For each $P \in X = \{A_1, A_2, B_1, B_2, B_3\}$ let $\mathcal{F}(P) \to I_P$ be an injection to an injective $\mathcal{O}(P)$-module, I_P. Then

$$\mathcal{F} \to \mathcal{I} \stackrel{\text{def}}{=} \bigoplus_{P \in X} \mathrm{Sky}(P, I_P)$$

be the natural map using (52). The map $\mathcal{F} \to \mathcal{I}$ is injective since $\mathcal{F} \to \mathrm{Sky}(P, I_P)$ is injective at P. Since I_P is injective, then (52) shows that $\mathrm{Sky}(P, I_P)$ is an injective $\mathcal{O}$-module; hence $\mathcal{I}$ an injective $\mathcal{O}$-module.

Hence the category of $\mathcal{O}$-modules has enough injectives. Similarly any $\mathcal{O}$-module, $\mathcal{F}$, has a surjective map $\mathcal{P} \to \mathcal{F}$ where

$$\mathcal{P} = \bigoplus_{P \in X} \mathrm{CoSky}(P, J_P)$$

where $J_P \to \mathcal{F}(P)$ is a surjection and J_P is a projective $\mathcal{O}(P)$ module. Hence $\mathcal{P}$ is a projective $\mathcal{O}$-module. Hence the category of $\mathcal{O}$ also has enough projectives.

A.4 A Projective Resolution of $\mathcal{O}$

In this subsection we build a useful projective resolution of $\mathcal{O}$.

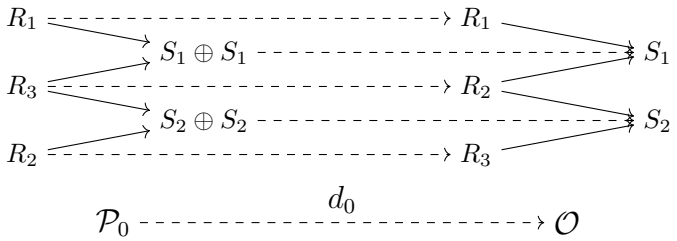
To do so, for each $i \in [3]$, the identity map $R_i \to R_i = \mathcal{O}(B_i)$ determines via (50) a map

$$\mathrm{CoSky}(B_i, R_i) \xrightarrow{\phi_i} \mathcal{O}.$$

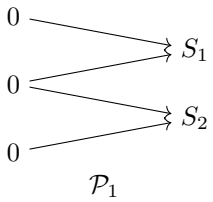
Taking direct sums, we get a morphism $d_0 = \phi_1 \oplus \phi_2 \oplus \phi_3$ from

$$\mathcal{P}_0 = \mathrm{CoSky}(B_1, R_1) \oplus \mathrm{CoSky}(B_2, R_2) \oplus \mathrm{CoSky}(B_3, R_3) \xrightarrow{d_0 = \phi_1 \oplus \phi_2 \oplus \phi_3} \mathcal{O},$$

which we illustrate as



(here the morphism d_0 and the $d_0(P)$ are illustrated in dashed lines, and the restriction maps of each of $\mathcal{P}_0$ and $\mathcal{O}$ are solid lines). We therefore see the kernel of d_0 is $\mathcal{P}_1$, given by the kernels of the individual $d_0(P)$:



Therefore

$$\mathcal{P}_1 \simeq \mathrm{CoSky}(A_1, S_1) \oplus \mathrm{CoSky}(A_2, S_2)$$

is projective, and we get the exact sequence of $\mathcal{O}$-modules:

$$0 \to \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \xrightarrow{d_0} \mathcal{O} \to 0 \tag{53}$$

where we may take $d_1(A_1)$ to map $1 \in S_1$ to $(1,-1) \in S_1 \oplus S_1$ (or to $(s,-s)$ for any s that is a unit in the ring S_1), and similarly we may take $d_1(A_2)$ to map $1 \in S_2$ to $(1,-1) \in S_2 \oplus S_2$.

A.5 Our First $H^i(\mathcal{F})$ Definition is Consistent

In this subsection, we prove that our definition of $H^i(\mathcal{F})$ for a k-diagram, $\mathcal{F}$, is actually $H^i(\mathcal{F}) \stackrel{\text{def}}{=} \mathrm{Ext}^i_{\mathcal{O}_r}(\mathcal{O}_r, \mathcal{F})$ as defined by the derived category.

For any $\mathcal{F}$, we may compute $H^i(\mathcal{F}) = \mathrm{Ext}^i_{\mathcal{O}_r}(\mathcal{O}_r, \mathcal{F})$ from any projective resolution of $\mathcal{O}$; taking the resolution (53), and hence $H^i(\mathcal{F})$ are the homology groups of

$$0 \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{P}_0, \mathcal{F}) \xrightarrow{\tau} \mathrm{Hom}_{\mathcal{O}}(\mathcal{P}_1, \mathcal{F}) \to 0, \tag{54}$$

i.e., the cokernel of τ. (This is a classical theorem about Ext; see [28], Theorem 2.7.6, page 63; to see its equality in the context of the derived category, see the discussion around (56) below.)

In view of (51) and (50) τ is canonically isomorphic to the map

$$\bigoplus_{i=1}^{3} \mathrm{Hom}_{R_i}\big(R_i, \mathcal{F}(B_i)\big) \to \bigoplus_{j=1}^{2} \mathrm{Hom}_{S_j}\big(S_j, \mathcal{F}(A_j)\big)$$

whose cokernel is that of the map

$$\bigoplus_{i=1}^{3} \mathcal{F}(B_i) \to \bigoplus_{j=1}^{2} \mathcal{F}(A_j). \tag{55}$$

In view of the fact that we may take for $j = 1, 2$ the maps

$$d_1(A_j)\colon S_j \to S_j^2 = \mathcal{P}_0(A_j) \simeq \mathcal{P}_0(B_j) \oplus \mathcal{P}_0(B_3)$$

to be the maps

$$d_1(A_j)1 = (1,-1),$$

the map in (55) can be taken to be the map

$$\bigoplus_{i=1}^{3} \mathcal{F}(B_i) \to \bigoplus_{j=1}^{2} \mathcal{F}(A_j)$$

which is the sum of maps taking, for $j \in [2]$, $\mathcal{F}(B_j)$ to $\mathcal{F}(A_j)$ via the identity map, and taking $\mathcal{F}(B_3)$ to $\mathcal{F}(A_j)$ via minus the identity map.

But this agrees with our definition of $\mathcal{F}(\partial)$ in (6) in Definition 2.6. Hence the two definitions agree.

A.6 Our First Definition of the Yoneda Pairing is Consistent

In Subsection 5.1 we defined the Yoneda pairing. Here we show that this agrees with the Yoneda pairing as defined in Subsection 6.3.

So let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}$-modules, which we view as elements of the derived category, $\mathcal{D}(\mathcal{O})$, as complexes that live only in the 0-place (i.e., we write $\mathcal{F}, \mathcal{G}$ for $\mathcal{F}[0], \mathcal{G}[0]$[12]). Note that $\mathrm{Hom}_{\mathcal{D}(\mathcal{O})}(\mathcal{F}, \mathcal{G})$ is exactly the set of morphisms $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ in place 0 (see [17], Proposition 2, Section III.5, page 164), although we don't really need this fact.

First, we can compute $\mathrm{Hom}_{\mathcal{D}(\mathcal{O})}(\mathcal{O}, \mathcal{F})$ by taking a projective resolution of $\mathcal{O}$ (see [17], Theorem 21, page 179, and Exercise 1, page 183, in Section III.5, or [28], Corollary 10.4.7, page 388). Hence $H^1(\mathcal{F})$ is the set of chain maps up to homotopy of:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{56}$$

and each such chain map is determined by $\alpha\colon \mathcal{P}_1 \to \mathcal{F}$. Since the homotopy maps are determined by a map $\mathcal{P}_0 \to \mathcal{F}$, we conclude that the set of chain maps are precisely the cokernel of τ in (54).

Now if $\phi\colon \mathcal{F} \to \mathcal{G}$ is any map, then the Yoneda pairing gives a map in the derived category

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \tag{57}$$

Hence the Yoneda pairing composes (57) and (56), which gives the map

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \phi\alpha} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Hence the Yoneda pairing has ϕ operating by composition as a map

$$\phi \circ \cdot\colon \mathrm{Hom}(\mathcal{P}_1, \mathcal{F}) \to \mathrm{Hom}(\mathcal{P}_1, \mathcal{G}) \tag{58}$$

taking α to $\phi \circ \alpha$. Now we check that the isomorphisms (51) are functorial in $\mathcal{F}$, and therefore under the identifications

$$\mathrm{Hom}(\mathcal{P}_1, \mathcal{F}) \simeq \bigoplus_{j=1}^{2} \mathcal{F}(A_j), \qquad \mathrm{Hom}(\mathcal{P}_1, \mathcal{G}) \simeq \bigoplus_{j=1}^{2} \mathcal{G}(A_j),$$

composition in ϕ in (58) becomes the map

$$\bigoplus_{j=1}^{2} \mathcal{F}(A_j) \to \bigoplus_{j=1}^{2} \mathcal{G}(A_j)$$

that is just the sum of composition in $\phi(A_1)$ and $\phi(A_2)$.

But this is just how we defined the Yoneda pairing in Subsection 5.1. Hence this definition is consistent with how we are defining it in terms of the derived category.

A.7 The Yoneda Pairing and Ext Functoriality

To prove (47), we will need the following facts: if $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}$ are $\mathcal{O}$-modules, and $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism, then there is a map

$$f_*\colon \mathrm{Ext}^i(\mathcal{G}, \mathcal{F}_1) \to \mathrm{Ext}^i(\mathcal{G}, \mathcal{F}_2),$$

and a map

$$f^*\colon \mathrm{Ext}^i(\mathcal{F}_2, \mathcal{G}) \to \mathrm{Ext}^i(\mathcal{F}_1, \mathcal{G});$$

this is the usual functoriality of Ext groups. We will need to know that f_* agrees with the composition in the derived category,

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}[-i], \mathcal{F}_1) \times \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}[-i], \mathcal{F}_2); \tag{59}$$

this is true because (59) is the functoriality of $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}[-i], \mathcal{F})$ in the variable, $\mathcal{F}$, this is proven in [17], Subsection III.6.14. Similarly f^* agrees with the composition

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \mathcal{F}_2) \times \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_2, \mathcal{G}[i]) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \mathcal{G}[i])$$

(see the last paragraph of [17], Subsection III.6.14).

[12] This is the convention in [17].

A.8 The First Adjointness Property

In this subsection we prove (47).

So let $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of $\mathcal{O}$-modules, and let $\mathcal{G}$ be another $\mathcal{O}$-module. Then for $i = 0, 1$ we have a map

$$\mathrm{Ext}^i(\mathcal{O}, \mathcal{F}_1) \times \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) \times \mathrm{Ext}^{1-i}(\mathcal{F}_2, \mathcal{G}) \to \mathrm{Ext}^1(\mathcal{O}, \mathcal{G}) = H^1(\mathcal{G})$$

given by composing the Yoneda pairings which are a composition of Hom sets in the derived category $\mathcal{D}(\mathcal{O})$:

$$\mathrm{Hom}(\mathcal{O}[-i], \mathcal{F}_1) \times \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) \times \mathrm{Hom}(\mathcal{F}_2, \mathcal{G}[1-i]) \to \mathrm{Hom}(\mathcal{O}[-i], \mathcal{G}[1-i]) \simeq H^1(\mathcal{G}).$$

So let $f\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism (which we also view as a morphism in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}_1, \mathcal{F}_2)$), and let

$$\alpha \in \mathrm{Hom}(\mathcal{O}[-i], \mathcal{F}_1), \quad \gamma \in \mathrm{Hom}(\mathcal{F}_2, \mathcal{G}[1-i]).$$

From the associativity of Hom in the derived category we have that

$$(\gamma \circ f) \circ \alpha = \gamma \circ (f \circ \alpha). \tag{60}$$

Then from Subsection A.7 we know that $f \circ \alpha = f_*\alpha$ as f_* acts on Ext groups, and $\gamma \circ f = f^*\gamma$. Hence (60) implies that in the Yoneda pairing, we have (47).

A.9 Proof of (48)

To verify (48), we need a similar functoriality. Let us explain.

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence. Then there corresponds a distinguished triangle in the derived category

$$\mathcal{F}_1[0] \to \mathcal{F}_2[0] \to \mathcal{F}_3[0] \xrightarrow{\delta} \mathcal{F}_1[1].$$

The Yoneda pairing gives a map

$$\mathrm{Hom}(\mathcal{O}, \mathcal{F}_3) \times \mathrm{Hom}(\mathcal{F}_3, \mathcal{F}_1[1]) \times \mathrm{Hom}(\mathcal{F}[1], \mathcal{G}[1])$$

by composition, which is associative, i.e., for any

$$\alpha \in \mathrm{Hom}(\mathcal{O}, \mathcal{F}_3),\ \delta \in \mathrm{Hom}(\mathcal{F}_3, \mathcal{F}_1[1]),\ \gamma \in \mathrm{Hom}(\mathcal{F}[1], \mathcal{G}[1]),$$

we have

$$\gamma \circ (\delta \circ \alpha) = (\gamma \circ \delta) \circ \alpha.$$

So it suffices to show that

1. $\delta \circ \alpha = \delta_*\alpha$ is the usual δ-connecting map in the long exact sequence

$$\cdots \to \mathrm{Ext}^0(\mathcal{O}, \mathcal{F}_3) \to \mathrm{Ext}^1(\mathcal{O}, \mathcal{F}_1) \to \cdots \tag{61}$$

   obtained from the short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$; and

2. similarly for $\gamma \circ \delta = \delta^*\gamma$.

We will prove (1); (2) is proven analogously. Note that (1) and (2) likely follow from the general theory of the long exact sequence for the Ext groups and that for distinguished triangles (could these two long exact sequences really be different?); however, we have not found a place in the literature that states this formally. Hence we include a proof.

In the short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, the map

$$\delta\colon \mathcal{F}_3[0] \to \mathcal{F}_1[1]$$

is given by the roof

$$\begin{array}{ccccc} & & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\ & \swarrow & & & \searrow \\ 0 & \longrightarrow & \mathcal{F}_3 & & \mathcal{F}_1 & \longrightarrow & 0 \end{array} \tag{62}$$

(see [22], just before Proposition 13.1.15, page 323, or [17] Lemma III.3.3, the map $\delta = \delta(f)$); the left arrow is a quasi-isomorphism and the right is a morphism. So given an

$$\alpha \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{O}, \mathcal{F}_3),$$

we take a projective resolution $0 \to \mathcal{P}_1 \to \mathcal{P}_0 \to 0$ of $\mathcal{O}$, and we may realize α as a map of complexes

$$\begin{array}{ccc} \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \alpha} \\ 0 & \longrightarrow & \mathcal{F}_3 \end{array}$$

let us show that the image of α in $\mathcal{F}_1 \to 0$ is just the usual connecting map. Since $\mathcal{P}_0$ is projective and $\mathcal{F}_2 \to \mathcal{F}_3$ is surjective, we have a map $\beta\colon \mathcal{P}_0 \to \mathcal{F}_2$ with a commutative diagram

$$\begin{array}{ccc} & & \mathcal{P}_0 \\ & {\scriptstyle \beta}\swarrow & \downarrow \\ \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 \end{array}$$

which gives a map τ

$$\begin{array}{ccccc} \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \\ {\scriptstyle \tau}\downarrow & \swarrow & \downarrow \\ \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 \end{array}$$

Since τ followed by $\mathcal{F}_2 \to \mathcal{F}_3$ is the map $\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{F}_3$ which is the zero map, the image of τ is in the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$, which therefore gives an arrow σ

$$\begin{array}{ccccc} & & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \\ & {\scriptstyle \sigma}\swarrow & {\scriptstyle \tau}\downarrow & \swarrow & \downarrow \\ \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 \end{array}$$

We therefore get a diagram

$$\begin{array}{ccccccc} & & & & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \\ & & & \swarrow & & \searrow & \\ & & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & & \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \\ & \swarrow & & \searrow & & \swarrow & \\ \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & & 0 & \longrightarrow & \mathcal{F}_3 \end{array}$$

whose leftward arrows are all quasi-isomorphisms. We can therefore compose the map $\mathcal{P}_1 \to \mathcal{P}_0$ to $0 \to \mathcal{F}_3$ to $\mathcal{F}_1 \to 0$ (the latter from (62)) in the derived category from the diagram:

$$\begin{array}{ccccccccc} & & & & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & & \\ & & & \swarrow & & \searrow & & & \\ & & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 & & 0 & \longrightarrow & \mathcal{F}_3 & & \mathcal{F}_1 \longrightarrow 0 \end{array}$$

The composition is therefore the map of complexes, from the top centre complex of the above diagram to the lower right complex, i.e., the map:

$$\begin{array}{ccc} \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathcal{F}_1 & \longrightarrow & 0 \end{array}$$

Now we have to check that the map $\alpha \mapsto \sigma$ is the usual connecting map (61) that we obtain from

$$\begin{array}{ccccc} \mathrm{Hom}(\mathcal{P}_1, \mathcal{F}_1) & \longrightarrow & \mathrm{Hom}(\mathcal{P}_1, \mathcal{F}_2) & \longrightarrow & \mathrm{Hom}(\mathcal{P}_1, \mathcal{F}_3) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Hom}(\mathcal{P}_0, \mathcal{F}_1) & \longrightarrow & \mathrm{Hom}(\mathcal{P}_0, \mathcal{F}_2) & \longrightarrow & \mathrm{Hom}(\mathcal{P}_0, \mathcal{F}_3) \end{array}$$

So consider the usual connecting map (or "snake lemma") procedure: if $\alpha \in \mathrm{Hom}(\mathcal{P}_0, \mathcal{F}_3)$ lies in $\mathrm{Ext}^0(\mathcal{O}, \mathcal{F}_3)$, then to α there corresponds and element of $\mathrm{Hom}(\mathcal{P}_0, \mathcal{F}_2)$, and β above is precisely one such example. Then β maps precisely to τ as above in $\mathrm{Hom}(\mathcal{P}_1, \mathcal{F}_2)$, which maps to 0 in $\mathrm{Hom}(\mathcal{P}_1, \mathcal{F}_3)$ and therefore comes precisely from what we above called $\sigma \in \mathrm{Hom}(\mathcal{P}_1, \mathcal{F}_1)$. Hence the map $\alpha \mapsto \sigma$ is precisely the usual connecting map or "snake lemma" construction.

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