

The Double Almost-Riordan Arrays and Their Sequence Characterization, Compression, and Total Positivity

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ABSTRACT: In this paper, we define double almost-Riordan arrays and show that the set of all double almost-Riordan arrays forms a group, called the double almost-Riordan group. We also obtain sequence characterizations of double almost-Riordan arrays and give their production matrices. We define compressed double almost-Riordan arrays and present their sequence characterization. Finally, we give a criterion for the total positivity of double Riordan arrays and use it to discuss the total positivity of several double almost-Riordan arrays.

Keywords: Double almost-Riordan arrays; Double almost-Riordan group; Generating function; Production matrix; Stieltjes matrix; Sequence characterization; Succession rule
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1 Introduction

Riordan arrays are infinite lower-triangular matrices defined by the generating functions of their columns. They form a group, denoted by \mathcal{R} and called the Riordan group; see Shapiro, Getu, Woan, and Woodson [36].

A Riordan array $(g(t), f(t))$ is a pair of formal power series

$$g(t) = \sum_{n \geq 0} g_n t^n, \quad f(t) = \sum_{n \geq 1} f_n t^n,$$

with $g_0 \neq 0$ and $f_1 \neq 0$. It defines an infinite lower-triangular array $[d_{n,k}]_{n,k \geq 0}$ according to

$$d_{n,k} = [t^n]g(t)f(t)^k.$$

The set of all Riordan arrays forms a group under matrix multiplication,

$$(g(t), f(t))(h(t), \ell(t)) = (g(t)h(f(t)), \ell(f(t))).$$

An almost-Riordan array is defined by an ordered triple $(a \mid g, f)$ of power series

$$a(t) = \sum_{n \geq 0} a_n t^n, \quad g(t) = \sum_{n \geq 0} g_n t^n, \quad f(t) = \sum_{n \geq 1} f_n t^n,$$

with $a_0, g_0, f_1 \neq 0$. The array is the lower-triangular matrix whose first column is generated by $a(t)$ and whose remaining entries coincide, after the appropriate shift, with the Riordan array $(g(t), f(t))$.

An infinite lower-triangular matrix $D = [d_{n,k}]_{n,k \geq 0}$ is called a double Riordan array if $g(t)$ gives column zero and $f_1(t)$ and $f_2(t)$ are multiplier functions, where

$$g(t) = \sum_{n \geq 0} g_{2n} t^{2n}, \quad f_i(t) = \sum_{n \geq 0} f_{i,2n+1} t^{2n+1} \quad (i = 1, 2),$$

with $g_0 f_{1,1} f_{2,1} \neq 0$. The double Riordan array related to g, f_1, f_2 , denoted by $(g; f_1, f_2)$, has column generating functions

$$(g, g f_1, g f_1 f_2, g f_1^2 f_2, g f_1^2 f_2^2, \dots).$$

Riordan arrays, almost-Riordan arrays, and double Riordan arrays have emerged as powerful tools with broad applications in combinatorics, group theory, matrix theory, and number theory. This paper studies the double almost-Riordan arrays and the double almost-Riordan group defined in [22]. Specifically, it treats their sequence characterization, compression, and total positivity.

More formally, let $\mathcal{F} = \mathbb{K}[[t]]$ be the ring of formal power series over a field \mathbb{K} of characteristic zero. The order of $f(t) = \sum_{k \geq 0} f_k t^k \in \mathcal{F}$ is the least $r \in \mathbb{N}$ such that $f_r \neq 0$, and \mathcal{F}_r denotes the set of formal power series of order r . Let $g(t) \in \mathcal{F}_0$ and $f(t) \in \mathcal{F}_1$. Then the pair $(g(t), f(t))$ defines the proper Riordan array $D = (d_{n,k})_{n,k \in \mathbb{N}}$ with

$$d_{n,k} = [t^n]g(t)f(t)^k. \tag{1}$$

Equivalently, $g(t)f(t)^k$ is the generating function for column k .

From the fundamental theorem of Riordan arrays [35], the usual row-by-column product of two Riordan arrays is again a Riordan array:

$$(g_1(t), f_1(t))(g_2(t), f_2(t)) = (g_1(t)g_2(f_1(t)), f_2(f_1(t))). \tag{2}$$

The Riordan array $I = (1, t)$ is the identity. Thus, all Riordan arrays form the Riordan group \mathcal{R} .

Several subgroups of \mathcal{R} are important:

- the Appell subgroup \mathcal{A} , consisting of $(g(t), t)$;
- the Lagrange subgroup \mathcal{L} , consisting of $(1, f(t))$;
- the Bell or renewal subgroup \mathcal{B} , consisting of $(g(t), tg(t))$;
- the hitting-time subgroup \mathcal{H} , consisting of $(tf'(t)/f(t), f(t))$;
- the derivative subgroup \mathcal{D} , consisting of $(f'(t), f(t))$;
- the Luzón–Merlini–Morón–Sprugnoli subgroup

$$\mathcal{E}[r, s] = \left\{ \left(\left(\frac{f(t)}{t} \right)^r f'(t)^s, f(t) \right) \right\},$$

for real or complex r, s , which includes $\mathcal{D} = \mathcal{E}[0, 1]$ as a special case [30].

From [34], an infinite lower-triangular array $[d_{n,k}]_{n,k \in \mathbb{N}} = (g(t), f(t))$ is a Riordan array if and only if an A -sequence $A = (a_0 \neq 0, a_1, a_2, \dots)$ exists such that

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + \dots + a_n d_{n,n}, \tag{3}$$

which is equivalent [26] to

$$f(t) = tA(f(t)). \tag{4}$$

Here $A(t)$ is the generating function of the A -sequence. It is also known [32] that a unique Z -sequence $Z = (z_0, z_1, z_2, \dots)$ exists such that

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + \dots + z_n d_{n,n}, \tag{5}$$

or equivalently,

$$g(t) = \frac{d_{0,0}}{1 - tZ(f(t))}. \tag{6}$$

Let $U = (\delta_{i+1,j})_{i,j \geq 0}$ be the upper shift matrix,

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and let

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 & \dots \\ z_1 & a_1 & a_0 & 0 & 0 & \dots \\ z_2 & a_2 & a_1 & a_0 & 0 & \dots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (Z(t), A(t), tA(t), t^2A(t), \dots). \tag{7}$$

The last expression represents P by its column generating functions. The matrix P is called the production matrix, or P -matrix [15, 16]. If \overline{R} denotes R with its first row omitted, then

$$P = R^{-1}UR = R^{-1}\overline{R}. \tag{8}$$

The ECO method is a constructive method for producing all objects in a class according to the growth of a size parameter; see [5]. Its roots can be traced to [12]. When an ECO construction is sufficiently regular, it can often be described by a succession rule, a notion introduced by West [39]. Algebraic properties of succession rules were studied in [18].

A succession rule consists of an axiom (a) , $a \in \mathbb{N}^+$, and productions

$$\{(k_j) \longrightarrow (e_1(k_j))(e_2(k_j)) \cdots (e_j(k_j)) : j \in \mathbb{N}\},$$

where $e_i : \mathbb{N}^+ \rightarrow \mathbb{N}^+$. We write

$$\Omega : \begin{cases} (a), \\ (k) \longrightarrow (e_1(k))(e_2(k)) \cdots (e_k(k)). \end{cases} \tag{9}$$

The succession rule can be represented by a generating rooted tree. If f_n is the number of nodes at level n , then $(f_n)_{n \geq 0}$ is the sequence associated with Ω . The production matrix $P = (p_{i,j})_{i,j \geq 0}$ has $p_{i,j}$ equal to the number of labels ℓ_j produced by label ℓ_i .

Proposition 1.1 ([16]). *Let P be an infinite production matrix and let A_P be the ECO matrix induced by P . Then A_P is a Riordan matrix if and only if*

$$P = \begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 & \cdots \\ z_1 & a_1 & a_0 & 0 & 0 & \cdots \\ z_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ z_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{10}$$

where $z_j, a_j \geq 0$ for $j = 0, 1, \dots$

As an example, consider the Fibonacci tree F_1 of Horibe [27], which satisfies

$$\Omega : \begin{cases} (a), \\ a(k) \longrightarrow a(k)a(k+1). \end{cases} \tag{11}$$

Figure 1 presents its first few levels.

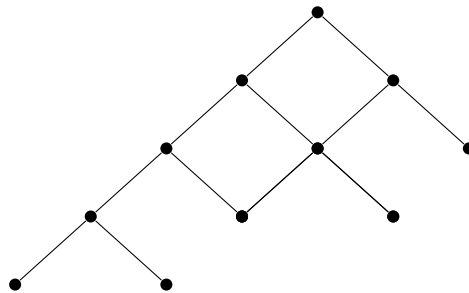


Figure 1: Fibonacci tree F_1 .

The production matrix is

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{12}$$

By (8), the corresponding Riordan array is the Pascal matrix

$$R = \left(\frac{1}{1-t}, \frac{t}{1-t} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{13}$$

The sums along the diagonals parallel to the direction $(1, 1)$ are the Fibonacci numbers $1, 1, 2, 3, 5, 8, \dots$

A succession rule can generate a wider class of lower-triangular matrices that do not belong to the Riordan class. Stanley's Fibonacci tree F_2 [37] has a root of degree one; the child of every vertex of degree one has degree two; and the two children of every vertex of degree two have degrees one and two. Its succession rule is

$$\Omega : \begin{cases} (a_{2\ell}) \longrightarrow (a_{2\ell})(a_{2\ell+1}), \\ (a_{2\ell+1}) \longrightarrow (a_{2\ell+2}), \end{cases} \quad \ell = 0, 1, 2, \dots \tag{14}$$

It generates the Fibonacci-Stanley array

$$\widehat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 1 & 3 & 2 & 1 & 0 & \dots \\ 1 & 1 & 4 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{15}$$

Its row sums are Fibonacci numbers, but \widehat{R} is not a Riordan array. It is, however, a compressed double Riordan array [20], which motivates the study below.

In the construction of a Riordan array, one multiplier function is used to pass from one column to the next. To model alternating rules, Davenport, Shapiro, and Woodson [17] introduced two multiplier functions. They require g to be even and f_1, f_2 to be odd. The columns of $(g; f_1, f_2)$ are

$$(g, gf_1, gf_1f_2, gf_1^2f_2, gf_1^2f_2^2, \dots).$$

There are two cases of the fundamental theorem of double Riordan arrays:

$$(g; f_1, f_2)A(t) = B(t), \tag{16}$$

where, for even $A(t)$, $B(t) = g(t)A(\sqrt{f_1(t)f_2(t)})$, while for odd $A(t)$,

$$B(t) = g(t)\sqrt{\frac{f_1(t)}{f_2(t)}}A(\sqrt{f_1(t)f_2(t)}).$$

The product is

$$(g; f_1, f_2)(d; h_1, h_2) = \left(gd(\sqrt{f_1f_2}); \sqrt{\frac{f_1}{f_2}}h_1(\sqrt{f_1f_2}), \sqrt{\frac{f_2}{f_1}}h_2(\sqrt{f_1f_2}) \right). \tag{17}$$

The collection forms the double Riordan group \mathcal{DR} , with identity $(1; t, t)$. Important subgroups include the associated subgroup $\{(1; f_1, f_2)\}$, the type-1 and type-2 Bell subgroups, and the Appell subgroup $\{(g; t, t)\}$. The Appell subgroup is normal, and \mathcal{DR} is a semidirect product of it with the associated subgroup [17].

If $(g; f_1, f_2)$ is a double Riordan array, define

$$\widehat{g}(t) = \sum_{k \geq 0} g_{2k}t^k, \quad \widehat{f}_i(t) = \sum_{k \geq 0} f_{i,2k+1}t^{k+1}.$$

Then $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is its compression. The matrix \widehat{R} in (15) is the compression of $(1/(1-t^2); t, t/(1-t^2))$.

Theorem 1.2 ([20]). *A double Riordan array $(g; f_1, f_2) = (d_{n,k})_{n,k \geq 0}$ with $d_{n,n} \neq 0$ exists if and only if there are sequences*

$$A_1 = (a_{1,0}, a_{1,1}, \dots), \quad A_2 = (a_{2,0} \neq 0, a_{2,1}, \dots), \quad Z = (z_0, z_1, \dots)$$

such that

$$d_{n,2k-1} = a_{1,0}d_{n-1,2k-2} + a_{1,1}d_{n-1,2k} + a_{1,2}d_{n-1,2k+2} + \dots, \tag{18}$$

$$d_{n,2k} = a_{2,0}d_{n-2,2k-2} + a_{2,1}d_{n-2,2k} + a_{2,2}d_{n-2,2k+2} + \dots, \tag{1}$$

$$d_{n,0} = z_0d_{n-2,0} + z_1d_{n-2,2} + z_2d_{n-2,4} + \dots. \tag{2}$$

If $A_i(t) = \sum_{k \geq 0} a_{i,k}t^{2k}$ and $Z(t) = \sum_{k \geq 0} z_k t^{2k}$, then equivalently

$$f_1 = tA_1(\sqrt{f_1f_2}), \tag{19}$$

$$f_1f_2 = t^2A_2(\sqrt{f_1f_2}), \tag{20}$$

$$g(t) = \frac{g(0)}{1 - t^2Z(\sqrt{f_1f_2})}. \tag{21}$$

We next recall almost-Riordan arrays.

Definition 1.3 ([6]). Let $d, g \in \mathcal{F}_0$ with $d(0) = g(0) = 1$ and let $f \in \mathcal{F}_1$ with $f'(0) = 1$. The almost-Riordan array with respect to d, g, f is

$$(d \mid g, f) = (d, tg, tgf, tgf^2, \dots). \tag{22}$$

Thus d, tg, tgf, tgf^2, \dots generate columns $0, 1, 2, 3, \dots$.

It has a block form

$$(d \mid g, f) = \begin{pmatrix} d(0) & 0 \\ (d - d(0))/t & (g, f) \end{pmatrix}. \tag{23}$$

If $d = g$ and $f = t$, it reduces to the Appell-type Riordan array (g, t) .

Theorem 1.4 ([7]). Almost-Riordan arrays form a group \mathcal{AR} under

$$(a \mid g, f)(b \mid d, h) = \left(a + \frac{tg}{f}(b(f) - 1) \mid gd(f), h(f) \right). \tag{24}$$

Alp and Koçer give a sequence characterization of almost-Riordan arrays and define the exponential almost-Riordan group [3, 4]. Slowik and the author discuss total positivity of almost-Riordan and quasi-Riordan arrays [24, 23].

Definition 1.5 ([21]). Let $g \in \mathcal{F}_0$ with $g(0) = 1$ and $f \in \mathcal{F}_1$. The quasi-Riordan array is

$$[g, f] = (g, f, tf, t^2f, \dots). \tag{25}$$

It has block form

$$[g, f] = \begin{pmatrix} g(0) & 0 \\ (g - g(0))/t & (f/t, t) \end{pmatrix}. \tag{26}$$

If $f = tg$, then $[g, tg] = (g, t)$, an Appell-type Riordan array. Also $[g, f] = (g \mid f/t, t)$. For square matrices A, B , their direct sum is

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \tag{27}$$

and

$$(g, f) = [g, f]([1] \oplus (g, f)). \tag{28}$$

Theorem 1.6 ([21]). The set \mathcal{QR} of all quasi-Riordan arrays is a group under

$$[g, f][d, h] = \left[g + \frac{f}{t}(d - 1), \frac{fh}{t} \right], \tag{29}$$

which follows from the fundamental theorem

$$[g, f]u = gu(0) + \frac{f}{t}(u - u(0)). \tag{30}$$

The identity is $[1, t]$.

If

$$Q = [g, f] = (g \mid f/t, t) = \begin{bmatrix} g_0 & 0 & 0 & 0 & \cdots \\ g_1 & f_1 & 0 & 0 & \cdots \\ g_2 & f_2 & f_1 & 0 & \cdots \\ g_3 & f_3 & f_2 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{31}$$

is totally positive, then so is the Riordan array (g, f) [31, 21, 23].

Theorem 1.7 ([24]). Every almost-Riordan array has the semidirect-product decomposition

$$(d \mid g, f) = [d, tg](1 \mid 1, f), \tag{32}$$

or equivalently

$$(d \mid g, f) = \begin{bmatrix} d_0 & 0 \\ (d - d_0)/t & (g, t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1, f) \end{bmatrix}. \tag{33}$$

The next section studies double almost-Riordan arrays and their group from a different viewpoint. Section 3 gives sequence characterizations and production matrices; Section 4 studies compression; and Section 5 gives criteria for total positivity.

2 Double almost-Riordan arrays

Definition 2.1. Let $b, g \in \mathcal{F}_0$ be even and $f_1, f_2 \in \mathcal{F}_1$ be odd:

$$b(t) = \sum_{n \geq 0} b_{2n} t^{2n}, \quad g(t) = \sum_{n \geq 0} g_{2n} t^{2n}, \quad f_i(t) = \sum_{n \geq 0} f_{i,2n+1} t^{2n+1}.$$

The double almost-Riordan array is

$$(b \mid g; f_1, f_2) = (b, tg, tg f_1, tg f_1 f_2, tg f_1^2 f_2, tg f_1^2 f_2^2, \dots). \tag{34}$$

Theorem 2.2. There are two cases of the first fundamental theorem of double almost-Riordan arrays:

$$(b \mid g; f_1, f_2)u(t) = v(t). \tag{35}$$

If $u(t) = \sum_{k \geq 0} u_{2k} t^{2k}$ is even, then

$$v(t) = u_0 b(t) + \frac{tg(t)}{f_2(t)} \left(u(\sqrt{f_1(t)f_2(t)}) - u_0 \right). \tag{36}$$

If $u(t) = \sum_{k \geq 0} u_{2k+1} t^{2k+1}$ is odd, then

$$v(t) = \frac{tg(t)}{\sqrt{f_1(t)f_2(t)}} u(\sqrt{f_1(t)f_2(t)}). \tag{37}$$

Proof. For even u ,

$$(b \mid g; f_1, f_2)u = u_0 b + tg f_1 \sum_{k \geq 1} u_{2k} (\sqrt{f_1 f_2})^{2k-2} = u_0 b + \frac{tg}{f_2} \sum_{k \geq 1} u_{2k} (\sqrt{f_1 f_2})^{2k},$$

which is (36). For odd u ,

$$(b \mid g; f_1, f_2)u = tg \sum_{k \geq 0} u_{2k+1} (\sqrt{f_1 f_2})^{2k} = \frac{tg}{\sqrt{f_1 f_2}} \sum_{k \geq 0} u_{2k+1} (\sqrt{f_1 f_2})^{2k+1},$$

which is (37). □

Theorem 2.3. Let $(b \mid g; f_1, f_2)$ and $(c \mid d; h_1, h_2)$ be double almost-Riordan arrays, with b and c even. Then

$$(b \mid g; f_1, f_2)(c \mid d; h_1, h_2) = \left(c_0 b + \frac{tg}{f_2} (c(\sqrt{f_1 f_2}) - c_0) \mid gd(\sqrt{f_1 f_2}); \sqrt{\frac{f_1}{f_2}} h_1(\sqrt{f_1 f_2}), \sqrt{\frac{f_2}{f_1}} h_2(\sqrt{f_1 f_2}) \right), \tag{38}$$

and

$$(b \mid g; f_1, f_2)^{-1} = \left(\frac{1}{b_0} + \frac{f_2(\bar{f}_{12})}{b_0 \bar{f}_{12} g(\bar{f}_{12})} (b_0 - b(\bar{f}_{12})) \mid \frac{1}{g(\bar{f}_{12})}; \frac{\bar{f}_{12}}{t} f_1(\bar{f}_{12}), \frac{\bar{f}_{12}}{t} f_2(\bar{f}_{12}) \right), \tag{39}$$

where \bar{f}_{12} is the compositional inverse of $\sqrt{f_1 f_2}$.

Proof. Formula (38) follows by applying the two cases of the fundamental theorem to the columns of the second factor. Substituting the right-hand side of (39) into (38) gives $(1 \mid 1; t, t)$; conversely,

$$(b \mid g; f_1, f_2)(1 \mid 1; t, t) = (b \mid g; f_1, f_2) = (1 \mid 1; t, t)(b \mid g; f_1, f_2).$$

Thus $(1 \mid 1; t, t)$ is the identity and (39) is the inverse. □

From now on, for $(b \mid g; f_1, f_2) \in \mathcal{D}\text{-}\mathcal{R}$, we assume b, g are even members of \mathcal{F}_0 and f_1, f_2 are odd members of \mathcal{F}_1 .

Theorem 2.4. The set of all double almost-Riordan arrays $(b \mid g; f_1, f_2)$ forms a group $\mathcal{D}\text{-}\mathcal{R}$ under (38), with identity $(1 \mid 1; t, t)$.

Proof. Closure and inverses follow from Theorem 2.3. Associativity follows either from the associativity of matrix multiplication or by substituting (38) into both

$$((b \mid g; f_1, f_2)(c \mid d; h_1, h_2))(e \mid u; v_1, v_2)$$

and

$$(b \mid g; f_1, f_2)((c \mid d; h_1, h_2)(e \mid u; v_1, v_2)),$$

which yield the same four generating functions. The group structure is developed further in [22]. □

3 Sequence characterization of double almost-Riordan arrays

We now give a sequence characterization of a double almost-Riordan array in $\mathcal{D}\text{-}\mathcal{R}$. Inspired by Branch, Davenport, Frankson, Jones, and Thorpe [8] and Davenport, Frankson, Shapiro, and Woodson [14], write

$$\begin{aligned} D = (b \mid g; f_1, f_2) &= (b, tg, tgf_1, tg(f_1f_2), tgf_1(f_1f_2), tg(f_1f_2)^2, \dots) \\ &= (b, 0, 0, \dots) + (0, tg, 0, tg(f_1f_2), 0, tg(f_1f_2)^2, 0, \dots) \\ &\quad + (0, 0, tgf_1, 0, tgf_1(f_1f_2), 0, tgf_1(f_1f_2)^2, 0, \dots) \\ &= D_0 + D_1 + D_2. \end{aligned} \tag{40}$$

After omitting zero columns and the top zero rows, denote the remaining arrays by D_1^* and D_2^* . Then

$$D_1^* = (g, f_1f_2), \quad D_2^* = (gf_1/t, f_1f_2)$$

are Riordan arrays. Hence $(b \mid g; f_1, f_2)$ has a W -sequence, two Z -sequences Z_1, Z_2 , and an A -sequence in this viewpoint. The viewpoint in [22] instead gives a W -sequence, one Z -sequence, and two A -sequences.

Theorem 3.1. *Let $(b \mid g; f_1, f_2)$ be a double almost-Riordan array. Let*

$$A(t) = \sum_{k \geq 0} a_k t^{2k}, \quad Z_i(t) = \sum_{k \geq 0} z_{i,k} t^{2k} \quad (i = 1, 2), \quad W(t) = \sum_{k \geq 0} w_k t^{2k}.$$

Then, with \bar{f}_{12} the compositional inverse of $\sqrt{f_1f_2}$,

$$A(t) = \frac{t^2}{\bar{f}_{12}(t)^2}, \tag{41}$$

$$Z_1(t) = \frac{1}{\bar{f}_{12}(t)^2} \left(1 - \frac{g_0}{g(\bar{f}_{12}(t))} \right), \tag{42}$$

$$Z_2(t) = \frac{g_0 f_{1,1}}{b_0} + \frac{t^2}{\bar{f}_{12}(t)^2} - \frac{g_0 f_{1,1} b(\bar{f}_{12}(t)) f_2(\bar{f}_{12}(t))}{b_0 \bar{f}_{12}(t) g(\bar{f}_{12}(t))}, \tag{43}$$

$$W(t) = \frac{f_2(\bar{f}_{12}(t))((1 - w_0 \bar{f}_{12}(t)^2) b(\bar{f}_{12}(t)) - b_0)}{\bar{f}_{12}(t)^3 g(\bar{f}_{12}(t))} + w_0, \quad w_0 = \frac{b_2}{b_0}. \tag{44}$$

Here $f_{1,1} = [t]f_1(t)$.

Proof. For $k \geq 2$, the common A -sequence of D_1^* and D_2^* gives

$$d_{n,k} = a_0 d_{n-2,k-2} + a_1 d_{n-2,k} + a_2 d_{n-2,k+2} + \dots = \sum_{j \geq 0} a_j d_{n-2,k+2(j-1)}. \tag{45}$$

For $k = 2\ell + 1$ and $k = 2\ell$, respectively,

$$[t^n]tg(f_1f_2)^\ell = [t^{n-2}]tg(f_1f_2)^{\ell-1}A(\sqrt{f_1f_2}), \tag{46}$$

$$[t^n]tgf_1(f_1f_2)^{\ell-1} = [t^{n-2}]tgf_1(f_1f_2)^{\ell-2}A(\sqrt{f_1f_2}). \tag{47}$$

Both imply

$$t^2A(\sqrt{f_1f_2}) = f_1f_2.$$

Putting $t = \bar{f}_{12}$ yields (41).

The recurrence

$$d_{n,1} = z_{1,0}d_{n-2,1} + z_{1,1}d_{n-2,3} + z_{1,2}d_{n-2,5} + \dots$$

gives

$$\frac{g(t) - g_0}{t^2} = g(t)Z_1(\sqrt{f_1f_2}),$$

and hence (42).

Similarly,

$$d_{n,2} = z_{2,0}d_{n-2,0} + z_{2,1}d_{n-2,2} + z_{2,2}d_{n-2,4} + \dots, \quad z_{2,0} = \frac{f_{1,1}g_0}{b_0},$$

yields

$$Z_2(\sqrt{f_1 f_2}) = \frac{f_{1,1} g_0}{b_0} + \frac{1}{b_0 t^2 g} (b_0 g f_1 f_2 - f_{1,1} g_0 t b f_2),$$

which is (43) after composition with \bar{f}_{12} .

Finally, for an infinite lower-triangular array with even first-column generating function $b(t)$, let

$$d_{n,0} = w_0 d_{n-2,0} + w_1 d_{n-2,2} + w_2 d_{n-2,4} + \dots \tag{48}$$

Then $w_0 = b_2/b_0$, and the remaining w_i are determined recursively; for example,

$$w_1 = \frac{d_{4,0} - w_0 d_{2,0}}{d_{2,2}} = \frac{d_{0,0} d_{4,0} - d_{2,0}^2}{d_{0,0} d_{2,2}}.$$

Equation (48) is equivalent to

$$\frac{b(t) - b_0}{t^2} = w_0 b + \frac{t g}{f_2} (W(\sqrt{f_1 f_2}) - w_0),$$

which gives (44). □

Theorem 3.2. *Let $(b \mid g; f_1, f_2) \in \mathcal{D}\text{-}\mathcal{R}$. Its production matrix is*

$$P = (W(t), tZ_1(t), Z_2(t), tA(t), t^2 A(t), t^3 A(t), \dots), \tag{49}$$

where

$$W(t) = \frac{b_2}{b_0} + \frac{f_2(\bar{f}_{12})}{b_0 \bar{f}_{12}^3 g(\bar{f}_{12})} (b(\bar{f}_{12})(b_0 - b_2 \bar{f}_{12}^2) - b_0^2), \tag{50}$$

$$Z_1(t) = \frac{1}{\bar{f}_{12}^2} \left(1 - \frac{g_0}{g(\bar{f}_{12})} \right), \tag{51}$$

$$Z_2(t) = \frac{g_0 f_{1,1}}{b_0} + \frac{t^2}{\bar{f}_{12}^2} - \frac{g_0 f_{1,1} b(\bar{f}_{12}) f_2(\bar{f}_{12})}{b_0 \bar{f}_{12} g(\bar{f}_{12})}, \tag{52}$$

$$A(t) = \frac{t^2}{\bar{f}_{12}^2}. \tag{53}$$

Proof. Let \bar{D} denote $(b \mid g; f_1, f_2)$ with its first two rows removed. Its column generating functions are

$$\left(\frac{b - b_0}{t^2}, \frac{g - g_0}{t}, \frac{g f_1}{t}, \frac{g f_1 f_2}{t}, \frac{g f_1 (f_1 f_2)}{t}, \frac{g (f_1 f_2)^2}{t}, \dots \right).$$

Thus $P = D^{-1} \bar{D}$. Apply the two cases of the fundamental theorem to each column. The first column gives (50), the second gives (51), the third gives (52), and the fourth gives

$$D^{-1} \frac{g f_1 f_2}{t} = \frac{t^3}{\bar{f}_{12}^2} = t A(t),$$

which is (53). In general, column $k - 1$, $k \geq 4$, has generating function $t^k / \bar{f}_{12}^2 = t^{k-2} A(t)$. □

Example 3.3. *The Fibonacci–Stanley array in (15) is the compression of the double almost-Riordan array*

$$\left(\frac{1}{1 - t^4} \mid \frac{1}{1 - t^2}; t, \frac{t}{1 - t^2} \right).$$

Here

$$f_1 = t, \quad f_2 = \frac{t}{1 - t^2}, \quad g = \frac{1}{1 - t^2}, \quad b = \frac{1}{1 - t^4},$$

and

$$\sqrt{f_1 f_2} = \frac{t}{\sqrt{1 - t^2}}, \quad \bar{f}_{12} = \frac{t}{\sqrt{1 + t^2}}.$$

Substitution into (50)–(53) gives

$$\begin{aligned} A(t) &= 1 + t^2, & Z_1(t) &= 1, \\ Z_2(t) &= \frac{1 + 3t^2 + t^4}{1 + 2t^2} = 1 + t^2 - t^4 + 2t^6 - 4t^8 + 8t^{10} + \dots, \end{aligned}$$

and

$$W(t) = \frac{t^2(1+t^2)}{1+2t^2} = t^2 - t^4 + 2t^6 - 4t^8 + 8t^{10} + \dots$$

The array begins

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 & \dots \\ 0 & 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{54}$$

Its production matrix is obtained directly from $(W, tZ_1, Z_2, tA, t^2A, \dots)$, and multiplying the array by this production matrix deletes the first two rows in the sense described above.

4 Compressions of double almost-Riordan arrays

Let $(b \mid g; f_1, f_2) = (d_{n,k})_{n \geq k \geq 0} \in \mathcal{D}\text{-}\mathcal{R}$. Define its compression $(\widehat{d}_{n,k})_{n \geq k \geq 0}$ by

$$\widehat{d}_{n,k} = d_{2n-k,k}, \quad n \geq k \geq 0. \tag{55}$$

Theorem 4.1. *Suppose*

$$b(t) = \sum_{k \geq 0} b_{2k} t^{2k}, \quad g(t) = \sum_{k \geq 0} g_{2k} t^{2k}, \quad f_i(t) = \sum_{k \geq 0} f_{i,2k+1} t^{2k+1}.$$

Define

$$\widehat{b}(t) = \sum_{k \geq 0} b_{2k} t^k, \quad \widehat{g}(t) = \sum_{k \geq 0} g_{2k} t^k, \quad \widehat{f}_i(t) = \sum_{k \geq 0} f_{i,2k+1} t^{k+1}. \tag{57}$$

Then

$$\widehat{d}_{n,0} = [t^n] \widehat{b}(t), \quad \widehat{d}_{n,k} = \begin{cases} [t^n] t \widehat{g}(\widehat{f}_1 \widehat{f}_2)^{(k-1)/2}, & k \text{ odd,} \\ [t^n] t \widehat{g} \widehat{f}_1(\widehat{f}_1 \widehat{f}_2)^{(k-2)/2}, & k \text{ even,} \end{cases} \tag{56}$$

for $k \geq 1$.

Proof. For column zero,

$$\widehat{d}_{n,0} = d_{2n,0} = [t^{2n}] b(t) = [t^n] \widehat{b}(t).$$

If $k = 2\ell + 1$, then

$$\widehat{d}_{n,2\ell+1} = d_{2n-2\ell-1,2\ell+1} = [t^{2n}] t^2 g(t f_1 t f_2)^\ell.$$

Under the transformation $t^2 \mapsto t$, $g(t^{1/2}) = \widehat{g}(t)$ and $t^{1/2} f_i(t^{1/2}) = \widehat{f}_i(t)$, giving

$$\widehat{d}_{n,2\ell+1} = [t^n] t \widehat{g}(\widehat{f}_1 \widehat{f}_2)^\ell.$$

The even case is similar:

$$\widehat{d}_{n,2\ell} = [t^n] t \widehat{g} \widehat{f}_1(\widehat{f}_1 \widehat{f}_2)^{\ell-1}.$$

□

Thus the Fibonacci–Stanley array $\widehat{R} = (1/(1-t); t, t/(1-t))$ is the compression of $(1/(1-t^2); t, t/(1-t^2))$.

Theorem 4.2. *Let $(b \mid g; f_1, f_2) = (d_{n,k})$ be a double almost-Riordan array and let $(\widehat{d}_{n,k})$ be its compression. Suppose the A -, Z_1 -, Z_2 -, and W -sequences have generating functions A, Z_1, Z_2, W . Then*

$$\widehat{d}_{n,k} = \sum_{j \geq 0} a_j \widehat{d}_{n+j-2,k+2(j-1)}, \quad k \geq 3, \tag{58}$$

$$\widehat{d}_{n,2} = \sum_{j \geq 0} z_{2,j} \widehat{d}_{n+j-2,2j}, \tag{59}$$

$$\widehat{d}_{n,1} = \sum_{j \geq 0} z_{1,j} \widehat{d}_{n+j-1,2j+1}, \tag{60}$$

$$\widehat{d}_{n,0} = \sum_{j \geq 0} w_j \widehat{d}_{n+j-1,2j}. \tag{61}$$

Equivalently,

$$\widehat{f}_1 \widehat{f}_2 = t^2 A \left(\sqrt{\frac{\widehat{f}_1 \widehat{f}_2}{t}} \right), \tag{62}$$

$$Z_2 \left(\sqrt{\frac{\widehat{f}_1 \widehat{f}_2}{t}} \right) = \frac{\widehat{f}_2}{t^2 \widehat{g}} (\widehat{g} \widehat{f}_1 - z_{2,0} t \widehat{b}) + z_{2,0}, \quad z_{2,0} = \frac{g_0 f_{1,1}}{b_0}, \tag{63}$$

$$Z_1 \left(\sqrt{\frac{\widehat{f}_1 \widehat{f}_2}{t}} \right) = \frac{1}{t} \left(1 - \frac{g_0}{\widehat{g}} \right), \tag{64}$$

$$W \left(\sqrt{\frac{\widehat{f}_1 \widehat{f}_2}{t}} \right) = \frac{\widehat{f}_2}{t^2 \widehat{g}} (\widehat{b}(1 - w_0 t) - b_0) + w_0, \quad w_0 = \frac{b_2}{b_0}. \tag{65}$$

Proof. From (55) and (45),

$$\widehat{d}_{n,k} = d_{2n-k,k} = \sum_{j \geq 0} a_j d_{2(n+j-2)-k-2(j-1),k+2(j-1)} = \sum_{j \geq 0} a_j \widehat{d}_{n+j-2,k+2(j-1)}, \tag{66}$$

which proves (58). Applying the same compression to the recurrences for columns 2, 1, and 0 gives (59)–(61). Multiplying each recurrence by t^n , summing over n , and using the column generating functions in (56) yields (62)–(65). \square

Remark 4.3. Let $\overline{\widehat{f}_1 \widehat{f}_2 / t}$ denote the compositional inverse of $(\widehat{f}_1 \widehat{f}_2) / t$. Equations (62)–(65) can be solved explicitly by composition with this inverse, recovering the formulas in Theorem 3.1 under compression. In particular, the A -sequence relation can also be written in the familiar form

$$\widehat{f}_1 = tA \left(\frac{\widehat{f}_1 \widehat{f}_2}{t} \right)$$

when the corresponding alternative A_1 -sequence viewpoint is used.

For the Fibonacci–Stanley example, the compression begins

$$\widehat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 2 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 3 & 2 & 1 & \cdots \\ 1 & 1 & 1 & 4 & 3 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sequence formulas specialize to

$$\widehat{d}_{n,k} = \widehat{d}_{n-2,k-2} + \widehat{d}_{n-1,k}, \quad \widehat{d}_{n,1} = \widehat{d}_{n-1,1},$$

and

$$\widehat{d}_{n,2} = \widehat{d}_{n-2,0} + \widehat{d}_{n-1,2} - \widehat{d}_{n,4} + \cdots, \quad \widehat{d}_{n,0} = \widehat{d}_{n,2} - \widehat{d}_{n+1,4} + \cdots.$$

We call the compression of a double Riordan array a compressed double Riordan array, and similarly for double almost-Riordan arrays.

5 Total positivity of compressed double almost-Riordan arrays

Following Karlin [28] and Pinkus [33], an infinite matrix is totally positive (TP) if all its minors are nonnegative. A nonnegative sequence $(a_n)_{n \geq 0}$ is a Pólya frequency (PF) sequence if its Toeplitz matrix $[a_{i-j}]_{i,j \geq 0}$ is TP. A finite sequence is PF when it is extended by zeros.

A fundamental characterization due to Aissen, Edrei, Schoenberg, and Whitney [1, 2, 28] states that (a_n) is PF if and only if

$$\sum_{n \geq 0} a_n t^n = C t^k e^{\gamma t} \frac{\prod_{j \geq 0} (1 + \alpha_j t)}{\prod_{j \geq 0} (1 - \beta_j t)}, \tag{68}$$

where $C > 0$, $k \in \mathbb{N}$, $\alpha_j, \beta_j, \gamma \geq 0$, and $\sum_j (\alpha_j + \beta_j) < \infty$. Such a generating function is called a PF formal power series.

Let $(b \mid g; f_1, f_2)$ be a double almost-Riordan array and let its compression be $(\widehat{b} \mid \widehat{g}; \widehat{f}_1, \widehat{f}_2)$. It has a block form

$$(\widehat{b} \mid \widehat{g}; \widehat{f}_1, \widehat{f}_2) = \begin{pmatrix} \widehat{b}(0) & 0 \\ (\widehat{b} - \widehat{b}(0))/t & (\widehat{g}; \widehat{f}_1, \widehat{f}_2) \end{pmatrix}. \tag{69}$$

For later use, recall the direct sum

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}. \tag{70}$$

5.1 Total positivity of compressed double Riordan arrays

We first study compressed double Riordan arrays and then apply the result to compressed double almost-Riordan arrays.

Theorem 5.1. *Let $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ be a compressed double Riordan array. If \widehat{g} , \widehat{f}_1 , and \widehat{f}_2 are PF formal power series, then $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is totally positive.*

Proof. A double Riordan array decomposes as

$$(\widehat{g}; \widehat{f}_1, \widehat{f}_2) = (\widehat{g}; t, t)(1; \widehat{f}_1, \widehat{f}_2). \tag{71}$$

The product of TP matrices is TP by Cauchy–Binet. If the coefficients of \widehat{g} form a PF sequence, then $(\widehat{g}; t, t)$ is a TP Toeplitz matrix. It remains to prove that $(1; \widehat{f}_1, \widehat{f}_2)$ is TP when both multiplier series are PF.

We have

$$(1; \widehat{f}_1, \widehat{f}_2) = \begin{bmatrix} 1 & 0 \\ 0 & (\widehat{f}_1/t; \widehat{f}_2, \widehat{f}_1) \end{bmatrix}. \tag{72}$$

Set

$$T_i = (\widehat{f}_i; t, t), \quad H_1 = (\widehat{f}_1/t; \widehat{f}_2, \widehat{f}_1), \quad H_2 = (\widehat{f}_2/t; \widehat{f}_1, \widehat{f}_2).$$

Then

$$H_1 = T_1 \begin{bmatrix} 1 & 0 \\ 0 & H_2 \end{bmatrix}, \tag{73}$$

$$H_2 = T_2 \begin{bmatrix} 1 & 0 \\ 0 & H_1 \end{bmatrix}. \tag{74}$$

Let $H_i[n]$ be the submatrix formed by the first n columns of H_i . Thus

$$H_1[n+1] = T_1 \begin{bmatrix} 1 & 0 \\ 0 & H_2[n] \end{bmatrix}, \quad H_2[n+1] = T_2 \begin{bmatrix} 1 & 0 \\ 0 & H_1[n] \end{bmatrix}. \tag{76}$$

The T_i are TP Toeplitz matrices. Induction on n now shows that all $H_i[n]$ are TP, and hence so are $H_1, H_2, (1; \widehat{f}_1, \widehat{f}_2)$, and finally $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$. \square

Proposition 5.2. *If a compressed double almost-Riordan array $(\widehat{b} \mid \widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is TP, then the corresponding compressed double Riordan array $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is TP.*

Proposition 5.3. *The compressed double almost-Riordan array $(1 \mid \widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is TP if and only if $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is TP.*

Example 5.4. Let $\widehat{b}(t) = (1+t)^2$, $\widehat{g}(t) = 1/(1-t)$, and $\widehat{f}_1(t) = \widehat{f}_2(t) = t$. All four series are PF, and $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is TP. Nevertheless,

$$\det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -1 < 0.$$

Thus $((1+t)^2 \mid 1/(1-t); t, t)$ is not TP.

5.2 Total positivity of compressed double almost-Riordan arrays

The central question is whether, for fixed $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$, one can choose \widehat{b} so that $(\widehat{b} \mid \widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is TP.

Theorem 5.5. Suppose $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is a TP compressed double Riordan array. Then

$$(t\widehat{g} + \alpha \mid \widehat{g}; \widehat{f}_1, \widehat{f}_2)$$

is TP for every $\alpha > 0$.

Proof. The array has block form

$$R = \begin{bmatrix} \alpha & 0 \\ \widehat{g} & (\widehat{g}; \widehat{f}_1, \widehat{f}_2) \end{bmatrix}.$$

A minor not containing column zero is a minor of the TP double Riordan array. A minor containing row zero expands to α times such a minor. A minor containing column zero but not row zero is a minor of a partitioned matrix whose first two columns are equal; if it uses both, its determinant is zero, and if it uses at most one, it is a minor of $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$. Hence, all minors are nonnegative. \square

Theorem 5.6. Let $\widehat{b}(t) = \widehat{b}_0 + \widehat{b}_1 t$ with $\widehat{b}_0, \widehat{b}_1 \geq 0$. If $(\widehat{g}; \widehat{f}_1, \widehat{f}_2)$ is TP, then

$$(\widehat{b}_0 + \widehat{b}_1 t \mid \widehat{g}; \widehat{f}_1, \widehat{f}_2)$$

is TP.

Proof. Write

$$R = \begin{bmatrix} \widehat{b}_0 & 0 \\ \widehat{b}_1 & (\widehat{g}; \widehat{f}_1, \widehat{f}_2) \\ 0 & \\ \vdots & \end{bmatrix}.$$

A minor not containing column zero is a minor of the TP double Riordan array. If it contains row zero, expansion gives \widehat{b}_0 times a nonnegative minor. If it omits row zero but contains row one, expansion along column zero gives \widehat{b}_1 times a nonnegative minor. If it contains neither row zero nor row one, its column-zero entries are all zero, so the minor is zero. Therefore, every minor is nonnegative. \square

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References

- [1] M. Aissen, A. Edrei, I. J. Schoenberg, and A. M. Whitney, *On the generating functions of totally positive sequences*, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 303–307.
- [2] M. Aissen, I. J. Schoenberg, and A. M. Whitney, *On the generating functions of totally positive sequences I*, J. Analyse Math. 2 (1952), 93–103.
- [3] Y. Alp and E. G. Koçer, *Sequence characterization of almost-Riordan arrays*, Linear Algebra Appl. 664 (2023), 1–23.
- [4] Y. Alp and E. G. Koçer, *Exponential almost-Riordan arrays*, Results Math. 79 (2024), Article 173.
- [5] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani, *ECO: a methodology for the enumeration of combinatorial objects*, J. Difference Equ. Appl. 5 (1999), 435–490.

- [6] P. Barry, *On the group of almost-Riordan arrays*, arXiv:1606.05077, 2016.
- [7] P. Barry, T.-X. He, and N. Pantelidis, *The quasi-Riordan group and the almost-Riordan group*, manuscript in preparation.
- [8] D. Branch, D. Davenport, S. Frankson, J. Jones, and G. Thorpe, *A- and Z-sequences for double Riordan arrays*, in Springer Proceedings in Mathematics and Statistics 388 (2022), 33–46.
- [9] F. Brenti, *Combinatorics and total positivity*, J. Combin. Theory Ser. A 71 (1995), 175–218.
- [10] X. Chen, H. Liang, and Y. Wang, *Total positivity of Riordan arrays*, European J. Combin. 46 (2015), 68–74.
- [11] X. Chen and Y. Wang, *Notes on the total positivity of Riordan arrays*, Linear Algebra Appl. 569 (2019), 156–161.
- [12] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, and M. Kleiman, *The number of Baxter permutations*, J. Combin. Theory Ser. A 24 (1978), 382–394.
- [13] L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, 1974.
- [14] D. E. Davenport, S. K. Frankson, L. W. Shapiro, and L. C. Woodson, *An invitation to the Riordan group*, Enumer. Combin. Appl. 4:3 (2024), Article S2S1.
- [15] E. Deutsch, L. Ferrari, and S. Rinaldi, *Production matrices*, Adv. in Appl. Math. 34 (2005), 101–122.
- [16] E. Deutsch, L. Ferrari, and S. Rinaldi, *Production matrices and Riordan arrays*, Ann. Comb. 13 (2009), 65–85.
- [17] D. E. Davenport, L. W. Shapiro, and L. C. Woodson, *The double Riordan array*, Electron. J. Combin. 18 (2011), Article P33.
- [18] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi, *An algebraic characterization of the set of succession rules*, Theoret. Comput. Sci. 281 (2002), 351–367.
- [19] T.-X. He, *Matrix characterizations of Riordan arrays*, Linear Algebra Appl. 465 (2015), 15–42.
- [20] T.-X. He, *Sequence characterizations of double Riordan arrays and their compressions*, Linear Algebra Appl. 549 (2018), 176–202.
- [21] T.-X. He, *The vertical recursive relation of Riordan arrays and its matrix representation*, J. Integer Seq. 25 (2022), Article 22.9.5.
- [22] T.-X. He, *The double almost-Riordan group*, Linear Algebra Appl. 705 (2025), 50–88.
- [23] T.-X. He and R. Słowik, *Total positivity of quasi-Riordan arrays and Riordan arrays*, arXiv:2406.07120.
- [24] T.-X. He and R. Słowik, *Total positivity of almost-Riordan arrays and Riordan arrays*, arXiv:2406.03774.
- [25] T.-X. He and L. Shapiro, *Sequence characterization of improper Riordan arrays and its application in bogus-involutions and pseudo-involutions*, manuscript, 2024.
- [26] T.-X. He and R. Sprugnoli, *Sequence characterization of Riordan arrays*, Discrete Math. 309 (2009), 3962–3974.
- [27] Y. Horibe, *Notes on Fibonacci trees and their optimality*, Fibonacci Quart. 21 (1983), 118–128.
- [28] S. Karlin, *Total Positivity*, Vol. 1, Stanford University Press, 1968.
- [29] A. Kuznetsov, I. Pak, and A. Postnikov, *Trees associated with the Motzkin numbers*, J. Combin. Theory Ser. A 76 (1996), 145–147.
- [30] A. Luzón, D. Merlini, M. A. Morón, and R. Sprugnoli, *Complementary Riordan arrays*, Discrete Appl. Math. 172 (2014), 75–87.
- [31] J. Mao, L. Mu, and Y. Wang, *Yet another criterion for the total positivity of Riordan arrays*, Linear Algebra Appl. 634 (2022), 106–111.
- [32] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, *On some alternative characterizations of Riordan arrays*, Canad. J. Math. 49 (1997), 301–320.

- [33] A. Pinkus, *Totally Positive Matrices*, Cambridge University Press, Cambridge, 2010.
- [34] D. G. Rogers, Pascal triangles, *Catalan numbers and renewal arrays*, Discrete Math. 22 (1978), 301–310.
- [35] L. W. Shapiro, *Some open questions about random walks, involutions, limiting distributions and generating functions*, Adv. in Appl. Math. 27 (2001), 585–596.
- [36] L. W. Shapiro, S. Getu, W. J. Woan, and L. C. Woodson, *The Riordan group*, Discrete Appl. Math. 34 (1991), 229–239.
- [37] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, 2015.
- [38] C. Sun and Y. Sun, *On the halves of double and 3-dimensional Riordan arrays*, Linear Algebra Appl. 679 (2023), 194–219.
- [39] J. West, *Generating trees and forbidden subsequences*, Discrete Math. 157 (1996), 363–374.
- [40] L. Zhang and X. Zhao, *q-double Riordan matrices*, Linear Algebra Appl. 603 (2020), 212–225.