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Central Trinomial Numbers, Motzkin Trees, and the Riordan Group

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Abstract: In this paper, we use properties of the Riordan group to discover identities involving the generating functions of the Motzkin numbers and central trinomial numbers. We provide combinatorial interpretations of some families of trees satisfying a uniform updegree requirement to prove the identities. We also answer a uniqueness question that arises in the Bell subgroup.

Keywords: Riordan group; Motzkin and central trinomial numbers; UUR trees
2020 Mathematics Subject Classification: 05A15; 05A05; 05A19; 11B83

Dedicated to the memory of Emanuele Munarini and Renzo Sprugnoli

1 Introduction

1.1 The Central Trinomial Numbers

The central trinomial number t_n is defined as the coefficient of z^n in the expansion of the polynomial $(1+z+z^2)^n$, and the generating function of these numbers is

$$E(z) = \frac{1}{\sqrt{1-2z-3z^2}}.$$

We can also think of the central trinomial number as the central (and largest) coefficient $\binom{n}{0}_2$ among the trinomial coefficients $\binom{n}{j}_2$ in the expansion of the Laurent polynomial

$$(1+z+z^{-1})^n = \sum_{j=-n}^{n} \binom{n}{j}_2 z^j, \qquad \binom{n}{j}_2 = \sum_{k\geq 0} \frac{n!}{k!(j+k)!(n-2k-j)!},$$

where terms with negative factorial arguments are interpreted as zero. These trinomial coefficients have properties similar to those of the binomial coefficients, namely,

$$\binom{n}{j}_2 = \binom{n-1}{j-1}_2 + \binom{n-1}{j}_2 + \binom{n-1}{j+1}_2.$$

For the first few values of n, the coefficients are

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & 1 & 1 & & & & \\
 & & & 1 & 2 & 3 & 2 & 1 & & & \\
 & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & \\
 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 & \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 \\
 & \ddots & & & & \vdots & & & & \ddots &
\end{array}$$

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The entries in the central column are the central trinomial numbers, and

$$t_n = \binom{n}{0}_2 = \sum_{k\geq 0} \frac{n!}{(k!)^2(n-2k)!}.$$

The study of these numbers dates back to Euler's time. According to Andrews [2], Euler wrote his short note titled "Exemplum Memorabile Inductionis Fallacis" [8] after discovering a mysterious property of these numbers:

$$3t_{n+1} - t_{n+2} = F_n(F_n + 1),$$

where F_n is the Fibonacci sequence defined by $F_{-1} = 1$, $F_0 = 0$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 1$. Unfortunately, as Euler noted, this property is valid only for $-1 \leq n \leq 7$.

1.2 The Motzkin Numbers

The Motzkin numbers m_n are the coefficients of z^n in the series expansion of $M(z)$, which satisfies

$$M(z) = 1 + zM(z) + z^2M^2(z).$$

It follows that

$$m_n = [z^n]\left(\frac{(1-z) - \sqrt{1-2z-3z^2}}{2z^2}\right), \qquad n = 0, 1, 2, \ldots,$$

and the first few entries are

$$1, 1, 2, 4, 9, 21, 51, 127, \ldots.$$

It is well known [1] that the Motzkin numbers satisfy

$$m_0 = 1, \qquad m_{n+1} = m_n + \sum_{k=0}^{n-1} m_k m_{n-1-k}.$$

Donaghey and Shapiro gave fourteen combinatorial interpretations of the Motzkin numbers in their seminal 1977 paper [7] and noted that there are many more situations in which the Motzkin numbers arise. One of their interpretations is in terms of ordered, or planar, trees in which every vertex has outdegree at most two. The number of such trees with n edges is the Motzkin number m_n. For example, the nine Motzkin trees with four edges are shown in Figure 1.

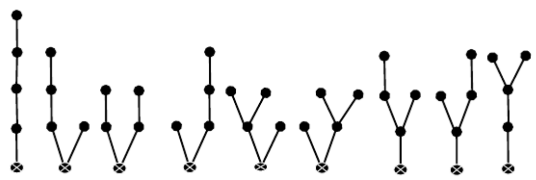

Figure 1: Motzkin trees with four edges.

2 Central Trinomial Numbers and Motzkin Trees

Barcucci, Pinzani, and Sprugnoli provide a detailed analysis of connections between central trinomial and Motzkin numbers in their 1991 paper [3] by deriving generating functions and recurrence relations that link central trinomial coefficients with Motzkin numbers. Recent articles by Liu and Sun [13, 18] discuss arithmetic properties of Motzkin and central trinomial numbers but do not provide a direct relationship. In this section, we obtain a direct relationship using the generating functions of the Motzkin and central trinomial numbers:

$$t_n = \sum_{k=0}^{\lfloor n/2 \rfloor} [z^n]M^{2k+1}(z), \qquad n = 0, 1, 2, \ldots.$$

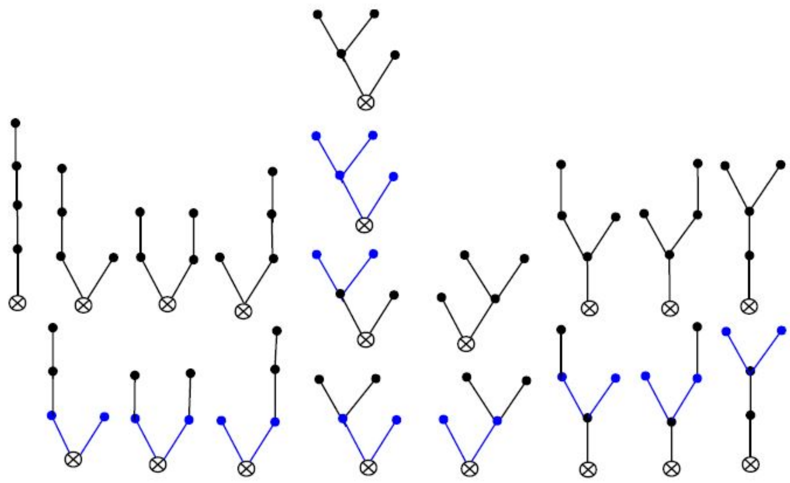

Figure 2: Bicolored Motzkin trees with four edges.

2.1 Bicolored Motzkin Trees

Consider a Motzkin tree with n edges, and bicolor the outgoing edges of each vertex of outdegree two on the leftmost path. Applying this coloring rule to the nine Motzkin trees shown in Figure 1, we obtain

$$1 + 2 + 2 + 2 + 4 + 2 + 2 + 2 + 2 = 19$$

different bicolored Motzkin trees, shown in Figure 2.

We claim that the number of Motzkin trees with n edges in which the outgoing edges at vertices of outdegree two on the leftmost path are bicolored is the central trinomial number t_n. Let $F(z)$ be the generating function for these bicolored Motzkin trees. Applying the symbolic method [10] to the decomposition by the outdegree of the root shown in Figure 3, we see that

$$F(z) = 1 + zF(z) + 2z^2M(z)F(z).$$

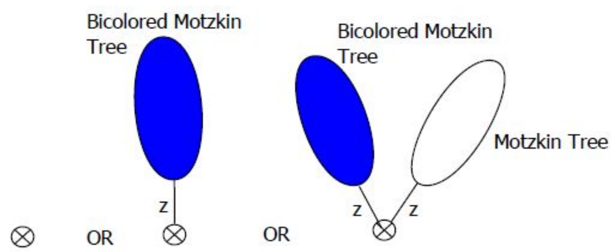

Figure 3: Decomposition of a bicolored Motzkin tree.

Solving for $F(z)$ and using

$$M(z) = \frac{(1-z) - \sqrt{1-2z-3z^2}}{2z^2},$$

we obtain

$$F(z) = \frac{1}{\sqrt{1-2z-3z^2}} = E(z),$$

which proves the claim.

Decomposing the bicolored Motzkin trees by the number of pairs of colored edges, as shown in Figure 4, and applying the symbolic method gives

$$E(z) = M(z) + z^2M^3(z) + z^4M^5(z) + z^6M^7(z) + \cdots = \frac{M(z)}{1 - z^2M^2(z)}.$$

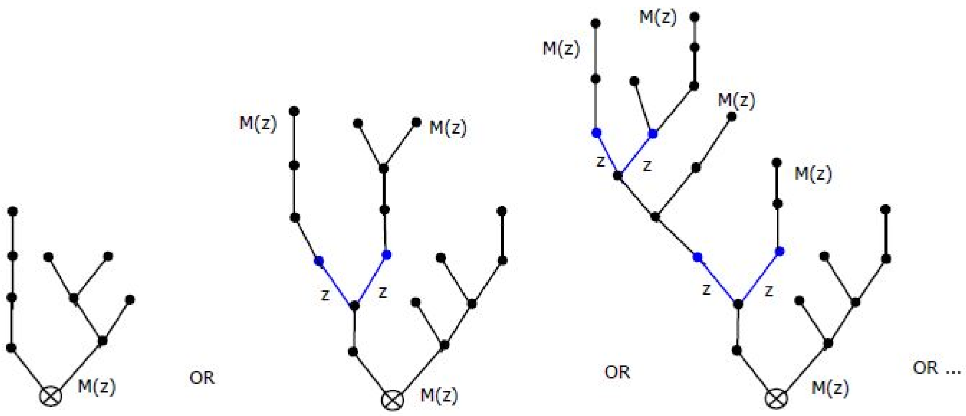

Figure 4: Decomposition of bicolored Motzkin trees by number of pairs of colored edges.

2.2 The Riordan Group Connection

Let

$$g(z) = 1 + \sum_{k\geq 1} g_k z^k, \qquad f(z) = \sum_{k\geq 1} f_k z^k, \qquad f_1 \neq 0.$$

A Riordan array $D = (g(z), f(z))$ is an infinite lower-triangular matrix whose kth column has generating function $g(z)f^k(z)$, for $k = 0, 1, 2, \ldots$. Its typical entry is

$$d_{n,k} = [z^n]g(z)f^k(z), \qquad n, k \geq 0.$$

Riordan arrays form a group under

$$(g(z), f(z)) * (h(z), k(z)) = (g(z)h(f(z)), k(f(z))).$$

The identity is $I = (1, z)$, and

$$(g(z), f(z))^{-1} = \left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z)\right),$$

where $\bar{f}(f(z)) = f(\bar{f}(z)) = z$. The Fundamental Theorem of Riordan Arrays [15, 16] states that if $A(z)$ and $B(z)$ are the generating functions of the column vectors

$$A = (a_0, a_1, a_2, \ldots)^T, \qquad B = (b_0, b_1, b_2, \ldots)^T,$$

then

$$(g, f)A = B \iff B(z) = g(z)A(f(z)).$$

The equation

$$E(z) = \frac{M(z)}{1 - z^2M^2(z)}$$

can therefore be rewritten as

$$(M(z), zM(z)) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 7 \\ 19 \\ 51 \\ \vdots \end{bmatrix},$$

where the beginning of the Riordan array is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 5 & 3 & 1 & 0 & 0 & \cdots \\ 9 & 12 & 9 & 4 & 1 & 0 & \cdots \\ 21 & 30 & 25 & 14 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We ask whether any other element $(g(z), zg(z))$ of the Bell subgroup satisfies

$$(g(z), zg(z))\left(\frac{1}{1-z^2}\right) = E(z).$$

By the Fundamental Theorem,

$$\frac{g(z)}{1-z^2g^2(z)} = E(z).$$

Letting $f(z) = zg(z)$ gives

$$f(z) = zE(z)(1-f^2(z)) \quad \Longleftrightarrow \quad zE(z)f^2(z) + f(z) - zE(z) = 0.$$

Solving for the branch with $f(0) = 0$ yields

$$f(z) = \frac{-\sqrt{1-2z-3z^2}+(1-z)}{2z} = zM(z), \qquad g(z) = M(z).$$

Thus $(M(z), zM(z))$ is the unique element of the Bell subgroup satisfying the equation.

If we count Motzkin trees by the number of double edges on the leftmost path, as shown in Figure 5, we obtain the Riordan array

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 7 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 15 & 5 & 0 & 0 & 0 & \cdots \\ 1 & 32 & 17 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

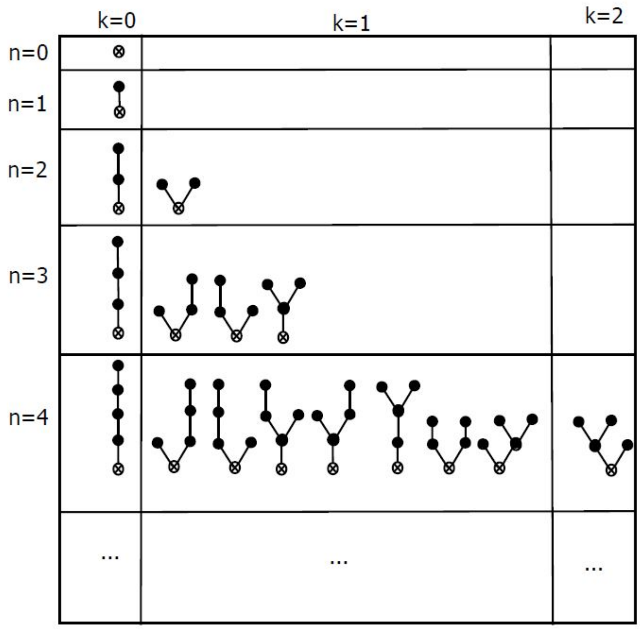

Figure 5: Motzkin trees with k double edges on the leftmost path.

The decomposition in Figure 6 shows that the generating function of the entries in column k is

$$\frac{z^{2k}M^k(z)}{(1-z)^{k+1}}.$$

Hence,

$$R = \left(\frac{1}{1-z}, \frac{z^2M(z)}{1-z}\right).$$

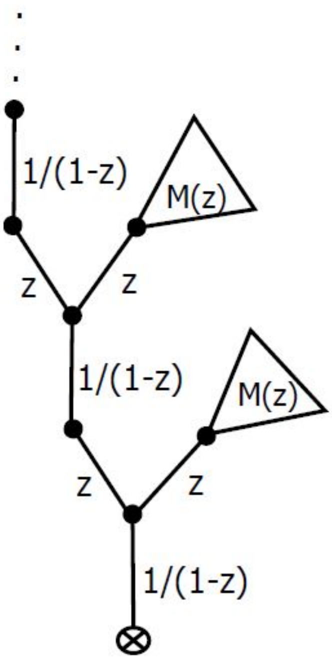

Figure 6: Decomposition of Motzkin trees by double edges on the leftmost path.

Multiplying R by $V = (1, 2, 4, 8, \ldots)^T$ gives

$$R \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 7 \\ 19 \\ 51 \\ \vdots \end{bmatrix}.$$

Proposition 1.

$$\left(\frac{1}{1-z}, \frac{z^2 M(z)}{1-z}\right)\left(\frac{1}{1-2z}\right) = E(z).$$

Proof. Applying the Fundamental Theorem of Riordan Arrays,

$$\left(\frac{1}{1-z}, \frac{z^2 M(z)}{1-z}\right)\left(\frac{1}{1-2z}\right) = \frac{1}{1-z}\frac{1}{1-\dfrac{2z^2 M(z)}{1-z}} = \frac{1}{1-z-2z^2 M(z)} = E(z).$$

□

3 Identities Involving $M'(z)$ and $E'(z)$

It is well known that the number of paths of length n from $(0,0)$ to $(n,0)$ with steps $U = (1,1)$, $H = (1,0)$, and $D = (1,-1)$ that never go below the x-axis is the Motzkin number m_n [7]. If we mark an edge of a Motzkin path, the generating function is

$$\sum_{n \geq 0} n m_n z^n = zM'(z).$$

Proposition 2. *Let $M^*(z) = zM'(z)$. Then*

$$M^*(z) = zM(z)(1 + 2zM(z))E(z).$$

Proof. There are three disjoint cases, according to the vertical location of the marked edge.

Case I. If the marked edge lies on the x-axis, the decomposition in Figure 7 has generating function $zM^2(z)$.

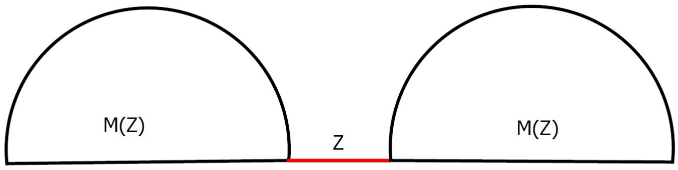

Figure 7: A marked edge on the x-axis.

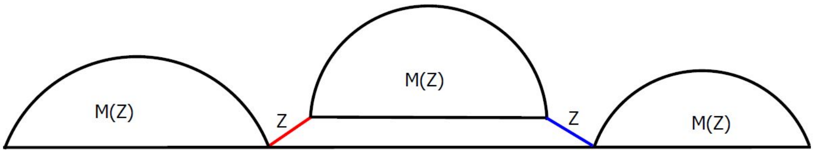

Figure 8: The marked edge is an up-step from the x-axis or a down-step to the x-axis.

Case II. If the marked edge is either an up-step from the x-axis or a down-step to the x-axis, the generating function is $2z^2M^3(z)$.
Case III. If the marked edge is on the elevated subpath and is neither the initial up-step nor the final down-step, the generating function is $z^2M^2(z)M^*(z)$.

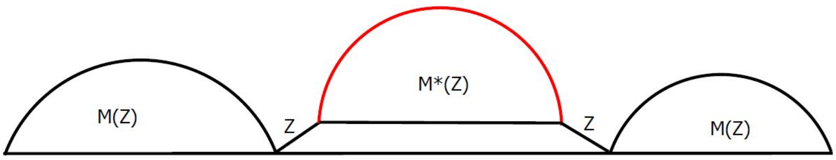

Figure 9: A marked edge on an elevated subpath.

Combining the three cases gives

$$M^*(z) = zM^2(z) + 2z^2M^3(z) + z^2M^2(z)M^*(z).$$

Therefore,

$$M^*(z) = \frac{zM^2(z)(1+2zM(z))}{1-z^2M^2(z)} = zM(z)(1+2zM(z))E(z).$$

□

Motzkin trees with a marked leaf satisfy the same decomposition as in Figure 3; hence, they are enumerated by the central trinomial numbers t_n. If we mark both a leaf and a vertex, the generating function is

$$\sum_{n\geq 0}(n+1)t_nz^n = (zE(z))'.$$

Proposition 3. *Let* $E^*(z) = zE'(z)$. *Then*

$$E^*(z) = (z+3z^2)E^3(z).$$

Proof. There are two disjoint cases.
Case I. The marked vertex v lies on the path from the root to the marked leaf. Figure 10 shows that the generating function is $E^2(z)$.
Case II. The marked vertex v does not lie on the path from the root to the marked leaf. Identifying the branching vertex w and examining the subtrees to its left and right, as in Figure 11, gives the generating function

$$2(E(z)zE(z)zE(z)M(z)) = 2z^2E^3(z)M(z).$$

Thus,

$$(zE(z))' = E^2(z) + 2z^2E^3(z)M(z)$$

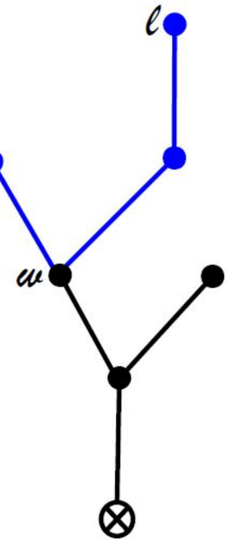

Figure 10: Decomposition of a Motzkin tree at the marked vertex.

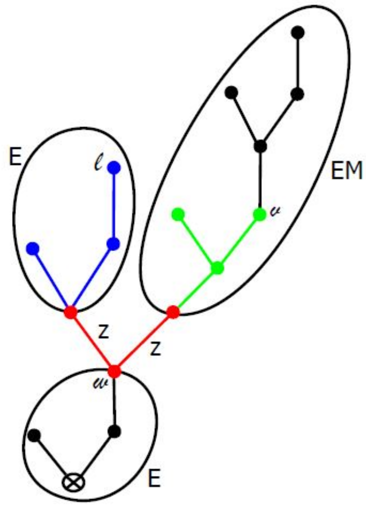

Figure 11: The marked vertex is not on the path from the root to the marked leaf.

$$\begin{aligned}
&= E^2(z)(1 + 2z^2E(z)M(z)) \\
&= E^2(z)(E(z) - zE(z)) \\
&= E(z)(E^2(z) - zE^2(z)) \\
&= E(z)(1 + zE^2(z) + 3z^2E^2(z)) \\
&= E(z) + (z + 3z^2)E^3(z),
\end{aligned}$$

where we used

$$E(z) = 1 + zE(z) + 2z^2E(z)M(z)$$

and

$$E^2(z) = 1 + 2zE^2(z) + 3z^2E^2(z).$$

Subtracting $E(z)$ gives the result. □

4 Trees Satisfying a Uniform Updegree Requirement

A class of trees satisfies the uniform updegree requirement (UUR) if every vertex has the same set of allowable updegrees [12, 16]. Let

$$T(z) = \sum_{n\geq 0} t_n z^n,$$

where t_n is the number of UUR trees with n edges. Then (T, zT) is an element of the Bell subgroup. Its A-sequence is $\{a_i\}_{i\geq 0}$, where a_i is the number of possibilities for a vertex of updegree i. Every Riordan array (g, f) has an A-sequence satisfying

$$f = z(a_0 + a_1 f + a_2 f^2 + \cdots) = zA(f),$$

where $A(z)$ is the generating function of the A-sequence. In the Bell subgroup $f = zg$, and when $g = T$ we have

$$T = A(zT).$$

4.1 UUR Trees with a Marked Leaf

Let L_k be the generating function for the number of leaves at height k in the class T, where $k = 1, 2, 3, \ldots$. Clearly,

$$L_k = L_1^k.$$

To obtain an equation for L_1, suppose that the root has updegree i. There are i choices for the distinguished edge on a path of length one, and UUR subtrees may be attached to the remaining $i-1$ children, as shown in Figure 12. Hence,

$$\begin{aligned} L_1 &= a_1 z + a_2(2z)zT + a_3(3z)z^2T^2 + a_4(4z)z^3T^3 + \cdots \\ &= z(a_1 + 2a_2 zT + 3a_3 z^2T^2 + 4a_4 z^3T^3 + \cdots) \\ &= zA'(zT). \end{aligned}$$

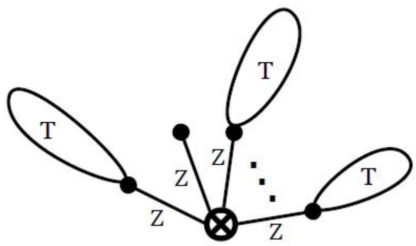

Figure 12: Decomposition of UUR trees at height one.

If L denotes the generating function for the total number of leaves, then

$$L = 1 + L_1 + L_2 + L_3 + \cdots = \sum_{k\geq 0} L_1^k = \frac{1}{1 - L_1} = \frac{1}{1 - zA'(zT)}.$$

A UUR tree with a marked vertex can be decomposed at the marked vertex into a UUR tree with a marked leaf and a UUR subtree. Thus,

$$V = LT = TL,$$

where V is the generating function for the total number of vertices. In summary,

$$T = A(zT), \qquad L_1 = zA'(zT), \qquad L = \frac{1}{1 - L_1}, \qquad V = TL.$$

Example 1. Let the UUR trees be the Motzkin trees, so $T = M$. Then $A(z) = 1 + z + z^2$ and $A'(z) = 1 + 2z$. Therefore,

$$L_1 = z(1 + 2zM(z)), \qquad L = \frac{1}{1 - z - 2z^2M(z)}.$$

Using the closed form for $M(z)$ gives

$$L = \frac{1}{\sqrt{1-2z-3z^2}} = E(z).$$

Hence, the total number of leaves in Motzkin trees is counted by the central trinomial numbers.

Example 2. Another class of UUR trees arising in data structures is the class of incomplete binary trees. These are binary trees in which each internal node has a left child, a right child, or both. Here

$$T = C^2, \qquad C = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} z^n.$$

Then

$$V = (zT)' = (zC^2)' = (C-1)' = BC^2, \qquad B = \frac{1}{\sqrt{1-4z}} = \sum_{n\geq 0} \binom{2n}{n} z^n.$$

Since $V = TL$, it follows that $L = B$; thus, the total number of leaves in incomplete binary trees is counted by the central binomial coefficients.

We can also obtain $L = B$ by noting that $A(z) = 1 + 2z + z^2 = (1+z)^2$ and

$$L_1 = zA'(zC^2) = 2z(1 + zC^2) = 2zC.$$

Multiplying the Riordan array $(1, L_1)$ by $U = (1, 1, 1, \ldots)^T$ gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 2 & 4 & 0 & \cdots \\ 0 & 4 & 8 & 8 & \cdots \\ 0 & 10 & 20 & 40 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \\ 20 \\ 70 \\ \vdots \end{bmatrix}.$$

The entries on the right are the row sums of $(1, L_1)$, and their generating function is

$$\frac{1}{1-L_1} = L = B.$$

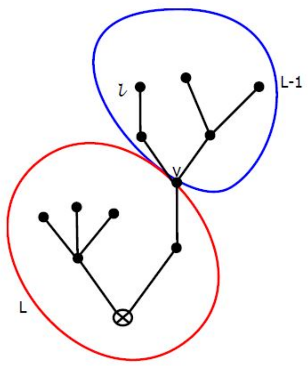

Figure 13: Clipping at an internal vertex.

Example 3. Let the UUR trees be ordered trees, so $T = C$. Then

$$V = \sum_{n\geq 0} (n+1)c_n z^n = B.$$

Since $V = TL$,

$$L = \frac{B}{C} = \frac{B+1}{2}, \qquad L - 1 = \frac{B-1}{2}.$$

The decomposition at an internal vertex v on a path to a leaf ℓ, shown in Figure 13, gives the generating function for the total path heights from the root to the leaves:

$$L(L-1) = \frac{B^2-1}{4} = \sum_{n\geq 1} 4^{n-1} z^n.$$

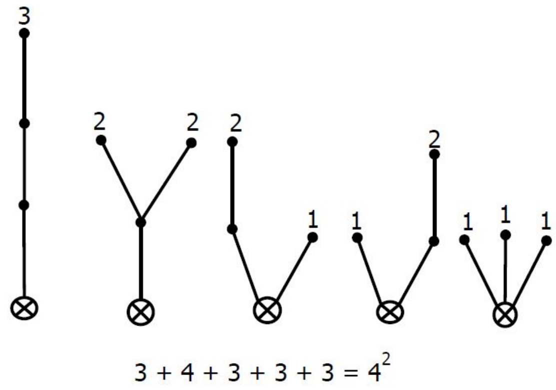

Figure 14: Trees with three edges.

For ordered trees, $L_1 = zC^2$, so the average number of leaves at height one is exactly one for $n \geq 1$.

Example 4. Let the UUR trees be ordered trees in which every vertex has even updegree. The A-sequence is

$$(1, 0, 1, 0, 1, 0, \ldots), \qquad A(z) = \frac{1}{1 - z^2}.$$

The decomposition in Figure 15 gives

$$T = 1 + z^2T^3, \qquad T = \sum_{n \geq 0} \frac{1}{2n+1} \binom{3n}{n} z^{2n},$$

as shown by Cheon, Kim, and Shapiro [5].

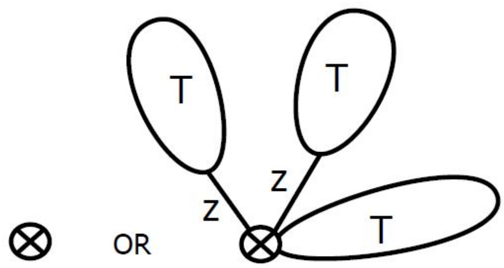

Figure 15: Decomposition of trees with even updegree.

The generating function for the total number of vertices satisfies

$$V = (zT)' = (z + z^3T^3)' = 1 + 3(zT)^2V.$$

Thus,

$$V = \frac{1}{1 - 3z^2T^2}, \qquad V - 1 = \frac{3z^2T^2}{1 - 3z^2T^2}.$$

Since

$$A'(z) = \frac{2z}{(1 - z^2)^2},$$

we have

$$L_1 = zA'(zT) = \frac{2z^2T}{(1 - z^2T^2)^2} = 2z^2T^3.$$

Hence,

$$L = \frac{1}{1 - 2z^2T^3}, \qquad L - 1 = \frac{2z^2T^3}{1 - 2z^2T^3}.$$

Proposition 4. *For ordered trees with even updegree,*

$$2(V - 1) = 3(L - 1).$$

Equivalently, two-thirds of the non-root vertices are leaves.

Proof. Using the formulas above,

$$\begin{aligned}
2(V-1) = 3(L-1) &\Longleftrightarrow 2\left(\frac{3z^2T^2}{1-3z^2T^2}\right) = 3\left(\frac{2z^2T^3}{1-2z^2T^3}\right) \\
&\Longleftrightarrow \frac{1}{1-3z^2T^2} = \frac{T}{1-2z^2T^3} \\
&\Longleftrightarrow 1-2z^2T^3 = T-3z^2T^3 \\
&\Longleftrightarrow 1+z^2T^3 = T.
\end{aligned}$$

The last identity is the defining equation for T. □

This is analogous to Shapiro's observation that half of the vertices among all ordered trees are leaves [11,14]. More generally, for UUR trees in which every vertex has updegree divisible by k ($k \geq 2$), the corresponding fraction is $k/(k+1)$. Since $L_1 = 2z^2T^3$ for ordered trees with even updegree, the average number of leaves at height one is exactly two for $n \geq 1$.

4.2 UUR Trees with Two Marked Edges

The generating function for UUR trees with two marked edges is

$$\sum_{n\geq 0} \binom{n}{2} t_n z^n = \frac{1}{2} z^2 T''(z).$$

Proposition 5.

$$\frac{1}{2} z^2 T''(z) = L^2 L_1^2 T + \frac{L}{2} A''(zT)(zV)^2.$$

Proof. There are two cases.

Case 1. The two marked edges lie on the same path from the root. The decomposition in Figure 16 gives the generating function $L^2L_1^2T$.

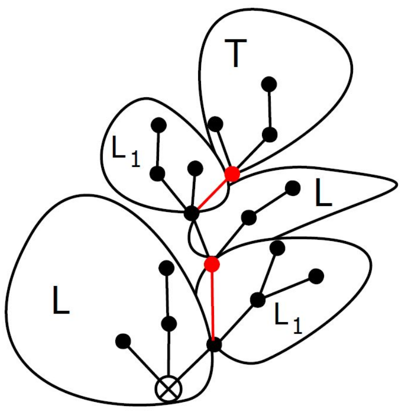

Figure 16: UUR trees with two marked edges on the same path.

Case 2. The two marked edges do not lie on the same path from the root. The generating function for UUR trees with two marked leaves at height one is $\frac{1}{2}z^2A''(zT)$. Using the decomposition in Figure 17, we obtain

$$\frac{L}{2} A''(zT)(zV)^2.$$

Adding the two cases proves the proposition. □

Example 1. Let $T = M$. Then

$$L = E = \frac{1}{\sqrt{1-2z-3z^2}}, \qquad L = \frac{1}{1-L_1},$$

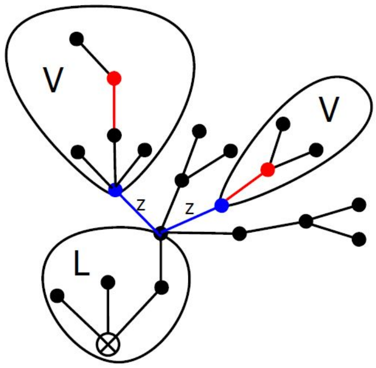

Figure 17: UUR trees with two marked edges not on the same path.

so $LL_1 = L - 1 = E - 1$. The generating function for Motzkin trees with two marked edges on the same path is

$$(LL_1)^2 T = \left(\frac{1}{\sqrt{1 - 2z - 3z^2}} - 1 \right)^2 M.$$

Since $A(z) = 1 + z + z^2$ and $V = LT = EM$, the generating function for Motzkin trees with two marked edges not on the same path is

$$\frac{L}{2} A''(zT)(zV)^2 = z^2 E^3 M^2.$$

Combining the cases gives

$$\left(\frac{1}{\sqrt{1 - 2z - 3z^2}} - 1 \right)^2 M + z^2 E^3 M^2.$$

The first few coefficients are

$$0, 0, 2, 12, 54, 210, 765, 2667, 9044, \ldots.$$

This sequence was not listed in the OEIS [17] at the time of writing.

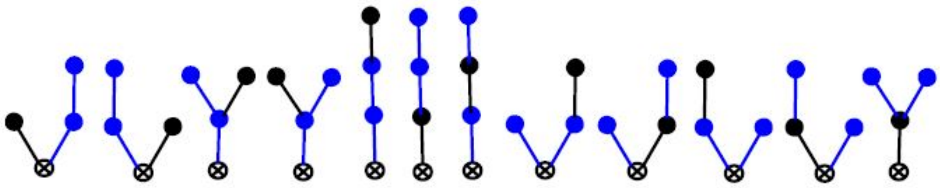

Figure 18: Three-edge Motzkin trees with two marked edges.

Example 2. Let $T = C$. Then

$$L = \frac{B}{C}, \qquad V = B, \qquad L_1 = zC^2.$$

The generating function for ordered trees with two marked edges on the same path is

$$(LL_1)^2 T = z^2 B^2 C^3 = (zBC)^2 C = \left(\frac{B - 1}{2} \right)^2 C.$$

Since $A(z) = 1/(1 - z)$ and $A''(z) = 2/(1 - z)^3$, the generating function for ordered trees with two marked edges not on the same path is

$$\frac{L}{2} A''(zT)(zV)^2 = z^2 C^2 B^3 = (zBC)^2 B = \left(\frac{B - 1}{2} \right)^2 B.$$

Thus, the generating function for ordered trees with two marked edges is

$$(zBC)^2 (C + B) = \left(\frac{B - 1}{2} \right)^2 (C + B).$$

The first few coefficients are

$$0, 0, 2, 15, 84, 420, \ldots.$$

This is OEIS sequence A002740 [17], which has many other combinatorial interpretations. Ferrari and Munarini observed that doubling each term gives the number of saturated chains of length two in Dyck lattices [9].

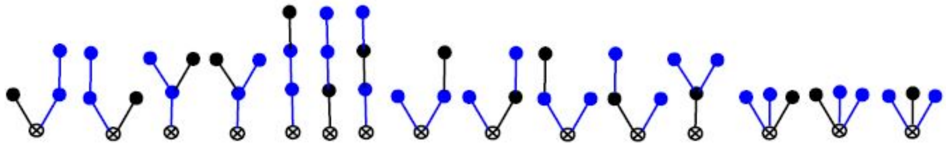

Figure 19: Three-edge ordered trees with two marked edges.

Concluding Remarks

Barcucci, Pinzani, and Sprugnoli [3] showed how central trinomial coefficients and Motzkin numbers can be interpreted in terms of lattice-path combinatorics, offering a perspective on how these numbers count certain types of paths. Our work provides additional combinatorial interpretations connecting Motzkin numbers and central trinomial coefficients. We have also shown how some of these connections can be obtained using properties of Riordan arrays, and interested readers may use this as a model for discovering similar results.

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