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Multicompositions via Formal Languages and Riordan Arrays

Brian Hopkins[†] and José L. Ramírez[‡]

[†] *Department of Mathematics and Statistics, Saint Peter's University, Jersey City, New Jersey 07306, USA*
Email: bhopkins@saintpeters.edu

[‡] *Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá D.C., Colombia*
Email: jlramirezr@unal.edu.co

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Abstract: We present a combinatorial study of multicompositions, an extension of the classical notion of integer compositions that allows some zeros between positive parts. Using the framework of formal languages, we derive generating functions to enumerate multicompositions under various constraints, including restrictions on parts and the avoidance of specific subwords. Additionally, we use Riordan arrays to count multicompositions with respect to several statistics. We establish connections to generalized Fibonacci and Pell polynomials and provide new combinatorial interpretations for many entries of the On-Line Encyclopedia of Integer Sequences.

Keywords: Multicompositions; Riordan group; Pell polynomials
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Dedicated to the memory of Professor Emanuele Munarini.

1. Introduction

A composition of a positive integer n is a sequence of positive integers, called parts, that sum to n. For example, the compositions of 4 are

$$(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,3),\ (1,2,1),\ (1,1,2),\ (1,1,1,1).$$

For more information on compositions, see the monograph of Heubach and Mansour [11].

In 2007, Andrews [2] introduced a generalization of compositions called k-compositions where each part can appear in one of k different colors except that the last part must have color 1. He was interested in the prime factorizations of the number of k-compositions of n with relatively prime parts.

In 2019, Ouvry and Polychronakos [18] introduced the concept of g-compositions in the context of closed lattice random walks that enclose a given area. A g-composition generalizes the concept of composition by allowing both positive integer and zero parts, subject to the condition that at most $g-2$ consecutive zeros may appear between positive parts. In 2021, the first named author and Ouvry [13] proved that k-compositions of Andrews are in bijection with g-compositions, with $g=k+1$. Recently, Dastidar and Wallner [7] have explored new connections between these objects and Dyck paths.

Here, we use the Ouvry–Polychronakos interpretation with Andrews's indexing: Let $C^k(n)$ denote the set of compositions of n such that at most $k-1$ consecutive zeros may appear between positive parts and $c^k(n) = |C^k(n)|$. In general, the elements of $C^k(n)$ are called multicompositions. Hopkins and Ouvry enumerated several statistics on multicompositions, such as the number of parts, the number of positive parts, and the number of zeros. They also studied multicompositions with parts restricted to particular sets.

In this paper, we determine various multicomposition statistics by means of regular languages (reviewed in §2), automata, and Riordan arrays (both discussed in §4), providing the first application of these tools to the topic. We use generating functions to derive explicit combinatorial formulas for the corresponding counting sequences, along with some combinatorial proofs. Additionally, we extend and generalize many of the results of Hopkins and Ouvry [13].

2. Preliminaries of formal languages

First, we recall some terminology and notation, mostly from Sipser [23]. Let Σ be an alphabet whose elements are called symbols. A word over Σ is a finite sequence of symbols from Σ. The set of all words over Σ is denoted by Σ^*. Each subset of Σ^* is called a formal language over Σ. For any word $w \in \Sigma^*$, let $|w|$ denote its length, which is the number of symbols occurring in w. The unique word of length zero is denoted by ϵ and is called the empty word.

Let L, L_1, and L_2 be languages over Σ. We define the union, concatenation, and Kleene star of languages as follows:

- $L_1 \cup L_2 = \{w \mid w \in L_1 \text{ or } w \in L_2\}$,
- $L_1 L_2 = \{w_1 w_2 \mid w_1 \in L_1, w_2 \in L_2\}$,
- $L^* = \bigcup_{i \geq 0} L^i = \{a_1 a_2 \cdots a_k \mid k \geq 0 \text{ and each } a_i \in L\}$.

Now, we describe an important family of languages that are closed under the above operations, called regular languages. A regular expression over Σ is any word obtained inductively from the alphabet $\Sigma \cup \{\emptyset, \epsilon, +, \cdot, *\}$ in the following manner:

- $\emptyset, \epsilon$, and a, for a in Σ, are all regular expressions;
- if α and β are regular expressions, then $(\alpha + \beta)$, $(\alpha \cdot \beta)$, and α^* are regular expressions.

For example, $\texttt{(1+01)}^*\texttt{00(0+1)}^*$ is a regular expression. Every regular expression α over Σ represents a language of Σ^*, denoted by $\mathcal{L}(\alpha)$, according to the following:

- $\mathcal{L}(\emptyset) = \emptyset$, $\mathcal{L}(\epsilon) = \{\epsilon\}$, and $\mathcal{L}(a) = \{a\}$ for all a in Σ;
- if α and β are regular expressions, then $\mathcal{L}((\alpha + \beta)) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta)$, $\mathcal{L}((\alpha \cdot \beta)) = \mathcal{L}(\alpha)\mathcal{L}(\beta)$, and $\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$.

For example, it is possible to verify that

$$\mathcal{L}((\texttt{1+01})^*\texttt{00}(\texttt{0+1})^*) = \{w \in \{\texttt{0}, \texttt{1}\}^* \mid w \text{ has at least one pair of consecutive zeros}\}.$$

A language L is called regular if there exists a regular expression α such that $\mathcal{L}(\alpha) = L$.

The ordinary generating function $H_L(x) = \sum_{n=0}^{\infty} h_n x^n$ with $h_n = |\{w \in L \mid |w| = n\}|$ corresponds to a formal language L, that is, the n-th coefficient h_n gives the number of words in L with length n. The procedure for finding the ordinary generating function corresponding to a regular language (and, more generally, to a context-free language) is known as the Chomsky–Schützenberger methodology [5, 9, 10]. The idea behind this procedure is to translate an unambiguous regular expression α into a generating function, denoted by $H_\alpha(x)$, following these rules:

- $H_\epsilon(x) = 1$ and $H_a(x) = x$ for all a in Σ;
- if α and β are regular expressions, then $H_{(\alpha+\beta)}(x) = H_\alpha(x) + H_\beta(x)$, $H_{(\alpha\cdot\beta)} = H_\alpha(x)H_\beta(x)$, and $H_{\alpha^*} = 1/(1 - H_\alpha(x))$ $(\epsilon \notin \mathcal{L}(\alpha))$.

The generating function of any regular language is a rational function. For example, if $\gamma = \texttt{(1+01)}^*\texttt{00(0+1)}^*$, then the generating function of the regular language $\mathcal{L}(\gamma)$ is

$$H_\gamma(x) = \frac{x^2}{(1 - x - x^2)(1 - 2x)} = \sum_{n \geq 0} (2^n - F_{n+2})x^n$$

where F_n is the n-th Fibonacci number. This means that there are $2^n - F_{n+2}$ words in $\mathcal{L}(\gamma)$ of length n [17, A008466]. We can also enumerate additional parameters over the regular expressions. For example, if we denote by $|w|_1$ the number of ones in the word w, then the bivariate generating function of the words in $\gamma = \texttt{(1+01)}^*\texttt{00(0+1)}^*$ with respect to the length and the number of ones is

$$H_\gamma(x, y) = \frac{x^2}{(1 - xy - x^2y)(1 - x - xy)}.$$

Expanding the first few terms gives

$$H_\gamma(x, y) = x^2 + (1 + 2y)x^3 + (\mathbf{1} + \mathbf{4y} + \mathbf{3y^2})x^4 + (1 + 5y + 9y^2 + 4y^3)x^5 + \mathcal{O}(x^6)$$

where the words corresponding to the bold expression are

$$\underbrace{\texttt{0000}}_{1}, \ \underbrace{\texttt{1000}, \texttt{0100}, \texttt{0010}, \texttt{0001}}_{4y}, \ \underbrace{\texttt{1100}, \texttt{1001}, \texttt{0011}}_{3y^2}.$$

3. The regular language of multicompositions

Let $\mathcal{U}$ denote the set of all nonempty words composed of positive integers. It is clear that $\mathcal{U}$ is a regular language associated with the integer compositions. For example, the composition $(3, 5, 3, 12)$ corresponds to the word $3 \cdot 5 \cdot 3 \cdot 12 \in \mathcal{U}^*$.

Multicompositions can be considered as words in the regular language associated with the regular expression

$$\mathcal{M}_k = \mathcal{U} \cdot \left((\epsilon + 0 + \cdots + 0^{k-1}) \cdot \mathcal{U} \right)^* . \tag{1}$$

For example, elements in $\mathcal{M}_2$ include

$$\{1, 2, 3, \ldots, 11, 101, 102, 103, \ldots, 21, 201, 202, 203, \ldots\}.$$

Given a multicomposition w, denote by $|w|$ the total sum of the parts in w, by $|w|_0$ the number of zeros in w, and by $|w|_+$ the total number of positive parts in w. Let $F^k(x, y, z)$ be the multivariate generating function

$$F^k(x, y, z) = \sum_{n \geq 1} \sum_{w \in C^k(n)} x^{|w|} y^{|w|_0} z^{|w|_+}.$$

Let $H_\gamma(x, y, z)$ be the multivariate generating function of the regular expression γ with respect to the above parameters, that is,

$$H_\gamma(x, y, z) = \sum_{w \in \mathcal{L}(\gamma)} x^{|w|} y^{|w|_0} z^{|w|_+}.$$

For example, for $\mathcal{U}$, we have

$$H_\mathcal{U}(x, y, z) = \sum_{i \geq 1} x^i z = \frac{xz}{1 - x}.$$

It is clear that $F^k(x, y, z) = H_{\mathcal{M}_k}(x, y, z)$.

Theorem 3.1. *The generating function $F^k(x, y, z)$ is*

$$F^k(x, y, z) = \frac{x(1 - y)z}{1 - y - x(1 - y + z - y^k z)}, \quad k \geq 1.$$

Proof. From the regular expression in (1), we obtain the generating function

$$\begin{aligned}
H_{\mathcal{M}_k}(x, y, z) &= H_\mathcal{U}(x, y, z) \frac{1}{1 - H_{(\epsilon + 0 + \cdots + 0^{k-1})}(x, y, z) H_\mathcal{U}(x, y, z)} \\
&= \left(\sum_{i \geq 1} x^i z \right) \frac{1}{1 - \left(\sum_{i=0}^{k-1} y^i \right) \left(\sum_{i \geq 1} x^i z \right)} \\
&= \frac{xz}{1 - x} \cdot \frac{1}{1 - \frac{1 - y^k}{1 - y} \frac{xz}{1 - x}} \\
&= \frac{x(1 - y)z}{1 - y - x(1 - y + z - y^k z)}.
\end{aligned}$$

□

The series expansion of the generating function $F^2(x, y, z)$ is

$$\begin{aligned}
F^2(x, y, z) = zx + (z + z^2 + yz^2)x^2 + (\boldsymbol{z + 2z^2 + 2yz^2 + z^3 + 2yz^2 + y^2 z^2})x^3 \\
+ (z + 3z^2 + 3yz^2 + 3z^3 + 6yz^3 + 3y^2 z^3 + z^4 + 3yz^4 + 3y^2 z^4 + y^3 z^4)x^4 + \mathcal{O}(x^5)
\end{aligned}$$

where the words corresponding to the bold expression are

$$\underbrace{3}_{z}, \underbrace{21, 12}_{2z^2}, \underbrace{201, 102}_{2yz^2}, \underbrace{111}_{z^3}, \underbrace{1101, 1011}_{2yz^3}, \underbrace{10101}_{y^2 z^3}.$$

From Theorem 3.1, we can derive several known results about multicompositions (cf. [2, 13]).

Corollary 3.1. *The generating function for $c^k(n)$, the number of multicompositions of n, is*

$$F^k(x, 1, 1) = \sum_{n \geq 1} c^k(n) x^n = \frac{x}{1 - (k + 1)x}, \quad k \geq 1.$$

Moreover, $c^k(n) = (k + 1)^{n-1}$.

Let $c^k(n,\ell)$ be the number of multicompositions of n with ℓ parts, including the zero summands. The generalized binomial coefficient $\binom{n}{\ell}_k$ is defined as the coefficient of x^ℓ in the expansion of $(1+x+x^2+\cdots+x^k)^n$, which we denote $[x^\ell](1+x+x^2+\cdots+x^k)^n$; Moghaddamfar et al. have given a linear algebra treatment of these coefficients [16].

Corollary 3.2 (Proposition 4 [13]). *The bivariate generating function for $c^k(n,\ell)$ is*

$$\begin{aligned} F^k(x,y,y) &= \sum_{n,\ell\geq 1} c^k(n,\ell)x^n y^\ell \\ &= \sum_{n\geq 1} y(1+y+y^2+\cdots+y^k)^{n-1}x^n \\ &= \frac{xy(1-y)}{1-y-x\left(1-y^{k+1}\right)} \end{aligned}$$

and $c^k(n,\ell)=\binom{n-1}{\ell-1}_k$. Moreover, the generating function for the total number of parts in $C^k(n)$ is

$$\sum_{n\geq 1}\sum_{\ell\geq 1}\ell c^k(n,\ell)x^n = \left.\frac{\partial F^k(x,y,y)}{\partial y}\right|_{y=1} = \frac{x(2-(2-k)(k+1)x)}{2(1-(1+k)x)^2}.$$

For example, since

$$C^2(3)=\{\mathtt{3},\mathtt{21},\mathtt{201},\mathtt{12},\mathtt{102},\mathtt{111},\mathtt{1101},\mathtt{1011},\mathtt{10101}\}, \tag{2}$$

the total number of parts is 27. In Table 1, we show the first few values for the total number of parts in $C^k(n)$ along with matches in [17].

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	OEIS
1	1	3	8	20	48	112	256	576	1280	2816	A001792
2	1	6	27	108	405	1458	5103	17496	59049	196830	A027471
3	1	10	64	352	1792	8704	40960	188416	851968	3801088	
4	1	15	125	875	5625	34375	203125	1171875	6640625	37109375	A171220

Table 1: Total number of parts in $C^k(n)$ for small k and n.

3.1 The total number of parts

Let $tc^k(n)$ be the total number of parts in $C^k(n)$. From Corollaries 3.1 and 3.2, we have

$$tc^k(n)=\sum_{\ell=1}^{nk-k+1}\ell c^k(n,\ell)=\sum_{\ell=1}^{nk-k+1}\ell\binom{n-1}{\ell-1}_k.$$

In the following corollary, we give another formula for this count.

Corollary 3.3. *For $n,k\geq 1$, we have*

$$tc^k(n)=\left(1+\frac{k}{2}(n-1)\right)(k+1)^{n-1}.$$

Proof. From Corollary 3.2 we have

$$\begin{aligned} tc^k(n) &= [x^n]\frac{x(2-(2-k)(k+1)x)}{2(1-(1+k)x)^2} \\ &= \frac{1}{2}[x^n]\sum_{\ell\geq 0}(2-(2-k)(k+1)x)(\ell+1)(k+1)^\ell x^{\ell+1} \\ &= n(k+1)^{n-1}-\frac{1}{2}(2-k)(k+1)^{n-1}(n-1) \\ &= \left(1+\frac{k}{2}(n-1)\right)(k+1)^{n-1}. \end{aligned}$$

□

We can conclude the following combinatorial formula:

$$\sum_{\ell=1}^{nk-k+1}\ell\binom{n-1}{\ell-1}_k=\left(1+\frac{k}{2}(n-1)\right)(k+1)^{n-1}.$$

3.2 Positive parts and zeros

Let $c_+^k(n,\ell)$ be the number of k-compositions of n with ℓ positive parts.

Corollary 3.4. *The bivariate generating function for $c_+^k(n,\ell)$ is*

$$F^k(x,1,z) = \sum_{n,\ell\geq 1} c_+^k(n,\ell)x^n z^\ell = \frac{xz}{1-(1+kz)x}$$

and $c_+^k(n,\ell) = \binom{n-1}{\ell-1}k^{\ell-1}$. Moreover, if $tc_+^k(n)$ denotes the total number of positive parts in $C^k(n)$, then

$$\sum_{n\geq 1} tc_+^k(n)x^n = \left.\frac{\partial F^k(x,1,z)}{\partial z}\right|_{z=1} = \frac{(1-x)x}{(1-(k+1)x)^2}$$

and $tc_+^k(n) = (1+k)^{n-2}(1+nk)$.

For example, from (2), we have $tc_+^2(3) = 21$. In Table 2, we show the first few values for the total number of positive parts in $C^k(n)$. Notice that the last two sequences in this table do not have any combinatorial interpretation in [17].

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	OEIS
1	1	3	8	20	48	112	256	576	1280	2816	A001792
2	1	5	21	81	297	1053	3645	12393	41553	137781	A081038
3	1	7	40	208	1024	4864	22528	102400	458752	2031616	A081039
4	1	9	65	425	2625	15625	90625	515625	2890625	16015625	A081040

Table 2: Total number of positive parts in $C^k(n)$ for small k and n.

Notice that, from Corollary 3.4, we obtain the combinatorial identity

$$\sum_{\ell=1}^{n} \ell c_+^k(n,\ell) = \sum_{\ell=1}^{n} \ell\binom{n-1}{\ell-1}k^{\ell-1} = (1+k)^{n-2}(1+nk).$$

Let $c_0^k(n,\ell)$ be the number of k-compositions of n with ℓ zeros.

Corollary 3.5. *The bivariate generating function for $c_0^k(n,\ell)$ is*

$$F^k(x,y,1) = \sum_{n\geq 1}\sum_{\ell\geq 0} c_0^k(n,\ell)x^n y^\ell = \frac{x(1-y)}{1-y-x\left(2-(y+y^k)\right)}.$$

Moreover, if $tc_0^k(n)$ denotes the total number of zeros in $C^k(n)$, then

$$\sum_{n\geq 1} tc_0^k(n)x^n = \left.\frac{\partial F^k(x,y,1)}{\partial y}\right|_{y=1} = \frac{(k-1)kx^2}{2(1-(k+1)x)^2}$$

and $tc_0^k(n) = \frac{1}{2}k(k-1)(k+1)^{n-2}(n-1)$.

For example, from (2), we have $tc_0^2(3) = 6$. In Table 3, we show the first few values for the total number of zero parts in $C^k(n)$ for small $k \geq 2$ (as standard compositions, the $k=1$ case, have no zeros). Our combinatorial interpretation is new for all of the corresponding OEIS sequences [17].

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	OEIS
2	0	1	6	27	108	405	1458	5103	17496	59049	A027471
3	0	3	24	144	768	3840	18432	86016	393216	1769472	A212698
4	0	6	60	450	3000	18750	112500	656250	3750000	21093750	A269760

Table 3: Total number of zero parts in $C^k(n)$ for small k and n.

In Tables 1–3, notice that one sequence is repeated: A027471 gives both the total number of parts in $C^2(n)$ and, with offset 1, the total number of zeros in $C^2(n+1)$. This follows from the corollaries, but we also provide the following combinatorial proof.

Proposition 3.1. *The total number of parts in $C^2(n)$ equals the total number of zeros in $C^2(n+1)$.*

Proof. We establish a bijection between the parts in $C^2(n)$ and the zeros in $C^2(n+1)$. For a positive part r in a 2-composition of n, replace it with the parts r01, keeping the rest of the composition intact, to produce a zero in a 2-composition of $n+1$. A zero in a 2-composition of n must be followed by a positive integer, say ℓ: Replace 0ℓ by 0$(\ell+1)$, keeping the rest of the composition intact, to produce a zero in a 2-composition of $n+1$. The map is easily reversed. □

As an example of the bijection, here is the correspondence between the six parts of $C^2(2)$ and the six zeros in $C^2(3)$:

$$\underline{2} \longleftrightarrow 2\underline{0}1, \quad \underline{1}1 \longleftrightarrow 1\underline{0}11, \quad 1\underline{1} \longleftrightarrow 11\underline{0}1,$$
$$\underline{1}01 \longleftrightarrow 1\underline{0}101, \quad 1\underline{0}1 \longleftrightarrow 1\underline{0}2, \quad 10\underline{1} \longleftrightarrow 101\underline{0}1.$$

In the correspondence between the parts of $C^2(3)$ and the zeros of $C^2(4)$, it is interesting that the two zeros of 101101 come from different 2-compositions, namely $\underline{1}101$ and $101\underline{1}$.

3.3 The total number of a given part

Given a multicomposition w, we denote by $|w|_p$ the number of times p occurs as a part in w. Let $G_p^k(x,y)$ be the bivariate generating function

$$G_p^k(x,y) = \sum_{n\geq 1}\sum_{w\in C^k(n)} x^{|w|}y^{|w|_p}.$$

Theorem 3.2. *The generating function $G_p^k(x,y)$ is*

$$G_p^k(x,y) = \frac{x - x^p(1-x)(1-y)}{1-(1+k)x+kx^p(1-x)(1-y)}, \quad k\geq 1,\ p\geq 1.$$

Moreover, the generating function for the total number of parts p in $C^k(n)$ is

$$\left.\frac{\partial G_p^k(x,y)}{\partial y}\right|_{y=1} = \frac{(1-x)^2x^p}{(1-(k+1)x)^2}. \tag{3}$$

Proof. From the regular expression given in (1),

$$G_p^k(x,y) = \left(\sum_{i=1}^{p-1}x^i + yx^p + \sum_{i\geq p+1}x^i\right)\frac{1}{1-k\left(\sum_{i=1}^{p-1}x^i + yx^p + \sum_{i\geq p+1}x^i\right)},$$

and the results follow by geometric series. □

Let $c_p^k(n,\ell)$ denote the number of k-compositions of n with ℓ parts p. In Table 4, we give particular values for $1\leq k,p\leq 3$. There are OEIS matches for the $k=1$ arrays A105422, A105114, A218796, respectively, and the $k=2$, $p=2$ array A120924.

Corollary 3.6. *The values $c_p^k(n,\ell)$ satisfy the following recurrence relations:*

- *for $n>p+1$ and $\ell\geq 0$,*

$$c_p^k(n,\ell) = (k+1)c_p^k(n-1,\ell) - kc_p^k(n-p,\ell) + kc_p^k(n-p,\ell-1) + kc_p^k(n-p-1,\ell) - kc_p^k(n-p-1,\ell-1),$$

- *for $n=p+1$ and $\ell\geq 0$,*

$$c_p^k(n,\ell) = (k+1)c_p^k(n-1,\ell) - k\delta_{\ell,1} + k\delta_{\ell,2} + a(\ell),$$

- *for $n=p$ and $\ell\geq 0$,*

$$c_p^k(n,\ell) = (k+1)c_p^k(n-1,\ell) - a(\ell),$$

- *for $2\leq n<p$ and $\ell\geq 0$,*

$$c_p^k(n,\ell) = (k+1)^{n-1}\delta_{\ell,0},$$

- *for $n=1$ and $p>1$, $c_p^k(1,\ell)=\delta_{\ell,0}$, and*

$k\backslash p$	1							2				3		
	0	1						1				1		
1	1	0	1					1	1			2		
	1	2	0	1				2	2			3	1	
	2	2	3	0	1			4	3	1		6	2	
	3	5	3	4	0	1		7	6	3		11	5	
	5	8	9	4	5	0	1	12	13	6	1	21	10	1
	0	1						1				1		
2	1	0	2					2	1			3		
	1	4	0	4				5	4			8	1	
	3	4	12	0	8			13	12	2		23	4	
	5	16	12	32	0	16		33	36	12		65	16	
	11	28	60	32	80	0	32	83	108	48	4	185	56	2
	0	1						1				1		
3	1	0	3					3	1			4		
	1	6	0	9				10	6			15	1	
	4	6	27	0	27			34	27	3		58	6	
	7	33	27	108	0	81		115	114	27		223	33	
	19	60	189	108	405	0	243	388	465	162	9	859	162	3

Table 4: Values of $c_p^k(n,\ell)$ for small k, ℓ, n, p.

- *for* $n = 1$ *and* $p = 1$, $c_p^k(1,\ell) = \delta_{\ell,1}$

where $a(0) = 1, a(1) = -1$ *and* $a(\ell) = 0$ *for* $\ell \geq 2$, *and* $\delta_{n,k}$ *is the Kronecker delta.*

Proof. From Theorem 3.2, we have the functional equation

$$G_p^k(x,y) - (k+1)xG_p^k(x,y) + kx^p(1-x)(1-y)G_p^k(x,y) = x - x^p(1-x)(1-y),$$

and the result follows from comparing coefficients of $x^n y^\ell$. □

Note that among the OEIS matches, only the $k = p = 1$ array A105422 includes a recurrence of the type given by Corollary 3.6.

Let $tc_p^k(n)$ be the total number of parts equal to p in the compositions in $C^k(n)$.

Corollary 3.7. *The sequence* $tc_p^k(n)$ *is*

$$tc_p^k(n) = \begin{cases} 0 & \textit{if } n < p, \\ 1 & \textit{if } n = p, \\ 2k & \textit{if } n = p+1, \\ k(k+1)^{n-p-2}(2 + k(1+n-p)), & \textit{if } n \geq p+2. \end{cases}$$

Proof. From (3), we have

$$\begin{aligned} tc_p^k(n) &= [x^n]\frac{(1-x)^2x^p}{(1-(k+1)x)^2} \\ &= [x^{n-p}](1-2x+x^2)\sum_{i\geq 0}\binom{i+1}{i}(k+1)^i x^i \\ &= [x^{n-p}](1-2x+x^2)\sum_{i\geq 0}(i+1)(k+1)^i x^i, \end{aligned}$$

and the result follows from comparing coefficients. □

4. Riordan arrays and restricted multicompositions

In this section, we study multicompositions where the parts are restricted to an ordered nonempty set $\mathcal{A} = \{a_1, a_2, \dots\} \subseteq \mathbb{Z}^+$. The set of k-compositions of n with parts in $\mathcal{A}$ is denoted by $C^k_{\mathcal{A}}(n)$ and its cardinality by $c^k_{\mathcal{A}}(n)$. In this context, multicompositions can be regarded as words in the regular language associated with the regular expression

$$\mathcal{M}^k_{\mathcal{A}} = \mathcal{A} \cdot \left(\left(\epsilon + \mathtt{0} + \cdots + \mathtt{0}^{k-1}\right) \cdot \mathcal{A}\right)^*.$$

Let $F^k_{\mathcal{A}}(x, y, z)$ be the multivariate generating function

$$\sum_{n\geq 1} \sum_{w\in C^k_{\mathcal{A}}(n)} x^{|w|} y^{|w|_0} z^{|w|_+}.$$

It follows that

$$F^k_{\mathcal{A}}(x, y, z) = \frac{z \sum_{i\in\mathcal{A}} x^i}{1 - z \sum_{i=0}^{k-1} y^i \sum_{i\in\mathcal{A}} x^i}, \quad k \geq 1. \tag{4}$$

Observe that if $\mathcal{A} = \mathcal{U}$, the generating function reduces to the expression given in Theorem 3.1.

We now give some background of Riordan arrays; see also the recent survey article by Davenport et al. [8]. Riordan arrays, introduced in 1991 by Shapiro et al. [22], are infinite lower triangular matrices characterized by their columns. Specifically, the kth column of a Riordan array is generated by the generating function $g(x)\,(f(x))^k$ for $k \geq 0$ where $g(x)$ and $f(x)$ are formal power series satisfying $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$ (where $f'(x)$ is the formal derivative of $f(x)$). A Riordan array is denoted by the pair $(g(x), f(x))$. When a Riordan array $(g(x), f(x))$ acts on a column vector $(c_0, c_1, \dots)^T$ with generating function $h(x)$, the resulting column vector has generating function $g(x)h(f(x))$. This property is known as the fundamental theorem of Riordan arrays or the summation property.

The product of two Riordan arrays, $(g(x), f(x))$ and $(h(x), j(x))$, is defined as

$$(g(x), f(x)) * (h(x), j(x)) = (g(x)h\,(f(x)), j(f(x))). \tag{5}$$

Under this operation, Riordan arrays form a group [22]. The identity element in this group is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ \overline{f}\right)(x), \overline{f}(x)\right)$$

where $\overline{f}(x)$ represents the compositional inverse of $f(x)$.

Let $c^k_{\mathcal{A}}(n, \ell)$ be the number of k-compositions of n with ℓ positive parts in $\mathcal{A}$. Clearly, $\sum_{\ell=1}^n c^k_{\mathcal{A}}(n, \ell) = c^k_{\mathcal{A}}(n)$. Consider the matrix

$$\mathcal{R}^k_{\mathcal{A}} = \left[c^k_{\mathcal{A}}(n+1, \ell+1)\right]_{n,\ell\geq 0}$$

where the (n, ℓ) entry corresponds to the number of k-compositions of $n + 1$ with $\ell + 1$ parts in $\mathcal{A}$.

Theorem 4.1. *If $1 \in \mathcal{A}$, then for any positive integer k, the matrix $\mathcal{R}^k_{\mathcal{A}}$ is a Riordan array*

$$\mathcal{R}^k_{\mathcal{A}} = \left(\sum_{i\in\mathcal{A}} x^{i-1}, kx \sum_{i\in\mathcal{A}} x^{i-1}\right).$$

Proof. Let $T_\ell(x) = \sum_{n\geq 0} c^k_{\mathcal{A}}(n + 1, \ell + 1)x^n$ be the generating function for the k-compositions of $n + 1$ with exactly $\ell + 1$ positive parts in $\mathcal{A}$. It is clear that $T_0(x) = \sum_{i\in\mathcal{A}} x^{i-1}$ and, since $1 \in \mathcal{A}$, we have $T_0(0) \neq 0$. For $\ell > 0$, there are $\ell + 1$ positive parts, and between them there are at most $k - 1$ zeros. Therefore,

$$T_\ell(x) = (kx)^\ell \left(\sum_{i\in\mathcal{A}} x^{i-1}\right)^{\ell+1}.$$

Using the definition of a Riordan array, we obtain the desired result. □

Among the combinatorial applications of Riordan arrays, the row sums and diagonal sums of an array can be derived from $(g(x), f(x))$, as we see now in our example. If $1 \in \mathcal{A}$, then from the summation property, the generating function for the row sum of the matrix $\mathcal{R}^k_{\mathcal{A}}$ is

$$\frac{\sum_{i\in\mathcal{A}} x^{i-1}}{1 - kx \sum_{i\in\mathcal{A}} x^{i-1}} = \frac{F^k_{\mathcal{A}}(x, 1, 1)}{x}.$$

Let $d^k_{\mathcal{A}}(n)$ denote the diagonal sum of the matrix $\mathcal{R}^k_{\mathcal{A}}$, namely

$$d^k_{\mathcal{A}}(n) = \sum_{i\geq 1} c^k_{\mathcal{A}}(n - (i - 1), i).$$

Corollary 4.1. *If $1 \in \mathcal{A}$, then for any positive integer k, the generating function for $d^k_{\mathcal{A}}(n)$ is*

$$\sum_{n\geq 1} d^k_{\mathcal{A}}(n)x^n = \frac{\sum_{i\in\mathcal{A}} x^i}{1 - kx^2\sum_{i\in\mathcal{A}} x^{i-1}}.$$

Proof. The diagonals of the Riordan array $\mathcal{R}^k_{\mathcal{A}}$ correspond to the rows of the Riordan array

$$\mathcal{T}^k_{\mathcal{A}} = \left(\sum_{i\in\mathcal{A}} x^{i-1}, kx^2\sum_{i\in\mathcal{A}} x^{i-1}\right).$$

By multiplying the right-hand side of this equation by the vector $(1, 1, 1, \dots)^T$, whose generating function is $1/(1-x)$, and applying the summation property, the resulting vector has generating function

$$\mathcal{T}^k_{\mathcal{A}}\left(\frac{1}{1-x}\right) = \frac{\sum_{i\in\mathcal{A}} x^{i-1}}{1 - kx^2\sum_{i\in\mathcal{A}} x^{i-1}} = \sum_{n\geq 0}\sum_{i\geq 0} c^k_{\mathcal{A}}(n+1-i, i+1)x^n = \sum_{n\geq 0} d^k_{\mathcal{A}}(n+1)x^n. \qquad \square$$

4.1 Restricting to consecutive small parts

Let $C^k_{\leq s}(n)$ denote the set of k-compositions of n with positive parts restricted to $\mathcal{A}_s = \{1, \dots, s\}$. From (4), the corresponding multivariate generating function is

$$F^k_{\mathcal{A}_s}(x, y, z) = \frac{z\sum_{i=1}^{s} x^i}{1 - z\sum_{i=0}^{k-1} y^i \sum_{i=1}^{s} x^i}, \quad k \geq 1.$$

Let $c^k_{\leq s}(n) = c_{\mathcal{A}_s}(n)$ denote the number of k-compositions of n with parts restricted to $\mathcal{A}_s$. The generating function of this counting sequence can be expressed as

$$F^k_{\mathcal{A}_s}(x, 1, 1) = \sum_{n\geq 0} c^k_{\leq s}(n)x^n = \frac{x(1-x^s)}{1-(1+k)x+kx^{s+1}}.$$

Table 5 provides examples of the sequence $c^k_{\leq s}(n)$ for $2 \leq s \leq 5$. In this and subsequent tables, we refer to well-known sequences by name rather than OEIS sequence number [17], such as tribonacci numbers A000073 and the Narayana's cows sequence A000930.

sequence\\n	1	2	3	4	5	6	7	8	9	10	OEIS
$c^1_{\leq 2}(n)$	1	2	3	5	8	13	21	34	55	89	Fibonacci
$c^1_{\leq 3}(n)$	1	2	4	7	13	24	44	81	149	274	tribonacci
$c^1_{\leq 4}(n)$	1	2	4	8	15	29	56	108	208	401	tetranacci
$c^1_{\leq 5}(n)$	1	2	4	8	16	31	61	120	236	464	pentanacci
$c^2_{\leq 2}(n)$	1	3	8	22	60	164	448	1224	3344	9136	A028859
$c^2_{\leq 3}(n)$	1	3	9	26	76	222	648	1892	5524	16128	A119826
$c^2_{\leq 4}(n)$	1	3	9	27	80	238	708	2106	6264	18632	A209239
$c^2_{\leq 5}(n)$	1	3	9	27	81	242	724	2166	6480	19386	
$c^3_{\leq 2}(n)$	1	4	15	57	216	819	3105	11772	44631	169209	A125145
$c^3_{\leq 3}(n)$	1	4	16	63	249	984	3888	15363	60705	239868	A282310
$c^3_{\leq 4}(n)$	1	4	16	64	255	1017	4056	16176	64512	257283	
$c^3_{\leq 5}(n)$	1	4	16	64	256	1023	4089	16344	65328	261120	
$c^4_{\leq 2}(n)$	1	5	24	116	560	2704	13056	63040	304384	1469696	A086347
$c^4_{\leq 3}(n)$	1	5	25	124	616	3060	15200	75504	375056	1863040	
$c^4_{\leq 4}(n)$	1	5	25	125	624	3116	15560	77700	388000	1937504	
$c^4_{\leq 5}(n)$	1	5	25	125	625	3124	15616	78060	390200	1950500	

Table 5: Total number of compositions in $C^k_{A_s}(n)$ for small k, n, s.

From the information proved in the OEIS for the sequences in Table 5, we can give the following alternative combinatorial interpretation for the sequence $c^k_{\leq s}(n)$ using deterministic finite automata.

Following [19], a deterministic finite automaton (DFA) can be considered as a directed graph with states represented by circles. One state is designated as the initial state, marked by an incoming arrow, while some states are identified as final states, represented by double circles. Each edge in the graph is labeled with a symbol from a given alphabet. A word is accepted or recognized by the automaton if there is a path starting from the initial state and following the labeled edges that corresponds to the sequence of symbols in the word, ending at a final state.

Theorem 4.2. *The sequence $c^k_{\le s}(n+1)$ gives the number of words of length n over the alphabet $\Sigma_k = \{\mathtt{0}, \dots, \mathtt{k}\}$ that avoid the subword $\mathtt{0}^s$.*

This result follows from a bijection described in [13, Prop. 2], which we outline here for completeness.

Combinatorial proof. Each multicomposition contributing to the count $c^k_{\le s}(n+1)$ can be considered as follows: Given a length $n+1$ board, there are n junctures that connect or separate the squares on either side. Working left to right, write J if the pair of squares are joined as part of a longer part and write S_m if the two squares belong to different parts and are separated by $m-1$ zeros. Note that $1 \le m \le k$ since we are working with k-compositions.

The conversion to words over the alphabet Σ_k is then $J \mapsto \mathtt{0}$ and $S_m \mapsto \mathtt{m}$. By the restriction on parts, the longest possible string of zeros is $\mathtt{0}^{s-1}$ coming from a part $\mathtt{s}$. The correspondence is clearly reversible. □

Proof using an automaton. From the description of the language, it is not difficult to show that

$$L^k_s = \{w \in \Sigma^*_k \mid w \text{ avoids the subword } \mathtt{0}^s\}$$

satisfies $L^k_s = L(\mathcal{M}_{s,k})$ where $\mathcal{M}_{s,k}$ is the automaton defined in Figure 1.

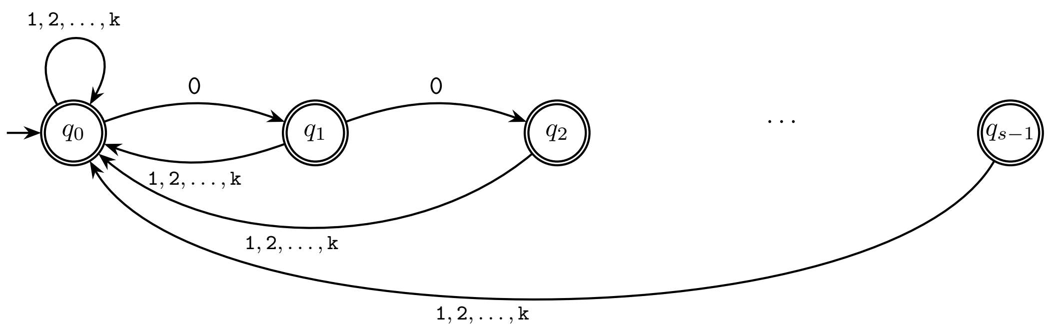

Figure 1: Transition diagram of $\mathcal{M}_{s,k}$.

For each state q_i of $\mathcal{M}_{s,k}$, let $L_i(x)$ be the generating function that counts accepted words by length when q_i serves as the initial state (with the same set of accepting states as in $\mathcal{M}_{s,k}$).

The automaton $\mathcal{M}_{s,k}$ gives rise to the following system of equations for the associated generating functions:

$$\begin{aligned}
L_0(x) &= kxL_0(x) + xL_1(x) + 1,\\
L_1(x) &= kxL_0(x) + xL_2(x) + 1,\\
L_2(x) &= kxL_0(x) + xL_3(x) + 1,\\
&\vdots\\
L_{s-2}(x) &= kxL_0(x) + xL_{s-1}(x) + 1,\\
L_{s-1}(x) &= kxL_0(x) + 1.
\end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix}
1-kx & -x & 0 & 0 & \cdots & 0 & 0 & 1\\
-kx & 1 & -x & 0 & \cdots & 0 & 0 & 1\\
-kx & 0 & 1 & -x & \cdots & 0 & 0 & 1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
-kx & 0 & 0 & 0 & \cdots & 1 & -x & 1\\
-kx & 0 & 0 & 0 & \cdots & 0 & 1 & 1
\end{bmatrix},$$

which, by Gaussian elimination, gives

$$L_0(x) = \frac{1-x^s}{1-(1+k)x+kx^{s+1}} = \sum_{n\geq 1} c^k_{\leq s}(n)x^{n-1},$$

and the result follows by comparing coefficients. □

Table 6 gives the first few words accepted by the automaton $\mathcal{M}_{2,3}$ defined by the transition graph in Figure 1.

length	words	count
0	ϵ	1
1	0, 1, 2, 3	4
2	01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33	15
3	010, 011, 012, 013, 020, 021, 022, 023, 030, 031, 032, 033, 101, 102, 103, 110, 111, 112, 113, 120, 121, 122, 123, 130, 131, 132, 133, 201, 202, 203, 210, 211, 212, 213, 220, 221, 222, 223, 230, 231, 232, 233, 301, 302, 303, 310, 311, 312, 313, 320, 321, 322, 323, 330, 331, 332, 333	57

Table 6: Short words accepted by the automaton $\mathcal{M}_{2,3}$.

Let $c^k_{\leq s}(n,\ell)$ denote the number of k-compositions of n with ℓ positive parts in $\mathcal{A}_s = \{1,\dots,s\}$. If $\mathcal{R}^k_{\leq s} = [c^k_{\leq s}(n+1,\ell+1)]_{n,\ell\geq 0}$, then, by Theorem 4.1, this is the Riordan array

$$\mathcal{R}^k_{\leq s} = \left(\frac{1-x^s}{1-x}, kx\frac{1-x^s}{1-x}\right).$$

For example, the first eight rows and columns of $\mathcal{R}^2_{\leq 2}$ and $\mathcal{R}^4_{\leq 3}$ are, respectively,

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 1&2&0&0&0&0&0&0\\ 0&4&4&0&0&0&0&0\\ 0&2&12&8&0&0&0&0\\ 0&0&12&32&16&0&0&0\\ 0&0&4&48&80&32&0&0\\ 0&0&0&32&160&192&64&0\\ 0&0&0&8&160&480&448&128 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 1&4&0&0&0&0&0&0\\ 1&8&16&0&0&0&0&0\\ 0&12&48&64&0&0&0&0\\ 0&8&96&256&256&0&0&0\\ 0&4&112&640&1280&1024&0&0\\ 0&0&96&1024&3840&6144&4096&0\\ 0&0&48&1216&7680&21504&28672&16384 \end{pmatrix}.$$

(This is how we present specific Riordan arrays, which are infinite objects.) Observe that the row sums of these matrices correspond to the sequences $\{c^2_{\leq 2}(n)\}_{n\geq 1}$ and $\{c^4_{\leq 3}(n)\}_{n\geq 1}$, respectively.

By Corollary 4.1, the generating function for the diagonal sums $d^k_{\leq s}(n) = d_{\mathcal{A}_s}(n)$ is

$$\sum_{n\geq 1} d^k_{\leq s}(n)x^n = \frac{x(1-x^s)}{1-x-kx^2+kx^{s+2}}.$$

Table 7 provides examples of the sequence $d^k_{\leq s}(n)$ for $2\leq s\leq 5$.

4.2 Parts from arithmetic progressions

In this section, we examine the number of k-compositions of n with positive parts restricted to the set $\mathcal{A}_{a,b} = \{ka+b \mid k \in \mathbb{Z}^{\geq 0}\}$ where a and b are positive integers. Similar families of compositions have been studied in [1,3,4,12,14,15,21]. Notice that for $a=2$ and $b=1$ (resp., $a=2$ and $b=2$), the set $\mathcal{A}_{a,b}$ corresponds to the odd positive integers (resp., the even positive integers). Let $L_{a,b}(x)$ denote the generating function

$$L_{a,b}(x) = \sum_{i\in\mathcal{A}_{a,b}} x^i = \frac{x^b}{1-x^a}.$$

Using (4), we obtain the generating function

$$F^k_{\mathcal{A}_{a,b}}(x,y,z) = \frac{zL_{a,b}(x)}{1-z\sum_{i=0}^{k-1} y^i L_{a,b}(x)} = \frac{x^b(1-y)z}{1-x^a(1-y)-y-x^b(1-y^k)z}, \quad k\geq 1.$$

sequence\\n	1	2	3	4	5	6	7	8	9	10	OEIS
$d^1_{\le 2}(n)$	1	1	1	2	2	3	4	5	7	9	Padovan
$d^1_{\le 3}(n)$	1	1	2	2	4	5	8	11	17	24	A013979
$d^1_{\le 4}(n)$	1	1	2	3	4	7	10	16	24	37	A013982
$d^1_{\le 5}(n)$	1	1	2	3	5	7	12	18	29	45	A013983
$d^2_{\le 2}(n)$	1	1	2	4	6	12	20	36	64	112	A107383
$d^2_{\le 3}(n)$	1	1	3	4	10	16	34	60	120	220	
$d^2_{\le 4}(n)$	1	1	3	5	10	20	38	76	146	288	
$d^2_{\le 5}(n)$	1	1	3	5	11	20	42	80	162	316	
$d^3_{\le 2}(n)$	1	1	3	6	12	27	54	117	243	513	
$d^3_{\le 3}(n)$	1	1	4	6	18	33	84	171	405	864	
$d^3_{\le 4}(n)$	1	1	4	7	18	39	90	204	462	1053	
$d^3_{\le 5}(n)$	1	1	4	7	19	39	96	210	495	1113	
$d^4_{\le 2}(n)$	1	1	4	8	20	48	112	272	640	1536	
$d^4_{\le 3}(n)$	1	1	5	8	28	56	164	368	992	2352	
$d^4_{\le 4}(n)$	1	1	5	9	28	64	172	424	1092	2752	
$d^4_{\le 5}(n)$	1	1	5	9	29	64	180	432	1148	2856	

Table 7: Diagonal sums $d^k_{\le s}(n)$ for small k, n, s.

Furthermore, the ordinary generating function for $c^k_{\mathcal{A}_{a,b}}(n)$ simplifies to the rational generating function

$$F^k_{\mathcal{A}_{a,b}}(x,1,1) = \sum_{n\ge 0} c^k_{\mathcal{A}_{a,b}}(n)x^n = \frac{x^b}{1-x^a-kx^b}.$$

Table 8 gives several examples of $c^k_{\mathcal{A}_{a,b}}(n)$, some with offsets in the corresponding OEIS matches.

Let $c^k_{\mathcal{A}_{a,b}}(n,\ell)$ denote the number of k-compositions of n with ℓ positive parts in $\mathcal{A}_{a,b}$ and let $\mathcal{R}^k_{a,1} = [c^k_{\mathcal{A}_{a,1}}(n+1,\ell+1)]_{n,\ell\ge 0}$. By Theorem 4.1, for any positive integers a and k, the matrix $\mathcal{R}^k_{a,1}$ is a Riordan array,

$$\mathcal{R}^k_{a,1} = \left(\frac{1}{1-x^a}, \frac{kx}{1-x^a}\right).$$

For example, for $(a,k) = (2,2)$ and $(a,k) = (3,2)$, we obtain the following matrices where the row sums correspond to the sequences $\{c^2_{\mathcal{A}_{2,1}}(n)\}_{n\ge 1}$ and $\{c^2_{\mathcal{A}_{3,1}}(n)\}_{n\ge 1}$, respectively.

$$\mathcal{R}^2_{\mathcal{A}_{2,1}} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0\\ 1&0&4&0&0&0&0&0\\ 0&4&0&8&0&0&0&0\\ 1&0&12&0&16&0&0&0\\ 0&6&0&32&0&32&0&0\\ 1&0&24&0&80&0&64&0\\ 0&8&0&80&0&192&0&128 \end{pmatrix}, \quad \mathcal{R}^2_{\mathcal{A}_{3,1}} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&2&0&0&0&0&0&0\\ 0&0&4&0&0&0&0&0\\ 1&0&0&8&0&0&0&0\\ 0&4&0&0&16&0&0&0\\ 0&0&12&0&0&32&0&0\\ 1&0&0&32&0&0&64&0\\ 0&6&0&0&80&0&0&128 \end{pmatrix}.$$

Note that the matrix $\mathcal{R}^2_{\mathcal{A}_{2,1}}$corresponds to the matrix of coefficients of Pell polynomials A115322; the Pell polynomials $P_n(x)$ are defined recursively as

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$$

for $n \ge 2$ with initial values $P_0(x) = 0$ and $P_1(x) = 1$. The next few are

$$P_2(x) = 2x,\ P_3(x) = 1+4x^2,\ P_4(x) = 4x+8x^3,\ P_5(x) = 1+12x^2+16x^4.$$

Thus, we have an automatic combinatorial interpretation for the coefficients of the Pell polynomials in terms of compositions. Additional combinatorial interpretations for generalized Fibonacci polynomials are given by Ramírez and Sirvent [20]. Also, $\mathcal{R}^2_{\mathcal{A}_{3,1}}$ matches the absolute value of A136334.

sequence\\n	1	2	3	4	5	6	7	8	9	10	OEIS
$c^1_{\mathcal{A}_{2,1}}(n)$	1	1	2	3	5	8	13	21	34	55	Fibonacci
$c^2_{\mathcal{A}_{2,1}}(n)$	1	2	5	12	29	70	169	408	985	2378	A000129
$c^3_{\mathcal{A}_{2,1}}(n)$	1	3	10	33	109	360	1189	3927	12970	42837	A006190
$c^4_{\mathcal{A}_{2,1}}(n)$	1	4	17	72	305	1292	5473	23184	98209	416020	A001076
$c^1_{\mathcal{A}_{2,3}}(n+4)$	1	1	1	2	2	3	4	5	7	9	Padovan
$c^2_{\mathcal{A}_{2,3}}(n+4)$	1	2	1	4	5	6	13	16	25	42	A052947
$c^3_{\mathcal{A}_{2,3}}(n+4)$	1	3	1	6	10	9	28	39	55	123	A106855
$c^4_{\mathcal{A}_{2,3}}(n+4)$	1	4	1	8	17	12	49	80	97	276	
$c^1_{\mathcal{A}_{3,1}}(n)$	1	1	1	2	3	4	6	9	13	19	Narayana's cows
$c^2_{\mathcal{A}_{3,1}}(n)$	1	2	4	9	20	44	97	214	472	1041	A008998
$c^3_{\mathcal{A}_{3,1}}(n)$	1	3	9	28	87	270	838	2601	8073	25057	A052541
$c^4_{\mathcal{A}_{3,1}}(n)$	1	4	16	65	264	1072	4353	17676	71776	291457	A052927
$c^1_{\mathcal{A}_{3,2}}(n+3)$	1	1	1	2	2	3	4	5	7	9	Padovan
$c^2_{\mathcal{A}_{3,2}}(n+3)$	2	1	4	4	9	12	22	33	56	88	A008346
$c^3_{\mathcal{A}_{3,2}}(n+3)$	3	1	9	6	28	27	90	109	297	417	A052931
$c^4_{\mathcal{A}_{3,2}}(n+3)$	4	1	16	8	65	48	268	257	1120	1296	
$c^1_{\mathcal{A}_{3,4}}(n+6)$	1	1	0	1	2	1	1	3	3	2	A017817
$c^2_{\mathcal{A}_{3,4}}(n+6)$	1	2	0	1	4	4	1	6	12	9	A077909
$c^3_{\mathcal{A}_{3,4}}(n+6)$	1	3	0	1	6	9	1	9	27	28	
$c^4_{\mathcal{A}_{3,4}}(n+6)$	1	4	0	1	8	16	1	12	48	65	
$c^1_{\mathcal{A}_{4,1}}(n)$	1	1	1	1	2	3	4	5	7	10	A003269
$c^2_{\mathcal{A}_{4,1}}(n)$	1	2	4	8	17	36	76	160	337	710	A008999
$c^3_{\mathcal{A}_{4,1}}(n)$	1	3	9	27	82	249	756	2295	6967	21150	A052917
$c^4_{\mathcal{A}_{4,1}}(n)$	1	4	16	64	257	1032	4144	16640	66817	268300	A098590
$c^1_{\mathcal{A}_{4,3}}(n+5)$	1	1	0	1	2	1	1	3	3	2	A017817
$c^2_{\mathcal{A}_{4,3}}(n+5)$	2	1	0	4	4	1	8	12	6	17	A052922
$c^3_{\mathcal{A}_{4,3}}(n+5)$	3	1	0	9	6	1	27	27	9	82	
$c^4_{\mathcal{A}_{4,3}}(n+5)$	4	1	0	16	8	1	64	48	12	257	

Table 8: Values related to $c^k_{\mathcal{A}_{a,b}}(n)$ for small a, b, k, n.

From the above relationship, we can introduce the following generalization of the Pell polynomials. Let

$$P^k_{a,n}(x) = \sum_{\ell=0}^{n} c^k_{\mathcal{A}_{a,1}}(n+1, \ell+1)x^\ell.$$

Proposition 4.1. *The generating function of the generalized Pell polynomials* $P^k_{a,n}(x)$ *is*

$$\sum_{n\geq 0} P^k_{a,n}(x)z^n = \frac{1}{1-kxz-z^a}.$$

Moreover, the polynomials $P^k_{a,n}(x)$ *satisfy the recurrence relation*

$$P^k_{a,n}(x) = kxP^k_{a,n-1}(x) + P^k_{a,n-a}(x).$$

Proof. The coefficients of $P^k_{a,n}(x)$ correspond to the rows of the Riordan array $\mathcal{R}^k_{a,1} = \left(\frac{1}{1-z^a}, \frac{kz}{1-z^a}\right)$. Multiplying the right-hand side of this equality by the vector $(1, x, x^2, \dots)^T$, which has generating function $1/(1-xz)$, and using the summation property, the resulting vector has generating function

$$\mathcal{R}^k_{a,1}\left(\frac{1}{1-xz}\right) = \frac{1}{1-z^a} \cdot \frac{1}{1-x\frac{kz}{1-z^a}} = \frac{1}{1-kxz-z^a}.$$

The recurrence relation follows directly from the generating function. □

Returning to the triangle of $c^k_{\mathcal{A}_{a,b}}(n,\ell)$ values, the generating function for the diagonal sum $d^k_{\mathcal{A}_{a,1}}(n)$ is

$$\sum_{n\geq 1} d^k_{\mathcal{A}_{a,1}}(n)x^{n-1} = \frac{1}{1-kx^2-x^a}$$

by Corollary 4.1. Table 9 describes several particular $d^k_{\mathcal{A}_{a,1}}(n)$ sequences.

k	a	description
1	2	2^n interspersed with zeros
1	3	Padovan numbers
1	4	Fibonacci numbers interspersed with zeros
2	2	3^n interspersed with zeros
2	3	A008346
2	4	Pell numbers interspersed with zeros
3	2	4^n interspersed with zeros
3	3	A052931
3	4	A006190 interspersed with zeros
4	2	5^n interspersed with zeros
4	3	A099781 for even n (no match for odd n)
4	4	A001076 interspersed with zeros

Table 9: Diagonal sum sequences $d^k_{\mathcal{A}_{a,1}}(n)$ for small a and k.

4.3 Avoiding single part sizes

Let $c^k_{\hat{t}}(n)$ denote the number of k-compositions of n that omit the part t. Theorem 3.2 gives the following corollary.

Corollary 4.2. *The generating function for $c^k_{\hat{t}}(n)$ is*

$$G^k_t(x,0) = \sum_{n\geq 1} c^k_{\hat{t}}(n)x^n = \frac{x-x^t+x^{t+1}}{1-x-kx+kx^t-kx^{t+1}}.$$

Table 10 gives several examples of $c^k_{\hat{t}}(n)$ sequences including Cayley's 1876 result that compositions with no parts 1 are counted by the Fibonacci numbers [6].

k	$t\backslash n$	1	2	3	4	5	6	7	8	9	10	OEIS
1	1	0	1	1	2	3	5	8	13	21	34	Fibonacci
1	2	1	1	2	4	7	12	21	37	65	114	A005251
1	3	1	2	3	6	11	21	39	73	136	254	A049856
1	4	1	2	4	7	14	27	52	101	195	377	A108758
2	1	0	1	1	3	5	11	21	43	85	171	Jacobsthal
2	2	1	2	5	13	33	83	209	527	1329	3351	A120925
2	3	1	3	8	23	65	185	525	1491	4233	12019	
2	4	1	3	9	26	77	227	669	1973	5817	17151	
3	1	0	1	1	4	7	19	40	97	217	508	A006130
3	2	1	3	10	34	115	388	1309	4417	14905	50296	A255813
3	3	1	4	15	58	223	859	3307	12733	49024	188752	
3	4	1	4	16	63	250	991	3928	15571	61723	244669	

Table 10: Counts $c^k_{\hat{t}}(n)$ for small k, n, t.

Proposition 4.2. *The sequence $c^k_{\hat{2}}(n+1)$ also gives the number of words of length n on the alphabet $\Sigma_k = \{0,\ldots,k\}$ with no isolated zeros.*

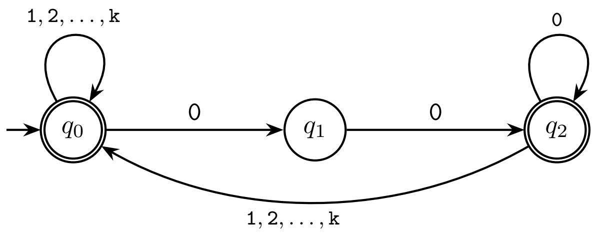

Figure 2: Transition diagram of $\mathcal{M}_k$.

Proof. Consider the language $L^k = \{w \in \Sigma_k^* \mid w \text{ has no isolated zeros}\}$. This language can be recognized by the automaton $\mathcal{M}_k$ shown in Figure 2.

The automaton $\mathcal{M}_k$ gives rise to the following system of equations for the generating functions associated with the states q_0, q_1, and q_2:

$$\begin{aligned}
L_0(x) &= kxL_0(x) + xL_1(x) + 1,\\
L_1(x) &= xL_2(x),\\
L_2(x) &= xL_2(x) + kxL_0(x) + 1.
\end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1-kx & -x & 0 & 1\\ 0 & 1 & -x & 0\\ -kx & 0 & 1-x & 1 \end{bmatrix},$$

which, by Gaussian elimination, gives

$$L_0(x) = \frac{1-x+x^2}{1-(1+k)x+kx^2-kx^3} = \sum_{n\geq 1} c_{\hat{2}}^k(n)x^{n-1},$$

and the result follows by comparing coefficients. □

Let $c_{\hat{t}}^k(n,\ell)$ denote the number of k-compositions of n with ℓ positive parts in $\mathbb{Z}^+ \setminus \{t\}$ and let $\mathcal{R}_{\hat{t}}^k = [c_{\hat{t}}^k(n+1,\ell+1)]_{n,\ell\geq 0}$. By Theorem 4.1, for any integers $t \geq 2$ and $k \geq 1$, the matrix $\mathcal{R}_{\hat{t}}^k$ is a Riordan array,

$$\mathcal{R}_{\hat{t}}^k = \left(\frac{1-x^{t-1}+x^t}{1-x}, kx\frac{1-x^{t-1}+x^t}{1-x}\right).$$

For example, for $(k,\hat{t}) = (2,2)$ and $(k,\hat{t}) = (3,2)$, we have

$$\mathcal{R}_{\hat{2}}^2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 4 & 0 & 0 & 0 & 0 & 0\\
1 & 4 & 0 & 8 & 0 & 0 & 0 & 0\\
1 & 4 & 12 & 0 & 16 & 0 & 0 & 0\\
1 & 6 & 12 & 32 & 0 & 32 & 0 & 0\\
1 & 8 & 24 & 32 & 80 & 0 & 64 & 0\\
1 & 10 & 36 & 80 & 80 & 192 & 0 & 128
\end{pmatrix}, \quad
\mathcal{R}_{\hat{2}}^3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 9 & 0 & 0 & 0 & 0 & 0\\
1 & 6 & 0 & 27 & 0 & 0 & 0 & 0\\
1 & 6 & 27 & 0 & 81 & 0 & 0 & 0\\
1 & 9 & 27 & 108 & 0 & 243 & 0 & 0\\
1 & 12 & 54 & 108 & 405 & 0 & 729 & 0\\
1 & 15 & 81 & 270 & 405 & 1458 & 0 & 2187
\end{pmatrix}.$$

These arrays are related to A115322, A135871, respectively.

From Corollary 4.1, the generating function for $d_{\hat{t}}^k(n)$ is

$$\sum_{n\geq 1} d_{\hat{t}}^k(n)x^{n-1} = \frac{1-x^{t-1}+x^t}{1-x-kx^2+kx^{t+1}-kx^{t+2}}.$$

Table 11 gives several examples of $d_{\hat{t}}^k(n)$ sequences.

5. Specified numbers of zeros in 2-compositions

In this last section, we examine several patterns among 2-compositions, which allow at most one zero between positive parts, focusing on the number of zeros and various restrictions on the positive parts allowed.

Let $c_{0,\mathcal{A}}^2(n,\ell)$ denote the number of 2-compositions of n with ℓ zeros and parts in $\mathcal{A}$. Consider the matrix $\mathcal{H}_{0,\mathcal{A}}^2 = \left[c_{0,\mathcal{A}}^2(n+1,\ell)\right]_{n,\ell\geq 0}$. The next result follows from Theorem 4.1.

k	$\hat{t}\backslash n$	1	2	3	4	5	6	7	8	9	10	OEIS
1	1	1	1	1	2	3	4	6	9	13	19	Narayana's cows
1	2	0	2	1	4	3	8	8	17	20	37	
1	3	1	1	3	3	6	9	13	22	32	51	
1	4	1	2	2	5	6	11	16	27	40	66	
2	1	1	1	1	3	5	7	13	23	37	63	A077949
2	2	0	3	1	9	5	27	21	83	81	259	
2	3	1	2	5	7	17	29	57	111	205	403	
2	4	1	3	4	11	17	39	69	145	269	547	
3	1	1	1	1	4	7	10	22	43	73	139	A084386
3	2	0	4	1	16	7	64	40	259	208	1057	
3	3	1	3	7	13	34	67	157	340	748	1669	
3	4	1	4	6	19	34	91	184	451	964	2272	
4	1	1	1	1	5	9	13	33	69	121	253	A089977
4	2	0	5	1	25	9	125	65	629	425	3181	
4	3	1	4	9	21	57	129	337	805	2009	4941	
4	4	1	5	8	29	57	173	385	1065	2521	6669	

Table 11: Diagonal sum sequences $d_{\hat{t}}^{k}(n)$ for small k, n, t.

Proposition 5.1. *If $1 \in \mathcal{A}$, then the matrix $\mathcal{H}_{0,\mathcal{A}}^2$ is a Riordan array*

$$\mathcal{H}_{0,\mathcal{A}}^2 = \left(\frac{\sum_{i\in\mathcal{A}} x^{i-1}}{1-\sum_{i\in\mathcal{A}} x^i}, \frac{\sum_{i\in\mathcal{A}} x^{i}}{1-\sum_{i\in\mathcal{A}} x^i} \right).$$

For example,

$$\mathcal{H}_{0,\mathbb{Z}^+}^2 = \left(\frac{1}{1-2x}, \frac{x}{1-2x} \right) = \left[\binom{n}{\ell} 2^{n-\ell} \right]_{n,\ell\geq 0},$$

which matches A038207. Note that the diagonal sums are the Pell numbers.

If $\mathcal{A}$ consists of the odd positive integers, then we have A037027 whose diagonal sums are the Jacobsthal numbers:

$$\mathcal{H}_{0,\text{odd}}^2 = \left(\frac{1}{1-x-x^2}, \frac{x}{1-x-x^2} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\ 5 & 10 & 9 & 4 & 1 & 0 & 0 & 0 \\ 8 & 20 & 22 & 14 & 5 & 1 & 0 & 0 \\ 13 & 38 & 51 & 40 & 20 & 6 & 1 & 0 \\ 21 & 71 & 111 & 105 & 65 & 27 & 7 & 1 \end{pmatrix}.$$

In general, if $\mathcal{A}_d = \{kd+1 \mid k \in \mathbb{Z}^{\geq 0}\}$, then let

$$\mathcal{H}_d = \mathcal{H}_{0,\mathcal{A}_d}^2 = \left(\frac{1}{1-x-x^d}, \frac{x}{1-x-x^d} \right),$$

so that the previous example with odd positive integers is $\mathcal{H}_2$. One can verify that $\mathcal{H}_3$ matches A202191 with diagonal sums the tribonacci numbers. Further, $\mathcal{H}_4$ matches A259074 with diagonal sums A060945. Note that these OEIS arrays and sequences do not currently include combinatorial interpretations.

We give a general result for $\mathcal{H}_d$ whose proof relies on the Riordan group structure.

Proposition 5.2. *The (n,k)-th entry of the matrix $\mathcal{H}_d$ is*

$$\sum_{\ell=0}^{n} u_d(n,\ell) \binom{\ell}{k}$$

where

$$u_d(n,\ell) = \begin{cases} \binom{\ell + \frac{n-\ell}{d}}{\ell} & \text{if } d \mid n-\ell, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We show that the matrix $\mathcal{H}_d$ factors as

$$\mathcal{H}_d = \left(\frac{1}{1-x^d}, \frac{x}{1-x^d}\right)\mathcal{P}$$

where $\mathcal{P} = \left[\binom{n}{k}\right]_{n,k\geq 0}$ is Pascal's matrix. It is easy to show that $\mathcal{P}$ is the Riordan array $\mathcal{P} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$. By the product rule (5),

$$\begin{aligned}\left(\frac{1}{1-x^d}, \frac{x}{1-x^d}\right)\mathcal{P} &= \left(\frac{1}{1-x^d}, \frac{x}{1-x^d}\right)\left(\frac{1}{1-x}, \frac{x}{1-x}\right)\\ &= \left(\frac{1}{1-x^d}\left(\frac{1}{1-\frac{x}{1-x^d}}\right), \frac{\frac{x}{1-x^d}}{1-\frac{x}{1-x^d}}\right)\\ &= \left(\frac{1}{1-x-x^d}, \frac{x}{1-x-x^d}\right) = \mathcal{H}_d.\end{aligned}$$

From the definition of a Riordan array, we have

$$\left(\frac{1}{1-x^d}, \frac{x}{1-x^d}\right) = \left[u_d(n,k)\right]_{n,k\geq 0}.$$

Therefore, the product of matrices gives the desired result. ☐

We conclude with one further example. Restricting positive parts to $\mathcal{A} = \{1,2\}$ gives

$$\mathcal{H}^2_{0,\{1,2\}} = \left(\frac{1+x}{1-x-x^2}, \frac{x(1+x)}{1-x-x^2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 3 & 4 & 1 & 0 & 0 & 0 & 0 & 0\\ 5 & 10 & 6 & 1 & 0 & 0 & 0 & 0\\ 8 & 22 & 21 & 8 & 1 & 0 & 0 & 0\\ 13 & 45 & 59 & 36 & 10 & 1 & 0 & 0\\ 21 & 88 & 147 & 124 & 55 & 12 & 1 & 0\\ 34 & 167 & 339 & 366 & 225 & 78 & 14 & 1 \end{pmatrix},$$

which matches A154929. Generating functions for the row sums and diagonal sums follow from the Riordan array structure. The row sums match A028859; compare the proof of [13, Proposition 8(a)]. The diagonal sums match A141015. As with several of our prior examples, these are apparently new combinatorial interpretations for these OEIS arrays and sequences.

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