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On Two Families of Generalized Stirling Numbers Representable by Bell Polynomials

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Abstract: We study two families related to the Stirling numbers: generalized Bessel numbers arising from truncation, and the coefficient terms in the expansion of the differential operator $(x^p \frac{d}{dx})^n$ first described by Scherk (1823). In this context, the partial Bell polynomials and their inverse counterparts prove to be a natural methodological framework for the investigation. With their help, we obtain new proofs of known results as well as some new findings concerning recurrences, explicit representations, and various inversion relations.

Keywords: Akiyama-Tanigawa identity; Bell polynomials; Generalized Bessel numbers; Generalized Stirling numbers; Multivariate Stirling polynomials; Scherk numbers
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1. Introduction and preliminaries

1.1 Exponential families

Let $a_1, a_2, a_3, \ldots$ be an arbitrary sequence of numbers and $\varphi(x) = a_1 x + a_2 x^2/2! + a_3 x^3/3! + \cdots$ the corresponding exponential generating function (EGF). For any integer $k \geq 1$, this defines the formal power series

$$\frac{\varphi(x)^k}{k!} = \sum_{n \geq k} b_{n,k} \frac{x^n}{n!}, \tag{1}$$

whose Taylor coefficients $b_{n,k}$ can, as is well known, be expressed as polynomials in the original coefficients:

$$b_{n,k} = B_{n,k}(a_1, a_2, \ldots, a_{n-k+1}). \tag{2}$$

Here, $B_{n,k}$ denotes the (partial) *exponential* polynomials described by Bell [4], which are already defined by the conditions (1) and (2) [10, p. 133]. Accordingly, in what follows, an *exponential family* is understood to be any doubly indexed sequence of expressions that can be represented by Bell polynomials according to the pattern in (2). In [38, Sect. 5], this form of representability is discussed in greater detail and proven for a number of polynomial families. In contrast, the present work focuses primarily on *exponential number families* (ENF), specifically those that can be interpreted as generalized Stirling numbers.

The Stirling numbers of the first and second kind, which we will denote by $s_1(n,k)$ and $s_2(n,k)$ respectively, are prominent examples of ENFs. Thus, the constant sequence $1, 1, 1, \ldots$ (represented by the EGF $\varphi(x) = e^x - 1$) directly yields $B_{n,k}(1, \ldots, 1) = s_2(n,k)$. The inverse function of φ, $\overline{\varphi}(x) = \log(1+x)$, has Taylor coefficients $\overline{a}_n = (-1)^{n-1}(n-1)!$, $n \geq 1$, which yields the signed Stirling numbers of the first kind:

$$B_{n,k}(\overline{a}_1, \ldots, \overline{a}_{n-k+1}) = B_{n,k}(0!, -1!, 2!, \ldots) = (-1)^{n-k} c(n,k) = s_1(n,k), \tag{3}$$

where $c(n,k)$ denotes the number of n-permutations with exactly k cycles [10, Thm. B (3g) & (3i), p. 135].

The classical Stirling numbers can be generalized in various ways, typically through certain modifications of a generating procedure and the associated parameters. This can be done, for example, by specifying an integer parameter $m \geq 1$ and modifying the generating sequence $a_n = 1$ $(n \geq 1)$ with the additional condition $a_n = 0$

for $n > m$. The ENFs of m-truncated Stirling numbers resulting in this way include the Bessel numbers (for $m = 2$) and are therefore also referred to here as *generalized Bessel numbers of level m*.

An alternative method of generalization starts from the following well-known formula for the expansion of the differential operator $(xD)^n$ (where $D(f)$ denotes the derivative of the function f):

$$(xD)^n = \sum_{k=0}^{n} s_2(n,k) x^k D^k. \tag{4}$$

In (4), Stirling numbers of the second kind appear as expansion coefficients; Mansour and Schork refer to them as *normal ordering coefficients*, in keeping with the focus of their monograph [23]. In Section 4.1.1 of that work, they consider the more general operator $(x^p D^q)^n$ and provide an overview of the investigations [6, 25, 45]—which began in the 1930s—regarding the resulting coefficients $S_{p,q}(n,k)$. Later on, we will show that this two-parameter family of generalized Stirling numbers (of the second kind) is an ENF only for $q = 1$. According to [23, p. 13], the first results on $S_{p,1}(n,k)$ were obtained as early as the 19th century, primarily by H. F. Scherk [34, 35], whom we wish to honor with our naming convention (in Section 4).

If one wishes to introduce or characterize the number families of the first kind by analogy with (4), one must, in the reverse direction, represent the operator $x^n D^n$ as a linear combination of the $(xD)^k$, $0 \le k \le n$, where $s_1(n,k)$ now appear as expansion coefficients. As Lang [22] has shown, this approach also leads to the inverse counterparts of the Scherk family $S_{p,1}(n,k)$ (see [23, p. 82–83] as well as the subsequent references to Toscano's preliminary work).

This approach can be reconciled with the method described at the beginning, which involves introducing an ENF via an EGF φ and its inverse $\overline{\varphi}$. The reasoning outlined below provides the key to this. Following the example of the classical Stirling numbers, let us call the $b_{n,k}$ defined by (1) a family of the second kind. Assuming $a_1 \neq 0$, φ has an inverse $\overline{\varphi}(x) = \overline{a}_1 x + \overline{a}_2 x^2/2! + \overline{a}_3 x^3/3! + \cdots$; the Taylor coefficients in

$$\frac{\overline{\varphi}(x)^k}{k!} = \sum_{n \ge k} \overline{b}_{n,k} \frac{x^n}{n!} \tag{5}$$

then form the corresponding ENF of the first kind

$$\overline{b}_{n,k} = B_{n,k}(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{n-k+1}). \tag{6}$$

The next and crucial step is to replace the right-hand side of (6) with a universal expression $A_{n,k}(a_1, a_2, \ldots, a_{n-k+1})$ that computes $\overline{b}_{n,k}$ directly from the Taylor coefficients of φ. As we will show in Section 1.3, this is in fact accomplished by a uniquely determined family of Laurent polynomials $A_{n,k}$ in the indeterminates $X_1^{-1}, X_2, \ldots, X_{n-k+1}$. In these expressions, the case-by-case determination of the inverse function and the evaluation of the Bell polynomials for their coefficients according to the pattern of (3) are merged into a single operation. In particular, the Stirling numbers of the first kind now follow directly as $s_1(n,k) = A_{n,k}(1, \ldots, 1)$. In total, we thus have two (as will be seen: mutually inverse) families of polynomials $A_{n,k}$ and $B_{n,k}$, whose coefficients sum to $s_1(n,k)$ and $s_2(n,k)$ respectively, and which can therefore justifiably be referred to as *multivariate Stirling polynomials* (MSP) *of the first and second kind*, respectively [37].

Finally, MSPs play a key role when it comes to representing the coefficients that appear in the expansions of certain differential operators (as in (4)). The basic idea is to restrict ourselves to derivations D_φ that are formed in a specific way from an invertible EGF φ. D_φ^n then turns out to be a linear combination of the D^k with coefficients $A_{n,k}(a_1, \ldots, a_{n-k+1})$, where $a_1 (\neq 0), a_2, a_3, \ldots$ are the Taylor coefficients of φ. Conversely, D^n can be represented in a similar manner by D_φ^k with coefficients $B_{n,k}(a_1, \ldots, a_{n-k+1})$. All coefficients appearing in this context are ENFs, including those formed with $A_{n,k}$, as can be immediately seen from $A_{n,k}(a_1, \ldots, a_{n-k+1}) = B_{n,k}(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_{n-k+1})$.

1.2 Content overview

To the extent necessary for continuing the line of reasoning presented so far in the introduction, the reader will find in Section 1.3 below a compilation of additional terms and theorems, as well as references to their systematic development. In Section 2, we build upon the work of E. Jabotinsky [16–18] on the composition of functions and, on this basis, formulate a general representation theorem for ENFs. In the same context, we obtain, among other things, new results on the substitution of binomial sequences in MSPs, as well as new proofs of known identities related to a lemma by Akiyama-Tanigawa. Section 3 deals with the generalization of the classical Bessel numbers in the form of truncated Stirling numbers. In Section 4, Scherk numbers of the first and second kind are introduced via the method outlined in the introduction, which involves expanding powers of derivations of the type D_φ into powers of D (and vice versa). The concluding Section 5 deals with various types of inverse relationships that are inherited from the MSPs—by specializing the indeterminates therein—to

the families of generalized Stirling numbers examined here: identities of the Schlömilch-Schläfli type, as well as the extension of the indices to the entire set of $\mathbb{Z}$ and the general reciprocity of the families of the first and second kind that can be expressed thereby.

1.3 Basic notions and facts

We will first outline the conceptual framework for everything that follows and establish some notation conventions as well as assumptions that will be used throughout. For background information, technical details, and proofs, the reader is referred to [37, 38] and [39].

Let $\mathcal{K}$ denote a commutative field of characteristic 0, here: $\mathcal{K} = \mathbb{R}$ or $\mathcal{K} = \mathbb{C}$, and let $\mathcal{F} = \mathcal{K}[[x]]$ denote the algebra of formal power series with coefficients in $\mathcal{K}$. We refer to the elements $f \in \mathcal{F}$ briefly as *functions* and denote them as the EGF of a sequence $f_0, f_1, f_2, \ldots \in \mathcal{K}$, i.e., in the form of a Taylor series at 0:

$$f = f(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}. \tag{7}$$

The functions f with $f(0) = f_0 = 0$ form a subalgebra $\mathcal{F}_0$ of $\mathcal{F}$ (without identity).

The composition $\circ$ of functions is a partial operation on $\mathcal{F}$; however, if we restrict it to the subalgebra $\mathcal{G}$ of compositionally invertible functions, we obtain a non-Abelian group $(\mathcal{G}, \circ)$ with identity element $\iota := x$. The inverse of $f \in \mathcal{G}$ is denoted by $\overline{f}$, and its corresponding Taylor coefficients by $\overline{f}_k$. Recall that a function (7) is invertible if and only if $f_0 = 0$ and $f_1 \neq 0$ [48, p. 29]. We have $\overline{f}_1 = 1/f_1$.

To be able to form differential operators on $\mathcal{F}$, we define a mapping $D : \mathcal{F} \to \mathcal{F}$ by

$$D(f)(x) := \sum_{n\geq 0} f_{n+1} \frac{x^n}{n!}.$$

D is a (the only!) normed derivation on $\mathcal{F}$, i.e., $D(\iota) = 1$, and the well-known rules for the sum and product of functions hold: $D(f+g) = D(f) + D(g)$ and $D(fg) = D(f)g + fD(g)$. In cases where composition is defined, the chain rule $D(f \circ g) = (D(f) \circ g)D(g)$ also holds. Where appropriate, we occasionally write the derivative $D(f)$ in the conventional form as f'. As usual, D^n denotes the operator D applied n times for all $n \geq 0$. For the coefficients in (7), this yields: $f_n = D^n(f)(0)$ $(n = 0, 1, 2, \ldots)$.

Let us now consider some special functions. The exponential $e^x = \exp(x) = 1 + x + x^2/2! + x^3/3! + \cdots$ belongs to $\mathcal{F}$, but not to $\mathcal{G}$. The common logarithm log, on the other hand, does not even have a counterpart in $\mathcal{F}$. Instead, we therefore use the modified exponential $\varepsilon(x) := e^x - 1$ and the modified logarithm function

$$\lambda(x) := \log(1+x) = \sum_{n\geq 1} (-1)^{n-1}(n-1)! \frac{x^n}{n!}.$$

ε and λ are (compositionally) inverse elements of each other in $\mathcal{G}$.

When studying generalized Bessel numbers, the following m-truncated versions prove to be useful:

$$\varepsilon_m(x) := \sum_{n=1}^{m} \frac{x^n}{n!} \quad \text{and} \quad \lambda_m := \overline{\varepsilon}_m \qquad (m \geq 1 \text{ integer}).$$

Unlike $\varepsilon_m(x)$, $\lambda_m(x)$ is not a polynomial in x, but a power series whose first m terms coincide with those of $\lambda(x)$.

Let $\varphi \in \mathcal{F}$ and Q be a polynomial or Laurent polynomial in a finite number of the variables $X_1^{-1}, X_1, X_2, X_3, \ldots$. We denote by Q^φ the result of replacing X_j with $D^j(\varphi)$ for all variable indices j occurring in Q. If the result is well-defined, we obtain a function $Q^\varphi := Q(D(\varphi), D^2(\varphi), D^3(\varphi), \ldots) \in \mathcal{F}$. For $\varphi \in \mathcal{G}$, this is guaranteed, since then $D(\varphi)$ is a unit in the ring $\mathcal{F}$ and thus the multiplicative inverse $D(\varphi)^{-1}$ exists. In this case, we define a mapping $D_\varphi : \mathcal{F} \to \mathcal{F}$ by

$$D_\varphi(f) := \frac{D(f)}{D(\varphi)}. \tag{8}$$

D_φ is a (generally unnormalized) derivation on $\mathcal{F}$; we call $D_\varphi(f)$ the *derivative of f with respect to φ*. Higher-order derivatives can be expressed as linear combinations whose coefficients are formed from the MSPs and are related to one another as follows:

$$D_\varphi^n = \sum_{k=0}^{n} A_{n,k}^{\varphi} D^k, \qquad D^n = \sum_{k=0}^{n} B_{n,k}^{\varphi} D_\varphi^k, \tag{9}$$

$$A_{n,k}^{\varphi} = B_{n,k}^{\overline{\varphi}} \circ \varphi, \qquad B_{n,k}^{\varphi} = A_{n,k}^{\overline{\varphi}} \circ \varphi. \tag{10}$$

The identities (10) have polynomial counterparts:

$$A_{n,k} = B_{n,k}(A_{1,1}, \ldots, A_{n-k+1,1}) \quad \text{and} \quad B_{n,k} = A_{n,k}(A_{1,1}, \ldots, A_{n-k+1,1}). \tag{11}$$

For the partial Bell polynomials, we have the long-known explicit representation [26, 32]:

$$B_{n,k} = \sum_{\mathbb{P}(n,k)} \frac{n!}{r_1! r_2! \, \ldots (1!)^{r_1} (2!)^{r_2} \cdots} X_1^{r_1} X_2^{r_2} \cdots X_{n-k+1}^{r_{n-k+1}}. \tag{12}$$

The sum extends over all elements $(r_1, \ldots, r_{n-k+1})$ of the set $\mathbb{P}(n,k)$ of (n,k)-partition types, i.e., the sequences of integers $r_1, r_2, r_3, \ldots \geq 0$ such that $r_1 + r_2 + r_3 + \cdots = k$ and $r_1 + 2r_2 + 3r_3 + \cdots = n$.

In addition, we note the following useful corollary:

$$\frac{\partial B_{n,k}}{\partial X_j} = \binom{n}{j} B_{n-j,k-1} \qquad (1 \leq j \leq n-k+1). \tag{13}$$

The inverse companions $A_{n,k}$ can also be described by a similarly constructed summation term, in which the special role of the indeterminate X_1 stands out:

$$A_{n,k} = X_1^{-(2n-1)} \sum_{\mathbb{P}(2n-1-k,n-1)} \frac{(-1)^{n-1-r_1}(2n-2-r_1)!}{(k-1)! r_2! r_3! \cdots (2!)^{r_2} (3!)^{r_3} \cdots} X_1^{r_1} X_2^{r_2} \cdots X_{n-k+1}^{r_{n-k+1}}. \tag{14}$$

Remark 1.1. *Formula* (14) *is derived in [37, Sect. 6] using differential recursions that arise in the context of the expansion theorems* (9)*. Previously, Todorov, who also uses the operator* (8)*, had expressed $A_{n,k}$ as a determinant [43, Thm. 6]; see also [39, p. 128]. Later, in [44], he proves a recurrence for $A_{n,n-k}$ and provides an evaluation for $k = 1, 2, 3$. All of these results were originally discovered independently of one another, and moreover without knowledge of Platonov's works [28, 29], published in Russian starting in 1975, in which he (possibly for the first time) explicitly represents the inverse companions of the Bell polynomials. In [30, Sect. 1.2], he cites the relevant (slightly different from* (14) *but equivalent) summation formula over the partitions $\mathbb{P}(2n-2k, n-k)$. In the special case $k = 1$, according to* (5)*, it yields the coefficients of inverted functions; see [37, Rem. 6.1, Cor. 7.2] and [39, p. 54–56].*

From (12) and (14), we can easily derive simple but useful extraction rules that hold for any $a, b \in \mathbb{Z}$:

$$B_{n,k}(t^{a+b}X_1, t^{2a+b}X_2, t^{3a+b}X_3, \ldots) = t^{an+bk} B_{n,k}(X_1, X_2, X_3, \ldots), \tag{15}$$

$$A_{n,k}(t^{a+b}X_1, t^{2a+b}X_2, t^{3a+b}X_3, \ldots) = t^{-ak-bn} A_{n,k}(X_1, X_2, X_3, \ldots). \tag{16}$$

Example 1.1. *To illustrate how the formulas presented so far work, let us consider the case $\varphi = \lambda$ and the corresponding derivation $D_\lambda = (1+\iota)D$. To express D_λ^n according to* (9)*, we first calculate*

$$D^j(\lambda) = (-1)^{j-1}(j-1)!(1+\iota)^{-j} = s_1(j,1)(1+\iota)^{-j} \qquad (j = 1, 2, 3, \ldots).$$

Thus, by (16) *(with $t = 1+\iota$, $a = -1$, $b = 0$) and by the second identity in* (11)*, the expansion coefficient is*

$$\begin{aligned} A_{n,k}^{\lambda} &= A_{n,k}(s_1(1,1)(1+\iota)^{-1}, s_1(2,1)(1+\iota)^{-2}, \ldots) \\ &= A_{n,k}(s_1(1,1), s_1(2,1), \ldots))(1+\iota)^k \\ &= A_{n,k}(A_{1,1}(1,\ldots,1), A_{2,1}(1,\ldots,1), \ldots)(1+\iota)^k \\ &= B_{n,k}(1,1,\ldots,1)(1+\iota)^k \\ &= s_2(n,k)(1+\iota)^k. \end{aligned}$$

The resulting equation $((1+\iota)D)^n = \sum_{k=0}^n s_2(n,k)(1+\iota)^k D^k$ obviously agrees with (4) *(except that there x takes on the role of the function $1+\iota$ used here).*

Using very similar reasoning, one verifies that $B_{n,k}^{\lambda} = s_1(n,k)(1+\iota)^{-n}$ thus obtaining the relation inverse to (4)*: $(1+\iota)^n D^n = \sum_{k=0}^n s_1(n,k)((1+\iota)D)^k$.*

The following *inversion law* for MSPs constitutes a maximally strong generalization of the well-known inversion relation between the classical Stirling numbers of the first and second kind:

$$\sum_{j=k}^{n} A_{n,j} B_{j,k} = \sum_{j=k}^{n} B_{n,j} A_{j,k} = \delta_{nk} \qquad \text{for all } n \geq k \geq 1. \tag{17}$$

It is clear that for every ENF and its associated inverse companion, one obtains an equally structured inversion law by specializing the indeterminates in (17).

We conclude this section with two fundamental rules concerning the composition of functions.
The *Faà di Bruno formula* generalizes the standard chain rule to higher-order derivatives:

$$D^n(f\circ\varphi)=\sum_{k=0}^{n}(D^k(f)\circ\varphi)\cdot B_{n,k}^{\varphi}\qquad(n\geq 0). \tag{18}$$

The *Jabotinsky formula* describes the behavior of Bell polynomials whose indeterminates are replaced by the Taylor coefficients of a composition product:

$$B_{n,k}^{\varphi\circ\psi}(0)=\sum_{j=k}^{n}B_{n,j}^{\psi}(0)B_{j,k}^{\varphi}(0). \tag{19}$$

See [38, Thm. 4.7, Rem. 4.5] for more details. From (19), one obtains the inversion law (17) specifically for $\psi=\overline{\varphi}$.

2. General identities involving exponential families

2.1 Representation of exponential number families

The ENF $f(n,k):=B_{n,k}(\varphi_1,\ldots,\varphi_{n-k+1})$ associated with any $\varphi\in\mathcal{F}_0$ has the form of a lower triangular matrix (since $B_{n,k}=0$ for $n<k$). For $\varphi\in\mathcal{G}$, $B_{n,n}(\varphi_1)=\varphi_1^n\neq 0$, and thus the matrix is regular. The set of these regular matrices forms a group (the *Jabotinsky group* representing $\mathcal{G}$, a historical precursor and subgroup of the Riordan group; see, for example, [23, p. 46]).

As a basis for the explicit determination of the matrix elements, we make use of the composition rule (19) in what follows. It turns out that many identities expressing a given ENF in terms of (in general two) other ENFs correspond to a matrix equation $F=GH$. In this setting, for instance, F and G may be prescribed, so that the matrix H can be obtained simply as $H=\overline{G}F$, provided that G is regular and $\overline{G}$ denotes its inverse.

The following *general representation theorem* describes the details of this procedure in the terminology of multivariate Stirling polynomials:

Theorem 2.1. *Let $f(n,k)$ and $g(n,k)$ be any two exponential number families, and let $g(n,n)\neq 0$ for all $n\geq 1$. Then there exists a unique exponential number family $h(n,k)$ such that*

$$f(n,k)=\sum_{j=k}^{n}g(n,j)h(j,k). \tag{20}$$

In this case, we have: $h(j,k)=B_{j,k}(a_1,\ldots,a_{j-k+1})$ *where*

$$a_r=\sum_{i=1}^{r}f(i,1)A_{r,i}(g(1,1),\ldots,g(r-i+1,1)),\ r\geq 1.$$

Proof. We write the ENFs in the form $f(n,k)=B_{n,k}^{\varphi}(0)$ and $g(n,k)=B_{n,k}^{\psi}(0)$ with $\varphi,\psi\in\mathcal{F}_0$. The function φ is uniquely determined, since $f(j,1)=\varphi_j$ for $j\geq 1$ and hence $\varphi(x)=\sum_{n\geq 1}f(n,1)x^n/n!$. The analogous statement holds for $\psi(x)=\sum_{n\geq 1}g(n,1)x^n/n!$. Moreover, by assumption, $0\neq g(n,n)=B_{n,n}^{\psi}(0)=\psi'(0)^n$ for $n\geq 1$, so that $\psi_1\neq 0$ and thus $\psi\in\mathcal{G}$. We now form the decomposition $\varphi=(\varphi\circ\overline{\psi})\circ\psi$ and apply (19). This yields

$$B_{n,k}^{\varphi}(0)=\sum_{j=k}^{n}B_{n,j}^{\psi}(0)B_{j,k}^{\varphi\circ\overline{\psi}}(0)$$

and thus

$$f(n,k)=\sum_{j=k}^{n}g(n,j)B_{j,k}(a_1,\ldots,a_{j-k+1}),$$

where $a_r=D^r(\varphi\circ\overline{\psi})(0)$ for $r\geq 1$. Thus, with $h(j,k):=B_{j,k}(a_1,\ldots,a_{j-k+1})$, we obtain the desired ENF depending solely on the uniquely determined functions φ and ψ.

It remains to determine the exact form of the arguments a_r by first evaluating the higher-order derivative of the composite function $\varphi \circ \overline{\psi}$:

$$\begin{aligned} D^r(\varphi \circ \overline{\psi}) &= \sum_{i=1}^{r} (D^i(\varphi) \circ \overline{\psi}) \cdot B_{r,i}^{\overline{\psi}} && \text{(by (18))} \\ &= \sum_{i=1}^{r} (D^i(\varphi) \circ \overline{\psi}) \cdot (A_{r,i}^{\psi} \circ \overline{\psi}) && \text{(by (10)).} \end{aligned}$$

Since $\varphi(0) = \overline{\psi}(0) = 0$, this yields

$$a_r = D^r(\varphi \circ \overline{\psi})(0) = \sum_{i=1}^{r} D^i(\varphi)(0) A_{r,i}^{\psi}(0)$$

and finally, taking into account $D^i(\varphi)(0) = f(i,1)$ and $D^i(\psi)(0) = g(i,1)$ for $i \geq 1$, we obtain the claimed expression for a_r. □

The application of Theorem 2.1 will be illustrated by two well-known examples.

Example 2.1. *The signed Lah numbers are given explicitly by* $l(n,k) = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}$ *(see [21] and [32, p. 43–44]). Their well-known representation in terms of Stirling numbers*

$$l(n,k) = \sum_{j=k}^{n} (-1)^j s_1(n,j) s_2(j,k) \tag{21}$$

has the same structure as (20) *and can in fact be derived from Theorem 2.1. To this end, we set* $f(n,k) = l(n,k)$ *and* $g(n,k) = (-1)^k s_1(n,k)$*. Both expressions are ENFs, since according to [10, p. 135]:* $l(n,k) = B_{n,k}(-1!, 2!, -3!, \ldots)$ *and* $(-1)^k s_1(n,k) = B_{n,k}(-0!, -1!, -2!, \ldots)$*; moreover,* $g(n,n) = (-1)^n \neq 0$*. Theorem 2.1 therefore yields a unique ENF* $h(n,k) = B_{n,k}(a_1, \ldots, a_{n-k+1})$ *satisfying* (20) *with*

$$a_r = \sum_{i=0}^{r} l(i,1) A_{r,i}(-s_1(1,1), -s_1(2,1), -s_1(3,1), \ldots).$$

We show $a_r = 1$ *for all* $r \geq 1$*, from which it follows that* $h(n,k) = B_{n,k}(1, \ldots, 1) = s_2(n,k)$*, and hence* (21)*.*

We have $l(i,1) = (-1)^i i!$*. Moreover, by* (16) *(for* $t = -1$*,* $a = 0$*,* $b = 1$*) and the second identity in* (11)

$$A_{r,i}(-s_1(1,1), -s_1(2,1), -s_1(3,1), \ldots) = (-1)^{-r} B_{r,i}(1,1,1,\ldots),$$

so that $a_r = \sum_{i=0}^{r} i!(-1)^{i-r} s_2(r,i)$*. For the cycle numbers* $c(r,i) = (-1)^{r-i} s_1(r,i)$*, their combinatorial interpretation implies* $\sum_{i=0}^{r} (-1)^{r-i} s_1(r,i) = r!$*, from which we get by Stirling inversion* $(-1)^r = \sum_{i=0}^{r} i!(-1)^i s_2(r,i) = (-1)^r a_r$*, and thus* $a_r = 1$*.*

Example 2.2. *Among the numerical specializations of the Bell polynomials listed by Comtet [10, p. 135] is the identity*

$$B_{n,k}(1,2,3,\ldots) = \binom{n}{k} k^{n-k}.$$

The right-hand side counts the number of idempotent self-maps of $\{1, \ldots, n\}$ *with exactly* k *cycles (*idempotent numbers*, [14]). We denote this ENF by* $g(n,k)$ *and consider the problem of determining its inverse companion* $h(n,k)$*, uniquely specified by Theorem 2.1, satisfying* $\sum_{j=k}^{n} g(n,j)h(j,k) = \delta_{nk}$*. Note that* $f(n,k) := \delta_{nk} = B_{n,k}(1,0,\ldots,0)$ *is itself an ENF.*

We first obtain the ENF in question in the form $h(n,k) = B_{n,k}(a_1, \ldots, a_{n-k+1})$*, where*

$$a_r = \sum_{i=0}^{r} \delta_{i1} A_{r,i}(g(1,1), g(2,1), g(3,1), \ldots) = A_{r,1}(1,2,3,\ldots).$$

Now, the sequence $1, 2, 3, \ldots$ *consists of the Taylor coefficients of* $\varphi(x) := xe^x$*. Hence, by* (10)*,* $a_r = A_{r,1}^{\varphi}(0) = B_{r,1}^{\overline{\varphi}}(0) = D^r(\overline{\varphi})(0)$*, that is,* a_r *is the* r*th Taylor coefficient of the inverse function* $\overline{\varphi}$ *(which is the well-known Lambert W-function). A straightforward computation using the Lagrange inversion formula yields*

$$a_r = D^{r-1}\left(\left(\frac{x}{\varphi(x)}\right)^r\right)\Big|_{x=0} = D^{r-1}(e^{-rx})\big|_{x=0} = (-1)^{r-1} r^{r-1}.$$

Khelifa and Cherruault [19] evaluated the Bell polynomials when the variables are specialized to the tree numbers r^{r-1}. The resulting expression for $h(n,k)$ is

$$h(n,k) = (-1)^{n-k} B_{n,k}(1^0, 2^1, 3^2, \ldots) = (-1)^{n-k} \binom{n-1}{k-1} n^{n-k}.$$

Here, $|h(n,k)|$ denotes the number of forests with k connected components on n labeled vertices.

Remark 2.1. *The inverse relationship between the ENFs g and h was already observed by Riordan [33] and studied in more detail by Wang and Wang [47]. Abbas and Bouroubi [1] provided a considerably shorter proof for the formula of Khelifa and Cherruault; see also [26, 27]. In [38, Sect. 5.4], all of these results are generalized to multivariate polynomials.*

2.2 Substitution of binomial sequences

It is a well-known fact that the Taylor coefficients $f_n(t) := [x^n/n!]e^{t\varphi(x)}$ (where $\varphi \in \mathcal{G}$) form a polynomial sequence of binomial type. That is, $f_0(t) \equiv 1$, $\deg f_n = n$, and $f_n(s+t) = \sum_{k=0}^{n} \binom{n}{k} f_{n-k}(s) f_k(t)$. Conversely, every sequence $f_n(t) \in \mathcal{K}[t]$ with these properties can be represented in this way as Taylor coefficients of $e^{t\varphi(x)}$ by choosing $\varphi(x) := \sum_{n\geq 1} f_n'(0)\frac{x^n}{n!}$. Note that $f_1'(0) \neq 0$.

In the study of generalized Stirling numbers, it is sometimes useful to be able to evaluate expressions obtained by substituting the indeterminates in MSPs with the terms of a polynomial sequence (in particular, a sequence of binomial type). An example is the formula

$$B_{n,k}(f_1(t), \ldots, f_{n-k+1}(t)) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f_n(jt). \tag{22}$$

Remark 2.2. *In this form, (22) was proved by Yang [49, Thm. 2]. A slightly more general version was given by Mihoubi [27]. In [39, Sect. 6.2], it is shown that (22), as well as numerous other substitution identities, follow directly from identities for multivariate Faà di Bruno polynomials.*

For our purposes, we require substitution results in which the variable t of the binomial sequence is extracted from the sequence elements $f_j(t)$ ($j = 1, 2, 3, \ldots$). The relevant identities are provided by the following

Theorem 2.2. *Let $(f_n(t))_{n\geq 0}$ be an arbitrary sequence of binomial type. Then, for all $n \geq k \geq 0$, the following identities hold:*

$$\text{(i)} \qquad B_{n,k}(f_1(t), \ldots, f_{n-k+1}(t)) = \sum_{j=k}^{n} t^j B_{n,j}(f_1'(0), \ldots, f_{n-j+1}'(0)) s_2(j,k),$$

$$\text{(ii)} \qquad A_{n,k}(f_1(t), \ldots, f_{n-k+1}(t)) = \sum_{j=k}^{n} t^{-j} s_1(n,j) A_{j,k}(f_1'(0), \ldots, f_{j-k+1}'(0)).$$

Proof. We express the general term of the binomial sequence as the Taylor coefficient $f_n(t) = D^n(e^{t\varphi})(0)$ for a suitable $\varphi \in \mathcal{G}$ (see above) and obtain, by applying the Jabotinsky formula (19):

$$\begin{aligned} B_{n,k}(f_1(t), f_2(t), \ldots) &= B_{n,k}(D^1(e^{t\varphi})(0), D^2(e^{t\varphi})(0), \ldots) \\ &= B_{n,k}^{\exp \circ t\varphi}(0) = \sum_{j=k}^{n} B_{n,j}^{t\varphi}(0) B_{j,k}^{\exp}(0). \end{aligned}$$

For $j \geq 1$, one has $D^j(t\varphi)(0) = tf_j'(0)$, and since $B_{n,j}$ is a homogeneous polynomial of degree j:

$$B_{n,j}^{t\varphi}(0) = B_{n,k}(tf_1'(0), tf_2'(0), \ ...) = t^j B_{n,j}(f_1'(0), f_2'(0), \ldots).$$

Finally, $B_{j,k}^{\exp}(0) = B_{j,k}(1, 1, \ldots) = s_2(j,k)$, which establishes (i).

Statement (ii) follows by the same pattern, with the difference that, for the inverse companions $A_{n,k}$, the contravariant product in (19) must be replaced by its covariant version [38, Thm. 4.7 (ii)]. This gives us

$$\begin{aligned} A_{n,k}(f_1(t), f_2(t), \ldots) &= A_{n,k}^{\exp \circ t\varphi}(0) = \sum_{j=k}^{n} A_{n,j}^{\exp}(0) A_{j,k}^{t\varphi}(0) \\ &= \sum_{j=k}^{n} A_{n,j}(1, 1, \ldots) A_{j,k}(tf_1'(0), tf_2'(0), \ldots). \end{aligned}$$

One has $A_{n,j}(1,1,\ldots) = s_1(n,j)$, and by (16) (for $a=0$, $b=1$) and noting that $f_1'(0) \neq 0$,

$$A_{j,k}(tf_1'(0), tf_2'(0), \ldots) = t^{-j} A_{j,k}(f_1'(0), f_2'(0), \ldots),$$

which establishes (ii). □

In the following sections (2.3 and 4), the *falling factorials*, defined by $(t)_n := t(t-1)\cdots(t-n+1)$ with $(t)_0 := 1$, play an important role. For this special case of a binomial sequence, we obtain the

Corollary 2.1 (Theorem 2.2)**.**

$$\text{(i)} \qquad B_{n,k}((t)_1, (t)_2, \ldots) = \sum_{j=k}^{n} t^j s_1(n,j) s_2(j,k)$$

$$\text{(ii)} \qquad A_{n,k}((t)_1, (t)_2, \ldots) = \sum_{j=k}^{n} t^{-j} s_1(n,j) s_2(j,k)$$

$$\text{(iii)} \qquad A_{n,k}((t)_1, (t)_2, \ldots) = B_{n,k}((\tfrac{1}{t})_1, (\tfrac{1}{t})_2, \ldots)$$

$$\text{(iv)} \qquad A_{n,k}((t)_1, (t)_2, \ldots) = \frac{1}{k!} \sum_{j=0}^{n} (-1)^{k-j} \binom{k}{j} \left(\frac{j}{t}\right)_n$$

Proof. (i) & (ii): Set $f_n(t) := (t)^n$. For $i \geq 1$, one has $f_i'(0) = (-1)^{i-1}(i-1)! = s_1(i,1)$, so that

$$B_{n,j}(f_1'(0), \ldots, f_{n-j+1}'(0)) = s_1(n,j) \quad \text{and} \quad A_{j,k}(f_1'(0), \ldots, f_{j-k+1}'(0)) = s_2(j,k).$$

(iii): Follows from (i) and (ii).
(iv): Follows from (iii) and (22). □

The summand on the right-hand side of equation (i) of the corollary is clearly an exponential family (depending on t), which can also be expressed as follows using (22):

$$\sum_{j=k}^{n} t^j s_1(n,j) s_2(j,k) = \frac{1}{k!} \sum_{r=0}^{n} (-1)^{k-r} \binom{k}{r} (rt)_n. \tag{23}$$

Remark 2.3. *Stenlund [42] considers various sums having the form of the left-hand side of* (23)*. In the general case (for arbitrary t), the evaluation given there (in Theorem 4) yields a result that still involves sums over partitions. In [46], the right-hand side of* (23) *is converted to the left-hand expression in just a few steps.*

2.3 An identity of Akiyama-Tanigawa and its companion

We formulate the identity under consideration in the following

Theorem 2.3. *Let n and k be integers with $n \geq k \geq 0$. Then one has*

$$\sum_{j=k}^{n} j s_2(n,j) s_1(j,k) = (-1)^{n-k} \binom{n}{k-1}. \tag{24}$$

The formula (24) was stated as Lemma 3 by Akiyama and Tanigawa in [2]. The authors omit the proof, as they do not make use of this identity. In this context, Vasuki et al. [46] derived a related *companion identity* by applying the operator $\frac{d}{dt}(\cdots)\big|_{t=1}$ to both sides of (23). The resulting expression is as follows:

Theorem 2.4. *Let n and k be integers with $n-1 \geq k \geq 0$. Then one has*

$$\sum_{j=k}^{n} j s_1(n,j) s_2(j,k) = (-1)^{n-k+1} (n-k-1)! \binom{n}{k-1}. \tag{25}$$

In what follows, we prove Theorem 2.3 and provide a new proof of Theorem 2.4. Both proofs are based on the differentiation of a parametrized exponential family. In contrast to [46], no use is made of explicit representations of the form (23).

We begin with an auxiliary statement concerning the action of the aforementioned differential operator.

Lemma 2.1. *Let $(p_r(t))_{r\geq 1}$ be a sequence of polynomials in $\mathcal{K}[t]$ such that $p_r(1) = \delta_{r1}$ for all $r \geq 1$. Then one has*

$$\frac{d}{dt} B_{n,k}(p_1(t), \ldots, p_{n-k+1}(t))\big|_{t=1} = \binom{n}{k-1} p_{n-k+1}'(1).$$

Proof. According to the chain rule, we first have

$$\frac{d}{dt}B_{n,k}(p_1(t),\dots,p_{n-k+1}(t)) = \sum_{r=1}^{n-k+1}\frac{\partial B_{n,k}}{\partial X_r}(p_1(t),\dots,p_{n-k+1}(t))\frac{d}{dt}p_r(t).$$

We evaluate the partial derivatives on the right-hand side using (13) and, by assumption, obtain for $t=1$

$$\begin{aligned}\sum_{r=1}^{n-k+1}\binom{n}{r}B_{n-r,k-1}(1,0,\dots,0)p_r'(1) &= \sum_{r=1}^{n-k+1}\binom{n}{r}\delta_{(n-r)\,(k-1)}p_r'(1)\\ &= \binom{n}{n-k+1}p_{n-k+1}'(1) = \binom{n}{k-1}p_{n-k+1}'(1).\end{aligned}$$

□

We now proceed to the proofs of Theorem 2.3 and Theorem 2.4.

Proof of Theorem 2.3. In analogy with (23), we define:

$$g_{n,k}(t) := \sum_{j=k}^{n} t^j s_2(n,j)s_1(j,k).$$

These polynomials form an exponential family, since $t^j s_2(n,j) = B_{n,j}(t,\dots,t)$ and by [38, Cor. 4.4 (ii)]

$$g_{n,k}(t) = B_{n,k}(L_1(t,\dots,t),\dots,L_{n-k+1}(t,\dots,t)),$$

where $L_r = \sum_{i=1}^{r} s_1(i,1)B_{r,i}$ are the logarithmic polynomials (see [10, p. 141], [38, Examples 3.1 (ii)]). We define $p_r(t) := L_r(t,\dots,t)$. Then the lemma's assumption $p_r(1) = \delta_{r1}$ $(r \geq 1)$ is satisfied, and moreover

$$p_r'(t)\big|_{t=1} = \sum_{i=1}^{r} it^{i-1}s_2(r,i)s_1(i,1)\big|_{t=1} = \sum_{i=1}^{r} is_2(r,i)s_1(i,1) = (-1)^{r-1}. \tag{26}$$

Proof of (26): For the cycle numbers $c(r,i) = (-1)^{r-i}s_1(r,i)$, one has: $\sum_{i=1}^{r} c(r,i) = r!$, that is, $\sum_{i=1}^{r}(-1)^{i-1}s_1(r,i) = (-1)^{r-1}r!$. Applying Stirling inversion to this identity and taking into account that $(-1)^{i-1}(i-1)! = s_1(i,1)$ yields $\sum_{i=1}^{r}(-1)^{i-1}i!s_2(r,i) = (-1)^{r-1}$ and hence (26).

Since $g_{n,k}'(1)$ represents the left-hand side of (24), Lemma 2.1 together with $p_{n-k+1}'(1) = (-1)^{n-k}$ and $n \geq k \geq 0$, finally yields the statement of the theorem. □

Along the same lines proceeds the following

Proof of Theorem 2.4. We start directly from statement (i) of the corollary to Theorem 2.2 and set $p_r(t) := (t)_r$ $(r \geq 1)$. Then $p_r(1) = \delta_{r1}$. Moreover, for $r \geq 2$

$$p_r'(t)\big|_{t=1} = \frac{d}{dt}\sum_{i=0}^{r} s_1(r,i)t^i\big|_{t=1} = \sum_{i=1}^{r} is_1(r,i) = (-1)^r(r-2)! \tag{27}$$

The last equation in (27) can be established by a simple induction (or taken directly from [40, Thm. 4.2, for $n=1$]). Lemma 2.1 now yields, together with (27) for $r = n-k+1 \geq 2$, that is, $n-1 \geq k \geq 0$, the claimed identity (25). □

3. Bessel numbers and their generalization

3.1 The classical Bessel numbers

The doubly indexed families of Bessel numbers of the first and second kind, denoted in the following by $b_1(n,k)$ and $b_2(n,k)$, can be defined as the coefficients of the polynomial solutions $y_n(x)$ to the second-order differential equation $x^2y_n'' + (2x+2)y_n' = n(n+1)y_n$ (with $y_n(0) = 1$):

$$\begin{aligned} b_1(n,k) &= (-1)^{n-k}[x^{n-k}]y_{n-1}(x) \quad \text{(according to [15])},\\ b_2(n,k) &= [x^{n-k}]y_k(x) \qquad\qquad\quad \text{(according to [9])}.\end{aligned}$$

The fundamental properties of both families are well known. Furthermore, their recurrence relations, EGFs, and inverse relationships reveal a close connection to the Stirling numbers [9, 15, 50]. This connection is particularly

evident from their combinatorial interpretations. Thus, $b_2(n,k)$ counts the number of partitions of $\{1,2,\ldots,n\}$ into k blocks of size 1 or 2. Without this restriction on block sizes, the number of such partitions is $s_2(n,k)$. A similar correspondence holds for $|b_1(n,k)|$ and $|s_1(n,k)|$, where the underlying set has $2n-k-1$ elements and is partitioned into $n-1$ blocks. Compare this with the partitions underlying the sums in (12) and (14).

The restriction on the blocks is also reflected in the (vertical) EGFs. In this setting, the functions ε and λ (corresponding to the Stirling numbers) are simply replaced by their truncated counterparts ε_2 and $\lambda_2(=\overline{\varepsilon}_2)$, respectively (see [50, Lemma 2.2, Lemma 2.1]):

$$\frac{\varepsilon_2(x)^k}{k!} = \frac{1}{k!}\left(x+\frac{x^2}{2!}\right)^k = \sum_{n\geq k} b_2(n,k)\frac{x^n}{n!}, \tag{28}$$

$$\frac{\lambda_2(x)^k}{k!} = \frac{1}{k!}(\sqrt{1+2x}-1)^k = \sum_{n\geq k} b_1(n,k)\frac{x^n}{n!}. \tag{29}$$

In view of (1) and (2), it follows that the Bessel numbers form exponential families:

$$b_2(n,k) = B_{n,k}^{\varepsilon_2}(0) = B_{n,k}(\varepsilon_2'(0),\varepsilon_2''(0),\ldots) = B_{n,k}(1,1,0,\ldots,0), \tag{30}$$

and, taking into account (10),

$$b_1(n,k) = B_{n,k}^{\lambda_2}(0) = B_{n,k}^{\overline{\varepsilon}_2}(0) = A_{n,k}^{\varepsilon_2}(0) = A_{n,k}(1,1,0,\ldots,0). \tag{31}$$

The fact revealed here—that the Bessel numbers of the first and second kind arise from the MSPs by the specialization $X_1 = X_2 = 1$, $X_j = 0$ $(j \geq 3)$—immediately entails, upon application to (17), their well-known inverse relationship. Likewise, the truncations of the MSPs obtained from (14) and (12) directly yield

$$\begin{aligned} A_{n,k}(X_1,X_2,0,\ldots,0) &= \frac{(-1)^{n-k}(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!}X_1^{-2n+k}X_2^{n-k} \quad (1\leq k\leq n),\\ B_{n,k}(X_1,X_2,0,\ldots,0) &= \frac{n!}{2^{n-k}(2k-n)!(n-k)!}X_1^{2k-n}X_2^{n-k} \quad (1\leq [n/2]\leq k) \end{aligned} \tag{32}$$

and hence, for $X_1 = X_2 = 1$, the explicit expressions for $b_1(n,k)$ and $b_2(n,k)$ (see, for example, [15], where these identities are proved combinatorially).

From (30)–(32) we get

$$\begin{aligned} A_{n,k}^{\varepsilon_2} &= A_{n,k}(1+\iota,1,0,\ldots,0) = (1+\iota)^{-2n+k}b_1(n,k) \quad \text{and}\\ B_{n,k}^{\varepsilon_2} &= B_{n,k}(1+\iota,1,0,\ldots,0) = (1+\iota)^{2k-n}b_2(n,k). \end{aligned}$$

The expansion formulas (9) for the operators D_φ^n and D^n reduce, for $\varphi = \varepsilon_2$, to mutually inverse relations in which the Bessel numbers appear as normal ordering coefficients [23, Prop. 8.125]. Indeed, by (8), one has $D_{\varepsilon_2}^n = ((1+\iota)^{-1}D)^n$, and hence

$$((1+\iota)^{-1}D)^n = \sum_{k=0}^n A_{n,k}^{\varepsilon_2}D^k = (1+\iota)^{-2n}\sum_{k=0}^n b_1(n,k)(1+\iota)^k D^k. \tag{33}$$

In a completely analogous manner, one obtains the inverse relation:

$$(1+\iota)^n D^n = \sum_{k=0}^n b_2(n,k)(1+\iota)^{2k}((1+\iota)^{-1}D)^k. \tag{34}$$

Remark 3.1. *As already observed in Example 1.1, the function $1+\iota$ appears in the last two identities in place of the otherwise customary operator variable x. The reason for this lies in the resulting ability to evaluate the function $A_{n,k}^{\varphi}$ at 0 (as, for instance, in the proof of Theorem 2.2 or in equation* (31)*). Of course, one could instead choose $\varphi(x) = x^2/2$ in place of $\varepsilon_2(x)$, which would correspond to the operator $D_\varphi = x^{-1}D$, whose iterates were already studied by Hadwiger [13]. However, this φ is not invertible and therefore cannot be used in place of ε_2 in* (28) *through* (31)*.*

3.2 Truncated Stirling numbers

The Bessel numbers can be generalized in various ways [3, 8, 23]. If one wishes the resulting family of numbers to form an exponential family, a natural approach is to introduce them as m-truncated Stirling numbers, in analogy with (30) and (31), where $m \geq 1$ is an integer. More precisely:

$$b_2^{[m]}(n,k) := B_{n,k}^{\varepsilon_m}(0) = B_{n,k}(1,\ldots,1,0,\ldots,0), \tag{35}$$

$$b_1^{[m]}(n,k) := B_{n,k}^{\lambda_m}(0) = B_{n,k}^{\overline{\varepsilon}_m}(0) = A_{n,k}^{\varepsilon_m}(0) = A_{n,k}(1,\ldots,1,0,\ldots,0). \tag{36}$$

Note that $D^j(\varepsilon_m) = 1+\varepsilon_{m-j}$ for $1 \le j \le m$ and $D^j(\varepsilon_m) = 0$ for $j > m$, so that $D^j(\varepsilon_m)(0) = 1$ for $j = 1,\ldots,m$; thus, exactly m ones appear in the initial arguments of the MSPs in (35) and (36). For the resulting *generalized Bessel numbers of level* m, this immediately yields their characterization via EGFs, in analogy with (28) and (29):

$$\frac{\varepsilon_m(x)^k}{k!} = \sum_{n\ge k} b_2^{[m]}(n,k)\frac{x^n}{n!} \quad \text{and} \quad \frac{\lambda_m(x)^k}{k!} = \sum_{n\ge k} b_1^{[m]}(n,k)\frac{x^n}{n!}.$$

The combinatorial interpretation is also immediate from (35) and (36). For instance, $b_2^{[m]}(n,k)$ counts the partitions of $\{1,2,\ldots,n\}$ into k blocks of at most m elements. Consequently, for fixed n and k, the generalized Bessel numbers of the second kind form an increasing sequence that 'interpolates' between δ_{nk} and $s_2(n,k)$:

$$\delta_{nk} = b_2^{[1]}(n,k) < b_2(n,k) < b_2^{[3]}(n,k) < \ldots < b_2^{[n-k+1]}(n,k) = s_2(n,k).$$

Mansour and Schork introduce generalized Bessel numbers in [23, Research Problem 8.3, p. 339] starting from $s_2(n,k)$ rather than from $b_2(n,k)$. The idea is as follows: If we replace the exponential term e^t in $x^n = [t^n/n!](e^t)^x$ with its truncated series—more precisely: the mth partial sum— we obtain a polynomial of degree n:

$$f_n^{[m]}(x) := \left[\frac{t^n}{n!}\right]\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^m}{m!}\right)^x.$$

In particular, $f_n^{\infty}(x) = x^n = \sum_{k=0}^n s_2(n,k)(x)^k$. For $m < \infty$, the polynomial can also be uniquely expanded in the basis $(x)_0, (x)_1, (x)_2, \ldots$ with certain (now parametrized) coefficients $S^{[m]}(n,k)$: $f_n^{[m]}(x) = \sum_{k=0}^n S^{[m]}(n,k)(x)_k$.

Indeed, for arbitrary $m \ge 1$ and $n \ge k \ge 0$ one has

$$S^{[m]}(n,k) = b_2^{[m]}(n,k). \tag{37}$$

Proof. We have

$$\begin{aligned}\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^m}{m!}\right)^x &= \exp(x\log(1+\varepsilon_m(t))) \\ &= e^{x(\lambda\circ\varepsilon_m)(t)} = \sum_{n\ge 0}\left(\sum_{k=0}^n B_{n,k}^{\lambda\circ\varepsilon_m}(0)x^k\right)\frac{t^n}{n!}\end{aligned}$$

and hence

$$\begin{aligned} f_n^{[m]}(x) = \sum_{k=0}^n B_{n,k}^{\lambda\circ\varepsilon_m}(0)x^k &= \sum_{k=0}^n\left(\sum_{j=k}^n B_{n,j}^{\varepsilon_m}(0)B_{j,k}^{\lambda}(0)\right)x^k \qquad \text{(by (19))} \\ &= \sum_{k=0}^n\left(\sum_{j=k}^n b_2^{[m]}(n,j)s_1(j,k)\right)x^k \quad \text{(cf. Exm. 1.1)} \\ &= \sum_{j=0}^n\left(b_2^{[m]}(n,j)\sum_{k=0}^j s_1(j,k)x^k\right).\end{aligned}$$

Since $\sum_{k=0}^j s_1(j,k)x^k = (x)_j$, we obtain $f_n^{[m]}(x) = \sum_{j=0}^n b_2^{[m]}(n,j)(x)_j$ and thus the claimed equality (37). □

Remark 3.2. *For generalized Bessel numbers of level $m \ge 3$, there are no operator expansions of the form given in* (33) *and* (34)*, in which $b_1^{[m]}(n,k)$ and $b_2^{[m]}(n,k)$ appear as normal ordering coefficients. Even for $m = 3$ one obtains an identity*

$$((1+\varepsilon_2)^{-1}D)^n = (1+\varepsilon_2)^{-2n}\sum_{k=0}^n \gamma_{n,k}\cdot(1+\varepsilon_2)^k D^k,$$

which looks quite similar to (33)*, but whose coefficients $\gamma_{n,k}$ do not represent numbers, but rather an intricately structured functional term.*

3.3 Recurrences

The classical Bessel numbers satisfy recurrence relations analogous to those for the Stirling numbers:

$$b_2(n+1,k) = b_2(n,k-1) + nb_2(n-1,k-1) \qquad \text{[9, Eq. (1.3), p. 46]}, \tag{38}$$

$$b_1(n+1,k) = b_1(n,k-1) - (2n-k)b_1(n,k) \qquad \text{[8, Thm. 3.2, p. 2192]}. \tag{39}$$

The initial conditions are: $b_i(r,0) = b_i(0,r) = \delta_{r0}$ for $r \geq 0$ $(i = 1, 2)$. In fact, (38) and (39) are equivalent to each other, which can be easily verified using the transformation formula $b_2(n,k) = (-1)^{n-k} b_1(k+1, 2k-n+1)$. For a combinatorial interpretation (of (38)), see, for example, [50, p. 629].

Recurrence relations of this standard form cannot be obtained for the generalized Bessel numbers of level $m \geq 3$. We first consider the numbers of the second kind, since the situation is simpler in this case and a satisfactory solution exists that includes (38) as a special case.

Theorem 3.1. *For integers $m \geq 1$ and $n \geq k \geq 1$, one has*

$$b_2^{[m]}(n,k) = \sum_{j=1}^{\min(m,n-k+1)} \binom{n-1}{j-1} b_2^{[m]}(n-j,k-1).$$

Proof. Given that $b_2^{[m]}(n,k)$ arises directly from instantiating the variables in $B_{n,k}$ according to (35), it is natural to start from a recurrence for the Bell polynomials. A suitable formula is

$$B_{n,k} = \sum_{j=1}^{n-k+1} \binom{n-1}{j-1} B_{j,1} B_{n-j,k-1}, \tag{40}$$

which is proved in [7, p. 415] and [37, Prop. 5.5 (ii)]. Under the assignment of variables in (35), one has $B_{j,1} = X_j = 1$ for $j \leq m$ and $B_{j,1} = 0$ for $j > m$. Consequently, (40) immediately reduces to the claimed identity. □

From Theorem 3.1 it follows that $b_2^{[m]}(n,n) = b_2^{[m]}(n-1,n-1) = \cdots = b_2^{[m]}(0,0) = 1$. The case $k = 1$ is also correctly reproduced: $b_2^{[m]}(n,1) = b_2^{[m]}(0,0) = 1$ for $1 \leq n \leq m$ and $b_2^{[m]}(n,1) = b_2^{[m]}(n-m,0) = 0$ for $n > m$.

For $m = 2$, we obtain $b_2(n,k) = b_2(n-1,k-1) + (n-1)b_2(n-2,k-1)$, which is precisely the recurrence noted in (38).

We now consider a variant of (40) that exploits the fact that the $B_{n,k}$ are homogeneous polynomials of degree k. By Euler's theorem on homogeneous functions and in view of (13), one then has:

$$k \cdot B_{n,k} = \sum_{j=1}^{n-k+1} X_j \frac{\partial B_{n,k}}{\partial X_j} = \sum_{j=1}^{n-k+1} \binom{n}{j} X_j B_{n-j,k-1}.$$

From this, using the same reasoning as in the proof of Theorem 3.1, one obtains the recurrence

$$b_2^{[m]}(n,k) = \frac{1}{k} \sum_{j=1}^{\min(m,n-k+1)} \binom{n}{j} b_2^{[m]}(n-j,k-1). \tag{41}$$

In the special case $m = 2$, (41) indeed yields a recurrence that is not equivalent to (38):

$$b_2(n,k) = \frac{n}{k} b_2(n-1,k-1) + \frac{n(n-1)}{2k} b_2(n-2,k-1).$$

We finally turn to the generalized Bessel numbers of the first kind. There is some good news: the $A_{n,k}$ satisfy the same recurrence relations as the $B_{n,k}$. To see this, one simply applies the first identity in (11) to both sides of (40) (see also [37, Prop. 5.5 (i)]). The bad news is that, since the X_j are replaced by $A_{j,1}$, after the instantiation of the variables the X_j no longer reduce to 1 or 0 (for $j > m$), but instead take the values $b_1^{[m]}(j,1)$ for $j = 1, 2, 3, \ldots$. Therefore, the result for the generalized Bessel numbers of the first kind, which corresponds to Theorem 3.1, is:

Theorem 3.2. *For integers $m \geq 1$ and $n \geq k \geq 1$, one has*

$$b_1^{[m]}(n,k) = \sum_{j=1}^{n-k+1} \binom{n-1}{j-1} b_1^{[m]}(j,1) b_1^{[m]}(n-j,k-1). \tag{42}$$

Equation (42) cannot, unlike the recurrence of Theorem 3.1, reduce the case $k = 1$; rather, all values $b_1^{[m]}(j,1)$ are required. For $1 \le j \le m$, we have $b_1^{[m]}(j,1) = s_1(j,1) = (-1)^{j-1}(j-1)!$, but for $j > m$, the $b_1^{[m]}(j,1)$ are complicated expressions. This becomes evident either by evaluating them in the form $b_1^{[m]}(j,1) = A_{j,1}(1,\dots,1,0,\dots,0)$ or by expressing them via a Schlömilch-Schläfli-type summation formula (to which we return in Section 5). Due to these properties, (42) also does not yield a simple recurrence of the type given in (39) for $m = 2$. Whether a recurrence relation for $b_1^{[m]}(n,k)$ that achieves this can be found at all must remain an open question at this point.

3.4 Representation by Stirling numbers

In Example 2.1 (Section 2.1), we examined the well-known identity in which the Lah numbers are expressed in terms of Stirling numbers. Analogous representations have also been found for the Bessel numbers of both kinds:

$$b_2(n,k) = \sum_{j=k}^{n} 2^{j-k} s_1(n,j) s_2(j,k), \tag{43}$$

$$b_1(n,k) = \sum_{j=k}^{n} 2^{n-j} s_1(n,j) s_2(j,k). \tag{44}$$

Proofs of both identities can be found in [50] and [24]. Within the ENF framework developed here, they can be deduced from Theorem 2.1 (see below) or, even more directly and simply, from the corollary to Theorem 2.2. We first note, by a simple device, that the sequence $1,1,0,0,0,\dots$ can be written as $\frac{1}{2}(2)_j$ $(j = 1,2,3,\dots)$; we then obtain from part (i) of that corollary:

$$\begin{aligned} b_2(n,k) &= B_{n,k}(1,1,0,\dots,0) = B_{n,k}(\tfrac{1}{2}(2)_1, \tfrac{1}{2}(2)_2, \dots) \\ &= 2^{-k} B_{n,k}((2)_1,(2)_2,\dots) = \sum_{j=k}^{n} 2^{j-k} s_1(n,j) s_2(j,k), \end{aligned}$$

and from part (ii):

$$\begin{aligned} b_1(n,k) &= A_{n,k}(1,1,0,\dots,0) = A_{n,k}(\tfrac{1}{2}(2)_1, \tfrac{1}{2}(2)_2, \dots) \\ &= \left(\frac{1}{2}\right)^{-n} A_{n,k}((2)_1,(2)_2,\dots) = \sum_{j=k}^{n} 2^{n-j} s_1(n,j) s_2(j,k). \end{aligned}$$

We now aim to extend (43) to generalized Bessel numbers of the second kind. To ease the formulation of the corresponding theorem, we first define, for integers $m \ge 1$,

$$b^{[m]}(r) := \sum_{i=0}^{\min(m,r)} s_2(r,i) \qquad (r = 0,1,2,\dots).$$

We call the $b^{[m]}(r)$ the *m-truncated Bell numbers*. In the case $m = \infty$, these reduce to the 'ordinary' Bell numbers $b(r)$ (= the total number of partitions of a set with r elements). This yields

Theorem 3.3.

$$b_2^{[m]}(n,k) = \sum_{j=k}^{n} s_1(n,j) B_{j,k}(b^{[m]}(1),\dots,b^{[m]}(j-k+1)).$$

Proof. We set $f(n,k) := b_2^{[m]}(n,k)$ and $g(n,k) := s_1(n,k)$. Since $g(n,n) = 1 \ne 0$, Theorem 2.1 yields a unique ENF h with $b_2^{[m]}(n,k) = \sum_{j=k}^{n} s_1(n,j) h(j,k)$, where $h(j,k) = B_{j,k}(a_1,\dots,a_{j-k+1})$ and, for $r \ge 1$:

$$a_r = \sum_{i=1}^{r} b_2^{[m]}(i,1) A_{r,i}(s_1(1,1),\dots,s_1(r-i+1,1)). \tag{45}$$

We have $A_{r,i}(s_1(1,1),\dots,s_1(r-i+1,1)) = s_2(r,i)$ (which is obtained by setting all variables equal to 1 in the second formula of (11)). Since the first factor in the summand of (45) is equal to 1 for $1 \le i \le m$ and vanishes for $i > m$, it follows that $a_r = b^{[m]}(r)$. □

Finally, we consider two special cases.

Case $m = 2$. We compute the 2-truncated Bell numbers: $b^{[2]}(r) = s_2(r,1) + s_2(r,2) = 1 + (2^{r-1} - 1) = 2^{r-1}$ and obtain for the h from the proof of Theorem 3.3:

$$h(j,k) = B_{j,k}(2^0, 2^1, \ldots, 2^{j-k}) = 2^{j-k} B_{j,k}(1,1,\ldots,1) = 2^{j-k} s_2(j,k).$$

This provides another proof of (43).

Case $m = \infty$. We have $b^{[\infty]}(r) = b(r)$ and $b_2^{[\infty]}(n,k) = s_2(n,k)$. Theorem 3.3 thus yields

$$s_2(n,k) = \sum_{j=k}^{n} s_1(n,j) B_{j,k}(b(1), \ldots, b(j-k+1))$$

and, by Stirling inversion,

$$B_{n,k}(b(1), \ldots, b(n-k+1)) = \sum_{j=k}^{n} s_2(n,j) s_2(j,k).$$

This identity was proved by Yang [49, Eq. (31)] by a different method.

4. Scherk numbers

4.1 Generalized Stirling numbers

In [23, Chap. 4], a number of different approaches to generalizing Stirling numbers are discussed, including the introduction—already mentioned in Section 1.1—of the family of numbers $S_{p,q}(n,k)$ as normal ordering coefficients in the expansion, for integers p, q with $p \geq q \geq 1$,

$$(x^p D^q)^n = x^{n(p-q)} \sum_{k=q}^{nq} S_{p,q}(n,k) x^k D^k. \tag{46}$$

The following explicit form of these coefficients, cited in [23, Thm. 4.3], is due to Blasiak and Flajolet [5] and is equivalent to earlier representations obtained by Carlitz [6] and McCoy [25]:

$$S_{p,q}(n,k) = \frac{(-1)^k}{k!} \sum_{j=q}^{k} (-1)^j \binom{k}{j} \prod_{i=1}^{n} (j + (i-1)(p-q))_q. \tag{47}$$

From (47), one can now readily exclude subfamilies of $S_{p,q}(n,k)$ that do not form ENFs, that is, which cannot be represented by Bell polynomials.

Proposition 4.1. *For $p \geq q \geq 2$, the generalized Stirling numbers $S_{p,q}(n,k)$ cannot be represented by Bell polynomials.*

Proof. We argue by contradiction and assume that the $S_{p,q}(n,k)$ form an ENF. Then, according to [38, Prop. 5.1 (iii)], we have $S_{p,q}(n,n) = S_{p,q}(1,1)^n$ for all $n \geq 1$. However, for $q \geq 2$ it is clear that $S_{p,q}(1,1) = 0$, whereas, for instance, for $n = q$ one obtains from (47) the contradiction

$$S_{p,q}(q,q) = \frac{1}{q!} \prod_{i=1}^{q} (q + (i-1)(p-q))_q > 0.$$

□

In the next section, we address the natural question of whether, conversely, the remaining $S_{p,1}(n,k)$ form an ENF.

4.2 Scherk numbers as expansion coefficients

Scherk has already published on the special case $q = 1$ of the operator equation (46) since 1823 [34, 35] (see also [23, p. 12]), which is why we shall refer to the corresponding expansion coefficients as *Scherk numbers of degree p*. Combinatorial interpretations of these numbers can be found in [5].

In the following, the previous assumption $p \geq 1$ is replaced here by $p \in \mathbb{Z}$.

To derive (46) along with its inverse, we use the expansion formulas (9) for a special invertible function $\varphi = \sigma_p \in \mathcal{G}$. This immediately yields the Scherk numbers of the first and second kind as mutually inverse exponential families, as well as their vertical EGFs.

We define σ_p as follows:

$$\sigma_p(x) := \begin{cases} \frac{(1+x)^{1-p}-1}{1-p} & \text{for } p \neq 1, \\ \lambda(x) & \text{for } p = 1. \end{cases} \tag{48}$$

As a consequence, the inverse function is given by

$$\overline{\sigma}_p(x) = \begin{cases} (1+(1-p)x)^{\frac{1}{1-p}} - 1 & \text{for } p \neq 1, \\ \varepsilon(x) & \text{for } p = 1. \end{cases} \tag{49}$$

Remark 4.1. *One has $\sigma_p \to \lambda$ and $\overline{\sigma}_p \to \varepsilon$ for $p \to 1$.*

Lang [22] presents a study of the coefficients there called 'p-Stirling numbers' (carried out in a different framework and by other methods). The functions given in (48) and (49) serve, under different notation, essentially to derive the EGFs. In contrast, in the approach presented here, σ_p additionally plays a key role in representing the expansion coefficients via Bell polynomials and, later (Section 4.3), also in explicit form.

Theorem 4.1. *For $p \in \mathbb{Z}$ and integers $n \geq 0$, we have*

$$\text{(i)} \qquad D_{\sigma_p}^n = \sum_{k=0}^{n} (-1)^{n-k} A_{n,k}(1, p, p(p+1), \ldots)(1+\iota)^{n(p-1)+k} D^k,$$

$$\text{(ii)} \qquad D^n = \sum_{k=0}^{n} (-1)^{n-k} B_{n,k}(1, p, p(p+1), \ldots)(1+\iota)^{-k(p-1)-n} D_{\sigma_p}^k.$$

Proof. (i): From (9), we first have

$$D_{\sigma_p}^n = \sum_{k=0}^{n} A_{n,k}^{\sigma_p} D^k = \sum_{k=0}^{n} A_{n,k}(D(\sigma_p), D^2(\sigma_p), D^3(\sigma_p), \ldots) D^k.$$

To evaluate $A_{n,k}(\ldots)$, we compute the higher derivatives of σ_p:

$$D^j(\sigma_p) = (-1)^{j-1}(p+j-2)_{j-1}(1+\iota)^{-p-j+1} \quad \text{for } j \geq 1.$$

Clearly, the factor $(p+j-2)_{j-1}$ generates the sequence $1, p, p(p+1), \ldots$, which will eventually be substituted for the variables $X_1, X_2, X_3, \ldots$ in $A_{n,k}$. Before that, we factor out the alternating sign $(-1)^{j-1}$ according to rule (16) (with $t = -1$, $a = 1$, $b = -1$) from $A_{n,k}$, giving an overall factor of $(-1)^{n-k}$. Finally, by the same rule (with $t = 1+\iota$, $a = -1$, $b = 1-p$), the term $(1+\iota)^{-p-j+1}$ produces the overall factor $(1+\iota)^{n(p-1)+k}$.

(ii): The inverse operator equation

$$D^n = \sum_{k=0}^{n} B_{n,k}^{\sigma_p} D_{\sigma_p}^k = \sum_{k=0}^{n} B_{n,k}(D(\sigma_p), D^2(\sigma_p), D^3(\sigma_p), \ldots) D_{\sigma_p}^k$$

from (9) can be evaluated in the same way as in (i) (using rule (15) instead of (16)) and thus leads to the claimed identity (ii). The straightforward details can be omitted here. □

For the derivation with respect to σ_p, we have, according to (8),

$$D_{\sigma_p} = D(\sigma_p)^{-1} D = (1+\iota)^p D \quad \text{for all } p \in \mathbb{Z}.$$

If, in addition, one formally replaces $1+\iota$ by an operator variable x in (i) and (ii) of Theorem 4.1, one immediately obtains the mutually inverse operator identities for the Scherk numbers of degree p in the classical form (cf. (46) with $q = 1$ as well as [22, Eqs. (12) & (34)]). According to their type, we may therefore denote the latter as *generalized Stirling numbers* as follows:

$$s_1^{[p]}(n,k) := (-1)^{n-k} B_{n,k}(1, p, p(p+1), \ldots), \tag{50}$$

$$s_2^{[p]}(n,k) := (-1)^{n-k} A_{n,k}(1, p, p(p+1), \ldots). \tag{51}$$

In accordance with (1) and (2), and taking into account (10) as well as $D^j(\sigma_p)(0) = (-1)^{j-1}(p+j-2)_{j-1}$, we immediately obtain the EGFs of these number families:

$$\frac{\sigma_p(x)^k}{k!} = \sum_{n\geq k} B^{\sigma_p}_{n,k}(0)\frac{x^n}{n!} = \sum_{n\geq k} s_1^{[p]}(n,k)\frac{x^n}{n!}, \tag{52}$$

$$\frac{\overline{\sigma}_p(x)^k}{k!} = \sum_{n\geq k} B^{\overline{\sigma}_p}_{n,k}(0)\frac{x^n}{n!} = \sum_{n\geq k} A^{\sigma_p}_{n,k}(0)\frac{x^n}{n!} = \sum_{n\geq k} s_2^{[p]}(n,k)\frac{x^n}{n!}. \tag{53}$$

We collect here several statements concerning special values of p.

Proposition 4.2.

(i) $s_1^{[0]}(n,k) = s_2^{[0]}(n,k) = \delta_{nk}$

(ii) $s_1^{[-1]}(n,k) = b_2(n,k)$ *and* $s_2^{[-1]}(n,k) = b_1(n,k)$

(iii) $s_1^{[1]}(n,k) = s_1(n,k)$ *and* $s_2^{[1]}(n,k) = s_2(n,k)$

(iv) $s_2^{[2]}(n,k) = (-1)^{n-k}s_1^{[2]}(n,k) = |l(n,k)|$

Proof. (i), (ii): Immediately from (52), (53), (28), and (29).

(iii): By (50), (15), and (3), we have $s_1^{[1]}(n,k) = B_{n,k}(0!,-1!,2!,\ldots) = s_1(n,k)$. Using (51) and (16), it follows that $s_2^{[1]}(n,k) = A_{n,k}(0!,-1!,2!,\ldots) = A_{n,k}(s_1(1,1),s_1(2,1),s_1(3,1),\ldots) = s_2(n,k)$.

(iv): The function $\varphi := -\iota/(1+\iota) \in \mathcal{G}$ is involutory, that is, $\overline{\varphi} = \varphi$. Hence, for its Taylor coefficients $\varphi_j = (-1)^j j!$, it follows from (10) that $B_{n,k}(\varphi_1,\varphi_2,\ldots) = A_{n,k}(\varphi_1,\varphi_2,\ldots)$. Thus, by (51), (50), and the rules (15), (16), we obtain

$$\begin{aligned} s_2^{[2]}(n,k) &= (-1)^{n-k}A_{n,k}(1!,2!,3!,4!,\ldots) = (-1)^n A_{n,k}(-1!,2!,-3!,4!\ldots) \\ &= (-1)^n B_{n,k}(-1!,2!,-3!,4!\ldots) = B_{n,k}(1!,2!,3!,4!\ldots) = (-1)^{n-k}s_1^{[2]}(n,k). \end{aligned}$$

On the other hand, it follows from Example 2.1 (Section 2.1) that $B_{n,k}(1!,2!,3!,\ldots) = |l(n,k)|$. □

Remark 4.2. *For $p=-1$, the statements of Theorem 4.1 reduce to the well-known operator identities* (33) *and* (34) *for the classical Bessel numbers of the first and second kind.*

Remark 4.3. *The well-known recurrence relations for the Scherk numbers of both kinds (see [23, Eqs. (4.5) & (4.14)]) can also be derived directly from Theorem 4.1. To this end, one compares—for instance, for the family $s_2^{[p]}$—the coefficients $s_2^{[p]}(n+1,k)$ appearing in $D_{\sigma_p}^{n+1}$ with those arising from applying the derivation D_{σ_p} to the expansion of $D_{\sigma_p}^n$. The same applies* mutatis mutandis *to the family $s_1^{[p]}$.*

4.3 Explicit representations

For $p \geq q = 1$, an explicit expression for the Scherk numbers of the second kind of degree p can be obtained from (47). In the following, we develop several new explicit representations for the Scherk numbers of both kinds for arbitrary parameter $p \in \mathbb{Z}$.

We begin with formulas that are structured analogously to the identities (43) and (44), and which moreover include the latter as special cases.

Theorem 4.2. *For arbitrary $p \in \mathbb{Z}$, we have:*

(i) $$s_1^{[p]}(n,k) = \sum_{j=k}^{n}(1-p)^{j-k}s_1(n,j)s_2(j,k)$$

(ii) $$s_2^{[p]}(n,k) = \sum_{j=k}^{n}(1-p)^{n-j}s_1(n,j)s_2(j,k)$$

Proof. First assume $p \neq 1$. (i): We start from (50) and represent the sequence $1, p, p(p+1), \ldots$ appearing in the argument of $B_{n,k}$ by $(-1)^{j-1}\frac{(1-p)_j}{1-p}$ $(j = 1,2,3,\ldots)$. Using (15), this leads to

$$s_1^{[p]}(n,k) = (1-p)^{-k}B_{n,k}((1-p)_1,(1-p)_2,(1-p)_3,\ldots). \tag{54}$$

We now apply statement (i) of the corollary to Theorem 2.2 to the term $B_{n,k}(\ldots)$ on the right-hand side and thus obtain claim (i).

(ii): Starting from (51), one obtains in the same way, using (16),

$$s_2^{[p]}(n,k) = (1-p)^n A_{n,k}((1-p)_1, (1-p)_2, (1-p)_3, \ldots). \tag{55}$$

In this case, statement (ii) of the corollary yields the asserted result.

Finally, let $p = 1$. With the convention $0^0 := 1$, statements (i) and (ii) then reduce to statement (iii) of Proposition 4.2. □

Remark 4.4. *Statement (ii) of Theorem 4.2 has been derived for $p \geq 1$ by a different method in [11, p. 119].*

Remark 4.5. *Using Theorem 4.2, one can easily verify for $p \in \{-1, 0, 2\}$ the statements (i), (ii), and (iv) of Proposition 4.2, including the identities* (43) *and* (44).

We conclude this section with explicit sum representations involving binomial coefficients and falling factorials.

Theorem 4.3. *For any integer $p \neq 1$, the following hold:*

$$\text{(i)} \qquad s_1^{[p]}(n,k) = \frac{(1-p)^{-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j(1-p))_n,$$

$$\text{(ii)} \qquad s_2^{[p]}(n,k) = \frac{(1-p)^n}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(\frac{j}{1-p}\right)_n.$$

Proof. (i) follows immediately by representing $B_{n,k}((1-p)_1, (1-p)_2, \ldots)$ in (54) using Yang's formula (22). For (ii), one starts from (55) and applies (iv) of the corollary to Theorem 2.2 to $A_{n,k}((1-p)_1, (1-p)_2, \ldots)$. □

Remark 4.6. *As expected, $S_{p,1}(n,k) = s_2^{[p]}(n,k)$ for all $p \geq 1$. Indeed, for $p = 1$, Proposition 4.2 (iii) gives $s_2^{[1]}(n,k) = s_2(n,k)$, and, on the other hand, by* (47) *and by the well-known explicit formula for $s_2(n,k)$ one gets*

$$S_{1,1}(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n = s_2(n,k).$$

For $p \geq 2$, the task (omitted here) is to simplify the product on the right-hand side of (47) *(with $q = 1$, equivalent to Eq. (4.10) in [23]) as follows:*

$$\prod_{i=1}^{n} (j + (i-1)(p-1)) = (1-p)^n \left(\frac{j}{1-p}\right)_n.$$

Using Theorem 4.3 (ii), one then obtains the asserted agreement.

5. Inversion relations

In this final section, we consider various types of inversion relations that are typical for generalized Stirling numbers and, more generally, for all ENFs—that is, all number families representable by partial Bell polynomials. The remarkable reason for this is the fundamental fact that the Bell polynomials $B_{n,k}$ and their inverse counterparts $A_{n,k}$ already satisfy the identities in question. Since ENFs arise by instantiation—i.e., by assigning specific values (or expressions) to the variables $X_1, X_2, X_3, \ldots$ in $B_{n,k}$ and $A_{n,k}$—their relational properties are *inherited directly* by the resulting exponential families (which is evidently sufficient to prove the results obtained in this way).

Let $e_1(n,k)$ be any ENF (here arbitrarily referred to as 'of the first kind'). From (2), by considering $k = 1$, we have

$$e_1(n,k) = B_{n,k}(e_1(1,1), \ldots, e_1(n-k+1,1)).$$

The corresponding ENF 'of the second kind' is then, according to (6),

$$e_2(n,k) = A_{n,k}(e_1(1,1), \ldots, e_1(n-k+1,1)).$$

Thus, the general inversion law for MSPs (17) (see [37, Thm. 5.1] and [38, Cor. 4.9]) immediately translates into

Theorem 5.1 (*Inversion Law*)**.**

$$\sum_{j=k}^{n} e_1(n,j) e_2(j,k) = \sum_{j=k}^{n} e_2(n,j) e_1(j,k) = \delta_{nk}.$$

Theorem 5.1 holds for all ENFs and therefore does not need to be proved separately for specific cases (such as $e_i(n,k) = b_i^{[m]}(n,k)$ or $e_i(n,k) = s_i^{[p]}(n,k)$ with $i = 1,2$), provided it is clear that the number family in question can be represented by Bell polynomials. The same applies to the other two types of inversion relations that we will discuss below.

The discovery that Stirling numbers of the first kind can be expressed in terms of Stirling numbers of the second kind goes back to Schlömilch [36]. Much later, Gould found that the converse also holds; cf. [31, Eq. (13.31)]. In fact, this mutual representability even holds for MSPs; cf. [37, Thm. 6.4]. Hence, from the Schläfli form of these polynomial identities given in [38, Thm. 5.5], we obtain for arbitrary ENFs

Theorem 5.2 (*Generalized Schlömilch-Schläfli Identities*)**.**

$$e_1(n,n-k) = (-1)^k \sum_{j=0}^{k} \binom{k+n}{k-j}\binom{k-n}{k+j} e_1(1,1)^{n+j} e_2(k+j,j),$$

$$e_2(n,n-k) = (-1)^k \sum_{j=0}^{k} \binom{k+n}{k-j}\binom{k-n}{k+j} e_2(1,1)^{n+j} e_1(k+j,j).$$

In many cases, in particular for the Scherk numbers and generalized Bessel numbers, we have $e_i(1,1) = 1$ ($i = 1,2$), which further simplifies the expressions on the right-hand side of the identities, as shown in the following

Corollary 5.1.

$$b_1^{[m]}(n,n-k) = (-1)^k \sum_{j=0}^{k} \binom{k+n}{k-j}\binom{k-n}{k+j} b_2^{[m]}(k+j,j),$$

$$s_1^{[p]}(n,n-k) = (-1)^k \sum_{j=0}^{k} \binom{k+n}{k-j}\binom{k-n}{k+j} s_2^{[p]}(k+j,j).$$

Swapping the indices 1 and 2 in both equations yields valid identities again.

In the remarks following Theorem 3.2, we pointed out that the recursion formula (42) given there only works if the values of $b_1^{[m]}(j,1)$ are already known, i.e., they must be computable independently of (42), even for $j > m$. Using the first equation of Corollary 5.1, applied to $b_1^{[m]}(j, j-(j-1))$, this gap can now be closed. After a few straightforward calculations, one obtains, for arbitrary integers $m, j \geq 1$:

$$b_1^{[m]}(j,1) = \sum_{r=0}^{j-1} (-1)^r \binom{2j-1}{j+r} b_2^{[m]}(j-1+r,r).$$

A more direct form of mutual representability is reciprocity, whose significance for the classical Stirling numbers was emphasized by Knuth [20] and further developed by Stanley [41] and Gessel [12]. In our case, it reveals that the two mutually inverse kinds of an ENF in fact belong to *a single united* exponential family.

Theorem 5.3 (*Reciprocity Law*)**.** *For all $n, k \in \mathbb{Z}$, one has*

$$e_1(n,k) = (-1)^{n-k} e_2(-k,-n).$$

Some remarks are in order regarding this remarkable identity.

The equation of Theorem 5.3 is equivalent to $e_2(n,k) = (-1)^{n-k} e_1(-k,-n)$.

The underlying polynomial identity reads $A_{n,k} = (-1)^{n-k} B_{-k,-n}$ for arbitrary $n, k \in \mathbb{Z}$. This extension of the arguments to integers requires explanation, since MSPs are initially defined only for non-negative n, k (cf. (12) and (14)). The key lies in the observation that the MSPs can be expressed using the *potential polynomials* introduced by Comtet [10, p. 141, Eq. (5f)] (here in the notation of [38, Eq. (3.10)]):

$$\widehat{P}_{n,k} = \sum_{j=0}^{n} (k)_j X_0^{k-j} B_{n,j}.$$

From [38, Cor. 3.8 & 3.10], we first obtain the following equations for non-negative n, k:

$$B_{n,k} = \binom{n}{k} \widehat{P}_{n-k,k}\left(\tfrac{X_1}{1}, \ldots, \tfrac{X_{n-k+1}}{n-k+1}\right), \tag{56}$$

$$A_{n,k} = \binom{n-1}{k-1} \widehat{P}_{n-k,-n}\left(\tfrac{X_1}{1}, \ldots, \tfrac{X_{n-k+1}}{n-k+1}\right). \tag{57}$$

Upon closer inspection, however, it readily turns out that the right-hand sides of these identities make also sense for negative values of n, k. Thus, the extension of the indices to the entire set of $\mathbb{Z}$ can be achieved by simply using (56) and (57) to redefine $B_{n,k}$ and $A_{n,k}$. For details and the proof of the general reciprocity for MSPs, the reader is referred to [38, Sect. 8].

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