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Positivity Properties of q-Hit Numbers in the Finite General Linear Group

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ABSTRACT: We consider the problem of counting matrices over a finite field with fixed rank and support contained in a fixed set. The count of such matrices gives a q-analogue of the classical rook and hit numbers, known as the q-rook and q-hit numbers. They are known not to be polynomial in q in general. We use inclusion-exclusion on the support of the matrices and the orbit counting method of Lewis et al. to show that the residues of these functions in low degrees are polynomial. We define a generalization of the classical rook and hit numbers which count placements of certain classes of graphs. These give us a formula for the residues of the q-rook and q-hit numbers in low degrees. We analyze the residues of the q-hit number and show that the coefficient of $q-1$ in the q-hit number is always non-negative.

Keywords: q-hit numbers; q-rook numbers; finite fields; matrix enumeration; rook theory
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1 Introduction

The rook numbers, first studied by Kaplansky and Riordan in [7], count placements of non-attacking rooks on a subset of the squares of a chessboard, while the hit numbers count full rook placements according to how many rooks lie on that subset. Garsia and Remmel introduced q-analogues of these numbers for Ferrers boards in [4]; their q-rook and q-hit numbers are polynomials in q with non-negative coefficients, and Haglund and Dworkin later gave combinatorial interpretations for these q-hit numbers [3,5]. A different q-rook number was introduced in [9] by counting matrices over a finite field $\mathbb{F}_q$ of a given rank whose nonzero entries are restricted to prescribed positions. For Ferrers boards, this matrix-counting definition agrees with the Garsia–Remmel theory by the work of Haglund [5], but for general boards, the behavior is much wilder: Stembridge showed that the smallest non-polynomial example occurs for a 7×7 board; see [15, p. 2]. These q-rook numbers have also appeared in coding theory [12], while q-hit numbers for Ferrers boards appear in work on chromatic quasisymmetric and chromatic symmetric functions [1,2,8,11,13].

Lewis and Morales defined q-hit numbers for arbitrary boards in [10] by inclusion-exclusion in terms of the matrix q-rook numbers. Their definition agrees with the Garsia–Remmel q-hit numbers on Ferrers boards, but in general these q-hit numbers can also be non-polynomial. Nevertheless, Lewis and Morales conjectured strong positivity properties for them, including non-negativity for fixed q and any board. The present paper studies the first few coefficients of these non-polynomial functions when they are expanded around $q = 1$.

Let $B \subseteq [m] \times [n]$ be a board. We think of B as an $m \times n$ 0-1 matrix, where $B_{ij} = 1$ if $(i, j) \in B$. The rook number $r_i(B)$ is the number of ways to place i non-attacking rooks on the cells of B, and the hit number $h_i(B)$ is the number of ways to place m non-attacking rooks on $[m] \times [n]$ with exactly i rooks in B [7]. They are related by the classical rook-hit formula

$$\sum_{i=0}^{n} h_i(B)t^i = \sum_{i=0}^{n} r_i(B)\frac{(n-i)!}{(n-m)!}(t-1)^i. \tag{1.1}$$

For a matrix A, let the support $\operatorname{supp}(A)$ be the board with $(i, j) \in \operatorname{supp}(A)$ if $A_{i,j} \neq 0$. For a board $B \subseteq [m] \times [n]$, let the matrix count $\mathfrak{m}_i(B, q)$ be the number of $m \times n$ matrices over $\mathbb{F}_q$ having rank i and support contained in B. Lewis et al. [9, Prop. 5.1] showed that

$$\mathfrak{m}_i(B,q) \equiv r_i(B)(q-1)^i \mod (q-1)^{i+1}.$$

Thus $\mathfrak{m}_i(B,q)$ is divisible by $(q-1)^i$ for fixed q, so we may define the q-rook number

$$M_i(B,q) := \mathfrak{m}_i(B,q)/(q-1)^i.$$

Then $M_i(B,q)$ is an integer for fixed q, and $M_i(B,q) \equiv r_i(B) \pmod{q-1}$. Although $\mathfrak{m}_i(B,q)$ need not be polynomial in q [15], it is polynomial for certain classes of boards; see [15, Theorem 8.2].

Lewis and Morales defined a q-analogue of the hit numbers in [10, Eq. (1.3)], by the generating function equality

$$\sum_{i=0}^{n} H_i(B,q)t^i = q^{\binom{m}{2}} \sum_{i=0}^{n} M_i(B,q) \cdot \frac{[n-i]!_q}{[n-m]!_q}(t-1)(tq^{-1}-1)\dots(tq^{-(i-1)}-1),$$

a q-analogue of Equation (1.1).

This paper is motivated by two conjectures of Lewis and Morales [10]: first, that $H_d(B,q)$ is non-negative for any board B and prime power q (see [10, Conj. 6.3]); second, that for boards where $H_d(B,q)$ is polynomial, the polynomial $H_d(B,x+1)$ has non-negative coefficients in x (see [10, Conj. 6.7]). Our goal is to prove unconditional low-degree positivity statements for $H_d(B,x+1)$ even when $H_d(B,q)$ is not polynomial.

To do this, we must explain what is meant by the "coefficient" of a non-polynomial function of $x = q-1$.

Let S be an unbounded subset of $\mathbb{Z}$, and let f be a function $S \to \mathbb{Q}$. When there is a polynomial $P \in \mathbb{Z}[x]$ of degree at most $k-1$ such that

$$P(x) \equiv f(x) \pmod{x^k}$$

for all x in S, we say that f is polynomial modulo x^k, and we say the coefficient of x^i in f is the coefficient of x^i in P for $i < k$. Note that there is at most one polynomial P satisfying these conditions for any fixed k, so this coefficient is well-defined. If R is an unbounded subset of S and $f|_R$ is polynomial modulo x^k, we say that f is polynomial modulo x^k *on* R, and we may talk about f's coefficients on R.

Our main result is

Theorem (Theorem 4.10)**.** For a board $B \subseteq [n] \times [n]$, $H_d(B,x+1)$ is polynomial modulo x^2, and its coefficient of x is non-negative.

In Section 3.3 we also show that $\mathfrak{m}_i(B,x+1)$, $M_i(B,x+1)$, and $H_i(B,x+1)$ are polynomial modulo x^6 for any board. The bound x^6 is best possible in general; see Remark 3.14.

In fact, though, we know of no counterexample to the stronger conjecture:

Conjecture (Conjecture 4.1)**.** Let B be a board, and let S be an unbounded subset of $\mathbb{Z}$ such that $H_d(B,x+1)$ is polynomial modulo x^k on S for some k. Then for $i < k$, the coefficient of x^i in $H_d(B,x+1)$ on S is non-negative.

If B is a board such that $H_d(B,x+1) = P(x)$ is polynomial, then $H_d(B,x+1)$ is polynomial modulo x^k for all k, and the coefficients of $H_d(B,x+1)$ modulo x^k agree with the first k coefficients of P. Thus Conjecture 4.1 would imply Conjecture 6.3 and Conjecture 6.7 in [10].

Our proofs refine the diagonal-group orbit-stabilizer method of [9] by introducing generalized rook and hit numbers associated with small bi-colored graphs. This produces explicit formulas for $M_i(B,x+1)$ and $H_i(B,x+1)$ modulo x^2, and it also explains why the matrix q-rook and q-hit numbers are polynomial modulo x^6 for every board.

2 Background information

In this section, we give the definitions and background information about the q-rook and q-hit numbers, and then we review important past results.

First, consider the q-analogues of the natural numbers

$$[i]_q = (q^i-1)/(q-1) = 1+q+\dots+q^{i-1}.$$

We define the q-factorial and q-binomial

$$[n]!_q = [n]_q[n-1]_q \dots [1]_q$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]!_q}{[m]!_q[n-m]!_q}.$$

Define the q-Pochhammer symbol as

$$(t;q)_k = \prod_{i=0}^{k-1}(1-tq^i) = (1-t)(1-tq)\dots(1-tq^{k-1}),$$

and define the q-hit numbers for a board $B \subseteq [m] \times [n]$, where $m \leq n$, with the equation [10]:

$$\sum_{i=0}^{n} H_i(B,q)t^i = q^{\binom{m}{2}} \sum_{i=0}^{m} M_i(B,q) \frac{[n-i]!_q}{[n-m]!_q}(-1)^i(t;q^{-1})_i. \tag{2.1}$$

The q-hit numbers are a q-analogue in the sense shown by Lewis and Morales in [10, Prop. 3.3], which states that $H_d(B,q) \equiv h_d(B) \pmod{q-1}$. When q is not 1, there is no known combinatorial interpretation of $H_d(B,q)$ for general boards B. However, the numbers are conjectured to be positive by Lewis and Morales in [10, Conjecture 6.3].

Lewis and Morales also showed the following.

Proposition 2.1 ([10, Prop. 3.5])**.** For any board B inside $[m] \times [n]$, we can compute the individual q-hit and q-rook numbers in the following way:

$$H_k(B,q) = q^{\binom{k+1}{2}+\binom{m}{2}} \sum_{i=k}^{m} M_i(B,q) \cdot \frac{[n-i]!_q}{[n-m]!_q} \begin{bmatrix} i \\ k \end{bmatrix}_q (-1)^{i+k} q^{-ik}$$

and

$$M_k(B,q) = q^{\binom{k}{2}-\binom{m}{2}} \frac{[n-m]!_q}{[n-k]!_q} \sum_{i=k}^{m} H_i(B,q) \begin{bmatrix} i \\ k \end{bmatrix}_q.$$

By this proposition, if $M_i(B,q)$ is an integer polynomial in q for all i such that $k \leq i \leq m$, then $H_k(B,q) \in \mathbb{Z}[q]$, and vice versa. In fact, this implies that all $M_i(B,q)$ are polynomials in q with integer coefficients if and only if the same is true for all $H_i(B,q)$.

Example 2.2. It is important to note that $H_d(B,q)$ is not a polynomial in q for all choices of B. A counterexample to this is the Fano board $F \subseteq [7] \times [7]$:

$$\begin{bmatrix} * & * & 0 & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 & 0 & * \\ * & 0 & * & * & 0 & 0 & 0 \\ 0 & * & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & * & 0 & * & * \\ * & 0 & 0 & 0 & * & 0 & * \end{bmatrix}.$$

In [15], Stembridge found that $M_7(F,q)$ is the following non-polynomial function of q:

$$\begin{aligned} M_7(F,q) = &(x+1)^3(x^{11}+17x^{10}+135x^9+650x^8+2043x^7+4236x^6 \\ &+5845x^5+5386x^4+3260x^3+1236x^2+264x+24-Z_2x^6), \end{aligned} \tag{2.2}$$

where $x = q-1$ and $Z_2 = 1$ for odd q and 0 for even q.

This shows that the q-rook number $M_d(B,q)$ is not always a polynomial in q, and consequently that $H_d(B,q)$ is not either. See [14] for further discussion on non-polynomiality of this and related counting problems over $\mathbb{F}_q$.

Remark 2.3. Stembridge's analysis implies that if $m, n \leq 6$, then $M_d(B,q)$ is a polynomial in q for every $m \times n$ board B; the Fano board gives the first non-polynomial example, of size 7×7 [15, p. 2].

3 Orbits of matrices

In this section, we study a group action on matrices with a fixed rank and support. Following [9], we find the size of the orbit of a given matrix with rank d. Many orbits have size divisible by $(q-1)^{d+2}$, and we are able to enumerate the orbits which do not, which gives a formula for $M_d(B,q)$ modulo $(q-1)^2$. We use the same technique to show polynomiality of $M_d(B,q)$ modulo $(q-1)^6$.

First, we define the graph of a board.

Definition 3.1. We define a bi-colored graph G as a graph together with a partition of its vertices into two sets V_1 and V_2, such that every edge is between V_1 and V_2. We call elements of V_1 row vertices, and elements of V_2 column vertices. A homomorphism of bi-colored graphs is required to take row vertices to row vertices, and column vertices to column vertices. In this paper, a graph means a bi-colored graph unless otherwise mentioned.

When B is a board (a subset of $[m] \times [n]$), we define $G(B)$ to be the graph with row vertices $[m]$, column vertices $[n]$, and edge set B.

3.1 Counting matrices by support

We show a relation between the maximal rook placement of a board and the maximal rank of a matrix with support in that board.

The following is a standard application of Hall's marriage theorem, which we include for completeness.

Proposition 3.2. Consider a board B, and let k be the maximum number of non-attacking rooks that can be placed in B. For any matrix M over $\mathbb{F}_q$ with support in B, the rank of M is at most k.

Proof. Let the rank of an m by n matrix M with support in B be r. We show that B admits a placement of r non-attacking rooks. Choose linearly independent rows $v_1, \dots, v_r$ of M. Let G be the bi-colored graph with row vertices $a_1, \dots, a_r$, column vertices $b_1, \dots, b_n$, and an edge $a_i b_j$ whenever the jth entry of v_i is nonzero.

A matching of size r in G corresponds to r cells of B, no two in the same row or column. By Hall's marriage theorem [6], it is enough to show that every subset $S \subseteq \{a_1, \dots, a_r\}$ has at least $\#S$ neighbors. Suppose instead that some subset S has only $i < \#S$ neighbors. Then the rows in S are supported on only i columns, so those rows span a subspace of dimension at most i. This contradicts the linear independence of $v_1, \dots, v_r$. Therefore G has a matching of size r, so $r \leq k$. $\square$

Definition 3.3. Define $S_d(B, q)$ as the set of m by n matrices A over $\mathbb{F}_q$ such that the rank of A is d and the support of A is *exactly* B.

Let $T_q(m, n, B, d)$ be the set of m by n matrices with support contained in B and rank d. We have the following relation.

$$T_q(m, n, B, d) = \bigcup_{C \subseteq B} S_d(C, q).$$

Therefore,

$$\mathfrak{m}_d(B, q) = \sum_{C \subseteq B} \#S_d(C, q).$$

By Möbius inversion, we get

$$\#S_d(B, q) = \sum_{C \subseteq B} (-1)^{(\#B - \#C)} \mathfrak{m}_d(C, q).$$

Define maxhit(B) as the maximum number of non-attacking rooks that can be placed on B. If maxhit$(C) < d$, then $\mathfrak{m}_d(C, q) = 0$ by Proposition 3.2. Thus, our two equations become:

$$T_q(m, n, B, d) = \bigcup_{C \subseteq B, \operatorname{maxhit}(C) \geq d} S_d(C, q)$$

and

$$\#S_d(B, q) = \sum_{C \subseteq B, \operatorname{maxhit}(C) \geq d} (-1)^{(\#B - \#C)} \mathfrak{m}_d(C, q).$$

Proposition 3.4. For fixed q, the number $\#S_d(B, q)$ is divisible by $(q-1)^{m+n-C(G(B))}$, where $C(G(B))$ is the number of connected components of $G(B)$.

Proof. We follow the proof of [9, Prop. 5.1]. Let $A \in S_d(B, q)$ be a matrix. Let $(\mathbb{F}_q^{\times})^l$ be the set of diagonal $l \times l$ matrices with each diagonal entry nonzero. Now, consider the group action $(\mathbb{F}_q^{\times})^m \times (\mathbb{F}_q^{\times})^n$ on S_d defined by $(X, Y) \cdot A = XAY^{-1}$. The support of XAY^{-1} is still exactly B because X and Y are diagonal matrices. Define $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_n$ as the diagonal entries of X and Y, in that order. We show that (X, Y) stabilizes A if, for each nonzero $a_{i,j}$, we have $x_i = y_j$. This is because, if $XAY^{-1} = A$, then $XA = AY$. Then $(XA)_{i,j}$ (the element on the ith row and jth column of XA) is $x_i a_{i,j}$, and similarly $(AY)_{i,j}$ is $a_{i,j} y_j$. Thus if $a_{i,j} \neq 0$, then $x_i = y_j$.

Thus the stabilizer has size $(q-1)^{C(G(B))}$, since for each connected component of $G(B)$ we may choose one common nonzero scalar for all row and column vertices in that component. By the orbit-stabilizer theorem, the orbit of A therefore has size $(q-1)^{m+n-C(G(B))}$. Since $S_d(B, q)$ is a disjoint union of such orbits, $\#S_d(B, q)$ is divisible by $(q-1)^{m+n-C(G(B))}$. $\square$

3.2 Classifying orbits of size $(q-1)^{d+1}$ in $\mathfrak{m}_d(B,q)$

We use the same diagonal action as in Proposition 3.4, now on $T_q(m,n,B,d)$. Let $\mathcal{O}$ be the set of its orbits. Since the action preserves support, each orbit $O \in \mathcal{O}$ has a well-defined support board $\operatorname{supp}(O)$. Therefore,

$$\mathfrak{m}_d(B,q) = \sum_{G} \sum_{\substack{B' \subseteq B \\ G(B') \cong G}} \sum_{\substack{O \in \mathcal{O} \\ \operatorname{supp}(O)=B'}} \#O,$$

where the outer sum is over all isomorphism classes of bi-colored graphs G, but can be restricted to a finite sum over just those isomorphism classes which appear as $G(B')$ for some $B' \subseteq B$.

By the orbit-stabilizer theorem,

$$\#O = (q-1)^{m+n-C(G)},$$

so

$$\mathfrak{m}_d(B,q) = \sum_{G} \sum_{\substack{B' \subseteq B \\ G(B') \cong G}} \sum_{\substack{O \in \mathcal{O} \\ \operatorname{supp}(O)=B'}} (q-1)^{m+n-C(G)},$$

and grouping terms with the same power of $(q-1)$ we get

$$\mathfrak{m}_d(B,q) = \sum_{i=d}^{m+n-1} (q-1)^i \sum_{\substack{G \\ C(G)=m+n-i}} \sum_{\substack{B' \subseteq B \\ G(B') \cong G}} \sum_{\substack{O \in \mathcal{O} \\ \operatorname{supp}(O)=B'}} 1. \tag{3.1}$$

Furthermore, if B_1 and B_2 are boards with $G(B_1) \cong G(B_2)$, then B_1 and B_2 differ only by permuting rows and columns. Hence $S_d(B_1,q)$ and $S_d(B_2,q)$ have the same number of orbits.

We can rewrite Equation (3.1) to remove the inner sums. Define $\mathcal{B}(B,G)$ to be the number of boards $B' \subseteq B$ such that $G(B') \cong G$, and, for a fixed rank d, define $\mathcal{O}_d(G,q)$ to be the number of orbits in $S_d(B',q)$ for any board B' satisfying $G(B') \cong G$. This is well defined by the previous paragraph, and equals $\#S_d(B',q)/(q-1)^{m+n-C(G)}$ for any such board B'. If G has x row vertices and y column vertices with $x < m$ or $y < n$, we adjoin isolated row and column vertices so that G can still be regarded as a support graph on $[m] \sqcup [n]$: if R is the completely disconnected bi-colored graph with $m-x$ row vertices and $n-y$ column vertices, then we set $\mathcal{B}(B,G) := \mathcal{B}(B, G \bigsqcup R)$ and $\mathcal{O}_d(G,q) := \mathcal{O}_d(G \bigsqcup R, q)$.

With these definitions, Equation (3.1) becomes

$$\mathfrak{m}_d(B,q) = \sum_{i=d}^{m+n-1} (q-1)^i \sum_{\substack{G \\ C(G)=m+n-i}} \mathcal{B}(B,G) \cdot \mathcal{O}_d(G,q).$$

A useful feature of this expansion is the convolution formula. If G_1, G_2 are bi-colored graphs on different vertex sets we have

$$\mathcal{O}_d(G_1 \sqcup G_2, q) = \sum_i \mathcal{O}_i(G_1,q)\mathcal{O}_{d-i}(G_2,q).$$

Therefore,

$$\mathfrak{m}_d(B,q) = \sum_{i=d}^{m+n-1} (q-1)^i \sum_{\substack{G \\ C(G)=m+n-i}} \mathcal{B}(B,G) \sum_{D_1+\cdots+D_{m+n-i}=d} \prod_{k=1}^{m+n-i} \mathcal{O}_{D_k}(G_k,q),$$

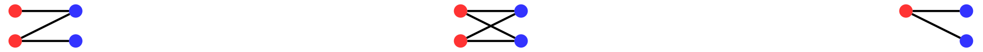
where $G_1, \ldots, G_{m+n-i}$ are the connected components of G. If G_k has x_k row vertices and y_k column vertices, we can rewrite the equation as

$$\mathfrak{m}_d(B,q) = (q-1)^d \sum_{i=1}^{m+n-d} \sum_{\substack{G \\ C(G)=i}} \mathcal{B}(B,G) \sum_{D_1+\cdots+D_i=d} \prod_{k=1}^{i} \mathcal{O}_{D_k}(G_k,q)(q-1)^{x_k+y_k-D_k-1}. \tag{3.2}$$

From Equation (3.2), the single edge graph E, with one row and one column vertex, takes on special importance for low-degree coefficients of $(q-1)$. The contributions to the $(q-1)^d$ coefficient come from indices where $\mathcal{O}_{D_k}(G_k,q)$ is nonzero while $D_k = x_k + y_k - 1$ for all k. For a connected graph G_k other than E, we have $x_k + y_k \geq 3$, so $D_k \leq \min(x_k, y_k) \leq x_k + y_k - 2 < x_k + y_k - 1$; hence $\mathcal{O}_{x_k+y_k-1}(G_k,q) = 0$. Thus each G_k must be the edge graph E. In fact, $\mathcal{O}_1(\mathsf{E},q) = 1$, so the lowest coefficient of Equation (3.2) is $\mathcal{B}(B,\mathsf{E}^d) = r_d(B)$, the classical rook number (see [9, Prop. 5.1]).

Motivated by this case, we view $\mathcal{B}(B, G)$ as a generalized rook number of the board B. We therefore introduce notation that more closely matches the usual rook numbers.

Modulo any fixed power of $(q-1)$, only finitely many isomorphism classes of connected bi-colored graphs give nonzero contributions to the inner sum in Equation (3.2). To understand $\mathfrak{m}_d(B, q)$ modulo $(q-1)^{d+2}$, it suffices to understand the connected bi-colored graphs G and ranks D such that $x+y-1 = D+1$ (and $\mathcal{O}_D(G, q)$ is nonzero), which we enumerate in Proposition 3.6. Then for each graph, and rank, we calculate $\mathcal{O}_D(G, q)$ in Proposition 3.7. The results are gathered into an expansion for the q-rook number $M_d(B, q)$ (modulo $(q-1)^2$) in Theorem 3.9.



(a) Z graph (Z)

(b) Shoelace graph (S)

(c) 1-wedge graph (WC)

Figure 1: Bi-colored graphs representing Z, S, WR, and WC

Definition 3.5. We define four bi-colored graphs as follows, illustrated by Figure 1:

1. Define Z as the bi-colored graph with 2 rows and 2 columns, which forms a path of length 3 (Figure 1a).
2. Define S as the bi-colored graph with 2 rows and 2 columns, which forms a $K_{2,2}$ (Figure 1b).
3. Define WC as the bi-colored graph with 1 row and 2 columns, which forms a row connected to two columns (Figure 1c).
4. Define WR as the bi-colored graph with 2 rows and 1 column, which forms a column connected to two rows (Figure 1c with red and blue flipped).

Proposition 3.6. If G is a connected bi-colored graph with x row vertices and y column vertices, D is a rank such that $\mathcal{O}_D(G, q)$ is nonzero, and $x + y - 1 = D + 1$, then (G, D) is one of: $(\mathsf{Z}, 2)$, $(\mathsf{S}, 2)$, $(\mathsf{WC}, 1)$, or $(\mathsf{WR}, 1)$.

Proof. Since $D \le \min(x, y)$, and $x + y = D + 2$, we must have $\max(x, y) \le 2$. The proof follows by checking the finite list of bi-colored graph isomorphism classes with at most two of each color of vertex. ☐

Proposition 3.7. The number of orbits of the appropriate rank for these graphs is

$$\mathcal{O}_2(\mathsf{Z}, q) = \mathcal{O}_1(\mathsf{WC}, q) = \mathcal{O}_1(\mathsf{WR}, q) = 1 \quad \text{and} \quad \mathcal{O}_2(\mathsf{S}, q) = q - 2.$$

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & * \\ * & * \end{bmatrix} \qquad \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$$

(a) Board for Z

(b) Board for S

(c) Board for WC

Figure 2: Boards used in the proof of Proposition 3.7

Proof. In each case we calculate $\#S_D(B, q)$ for a board B with the required graph and divide by the orbit size $(q-1)^{x+y-1}$.

For Z, let B be the board from Figure 2a. Any 2×2 matrix with this support has rank 2, so $\#S_2(B, q) = (q-1)^3$, and therefore $\mathcal{O}_2(\mathsf{Z}, q) = 1$.

For S, let B be the board from Figure 2b. There are $(q-1)^4$ matrices with that support over $\mathbb{F}_q$, of which $(q-1)^3$ have rank 1. Hence $\#S_2(B, q) = (q-1)^3(q-2)$, so $\mathcal{O}_2(\mathsf{S}, q) = q - 2$.

For WC, let B be the board from Figure 2c. Any matrix with this support has rank 1, so $\#S_1(B, q) = (q-1)^2$ and $\mathcal{O}_1(\mathsf{WC}, q) = 1$. The case of WR is symmetric. ☐

For convenience, we define generalized rook numbers.

Definition 3.8. For a board $B \subseteq [m] \times [n]$ and a bi-colored graph F with x row and y column vertices, we define $r_{F,i}(B)$ as $\mathcal{B}(B, F \bigsqcup \mathsf{E}^i)$. Equivalently, $r_{F,i}(B)$ is the number of boards $\sigma \subseteq B$ with $G(\sigma) \cong F \bigsqcup \mathsf{E}^i \bigsqcup R$, where R is a bi-colored graph with no edges. When F is the empty graph, $r_{F,i}(B)$ is the standard rook number $r_i(B)$. By convention, $r_{F,i}(B) = 0$ if $i < 0$.

We give a formula for the q-rook number $M_d(B, q)$ modulo $(q-1)^2$:

Theorem 3.9. For a board B and non-negative integer d, let

$$\Lambda_d(B) := r_{\mathsf{Z},d-2}(B) - r_{\mathsf{S},d-2}(B) + r_{\mathsf{WR},d-1}(B) + r_{\mathsf{WC},d-1}(B).$$

Then

$$M_d(B,q) \equiv r_d(B) + (q-1)\Lambda_d(B) \pmod{(q-1)^2}.$$

Proof. By Equation (3.2) and Proposition 3.6, we have

$$\mathfrak{m}_d(B,q) \equiv r_d(B)(q-1)^d + (q-1)^{d+1} \sum_{(G,D)} r_{G,d-D}(B)\mathcal{O}_D(G,q) \pmod{(q-1)^{d+2}},$$

where the sum ranges over $\{(\mathsf{Z},2),(\mathsf{S},2),(\mathsf{WC},1),(\mathsf{WR},1)\}$. Evaluating these orbit counts gives

$$\begin{aligned}\mathfrak{m}_d(B,q) \equiv r_d(B)(q-1)^d + (q-1)^{d+1}\Big(&r_{\mathsf{Z},d-2}(B) + r_{\mathsf{WR},d-1}(B) \\ &+ r_{\mathsf{WC},d-1}(B) + (q-2)r_{\mathsf{S},d-2}(B)\Big) \pmod{(q-1)^{d+2}}.\end{aligned} \tag{3.3}$$

Dividing by $(q-1)^d$ gives the desired equation. ☐

Remark 3.10. Since every copy of S contains a copy of Z, deleting the edge in the larger-indexed row and smaller-indexed column defines an injection from the copies counted by $r_{\mathsf{S},d-2}(B)$ to those counted by $r_{\mathsf{Z},d-2}(B)$, so

$$r_{\mathsf{Z},d-2}(B) - r_{\mathsf{S},d-2}(B) + r_{\mathsf{WR},d-1}(B) + r_{\mathsf{WC},d-1}(B) \geq 0,$$

so the coefficients of $M_d(B,q) \pmod{(q-1)^2}$ are nonnegative.

Example 3.11. For example, for the board $B = [2]\times[2]$ with $d=2$, we have $r_2(B)=2$, $r_{\mathsf{Z},0}(B)=4$, $r_{\mathsf{S},0}(B)=1$, and $r_{\mathsf{WR},1} = r_{\mathsf{WC},1} = 0$, so

$$\begin{aligned}M_2(B,q) &= (q^2-1)(q^2-q)/(q-1)^2 \equiv 2 + 3(q-1) \\ &\equiv r_2(B) + (q-1)\big(r_{\mathsf{Z},0}(B) - r_{\mathsf{S},0}(B) + r_{\mathsf{WR},1}(B) + r_{\mathsf{WC},1}(B)\big) \pmod{(q-1)^2}.\end{aligned}$$

The coefficient of $q-1$ is therefore nonnegative.

3.3 Polynomiality of $\mathfrak{m}_d(B,q)$ modulo $(q-1)^{d+6}$

Another immediate consequence of Equation (3.2) is the following:

Theorem 3.12. For any board B, $\mathfrak{m}_d(B,q)$ is polynomial modulo $(q-1)^{d+6}$, and its coefficients modulo $(q-1)^{d+6}$ are integers.

Proof. Consider Equation (3.2) modulo $(q-1)^{d+6}$. Let $G_1,\ldots,G_i$ be the connected components of G, and $D_1,\ldots,D_i$ index a nonzero term of the summation. Consider the kth factor of the product,

$$\mathcal{O}_{D_k}(G_k,q)(q-1)^{x_k+y_k-D_k-1}.$$

If either of x_i or y_i is at least 7, then $x_i+y_i-D_i-1$ is at least 6, as D_i can be at most $\min(x_i,y_i)$. In this case, the entire term of the summation is congruent to 0. Therefore, none of the connected components of G can have 7 or more of either row or column vertices. We know however by Remark 2.3 that $\mathcal{O}_{D_k}(G_k,q)$ is polynomial if G_k has at most 6 row vertices and at most 6 column vertices, and so $\mathfrak{m}_d(B,q)$ is congruent to a polynomial. ☐

Corollary 3.13. For a board B, $M_d(B,x+1)$ and $H_d(B,x+1)$ are both polynomial modulo x^6, and have integer coefficients.

Proof. By definition,

$$M_d(B,q) = \mathfrak{m}_d(B,q)/(q-1)^d,$$

so $M_d(B,q)$ is polynomial with integer coefficients modulo $(q-1)^6$ for any d. Then Proposition 2.1 shows $H_d(B,q)$ is too. ☐

Remark 3.14. The number 6 is the maximal c such that $\mathfrak{m}_d(B,q)$ is always polynomial modulo $(q-1)^{d+c}$. Consider the Fano board F with $d=7$. Using Equation (2.2), the coefficient of $(q-1)^{13}$ depends on the residue class of $q \pmod 2$. Therefore $\mathfrak{m}_d(F,q)$ is not polynomial modulo $(q-1)^{d+7}$.

4 Hit numbers modulo $(q-1)^2$

In this section, we consider the q-hit number $H_i(B,q)$. Motivated by [10, Conjecture 6.7], we conjecture the following.

Conjecture 4.1. Let B be a board, and $P \in \mathbb{Z}[x]$ be a polynomial of degree $k-1$ such that $P(x) \equiv H_i(B,x+1) \pmod{x^k}$ for all x in an unbounded subset of $\mathbb{Z}$. Then, P has non-negative coefficients.

In this section, we verify this conjecture for $k=1$ and $k=2$. By [10, Prop. 3.3], we know that $H_i(B,x+1) \equiv h_i(B) \pmod{x}$, which is manifestly non-negative. Next, we find an expression for the coefficient of x in $H_i(B,x+1)$ modulo x^2. Because $H_i(B,x+1)$ is a polynomial with integer coefficients modulo x^2 (by Corollary 3.13), we know that the coefficient does not depend on x and is an integer.

Therefore, let $C_i(B)$ be defined by

$$H_i(B,x+1) \equiv C_i(B)x + h_i(B) \pmod{x^2}.$$

We seek a formula for $C_i(B)$ by reducing Equation (2.1) modulo $(q-1)^2$ and then extracting the coefficient of t^i from both sides. We show that the coefficient $C_i(B)$ is non-negative for any board B.

4.1 Finding the q-hit residue modulo x^2

In this subsection, we transfer the results of Section 3.2 to the computation of q-hit numbers. We use our formula (3.3) to produce a formula for $H_d(B,x+1) \pmod{x^2}$.

We make the following definitions of generalized hit numbers, analogous to Definition 3.8.

Definition 4.2. Let F be a bi-colored graph with x row vertices and y column vertices, and assume that no connected component of F is isomorphic to E. For a board $B \subseteq [m]\times[n]$, define $h_{F,d}(B)$ to be the number of boards $\sigma \subseteq [m]\times[n]$ such that

$$G(\sigma) \cong F \bigsqcup \mathsf{E}^{\min(m-x,n-y)} \bigsqcup R$$

and

$$G(\sigma\cap B) \cong F \bigsqcup \mathsf{E}^{d} \bigsqcup R',$$

where R and R' are bi-colored graphs with no edges. When F is the empty graph, this is exactly the usual hit number $h_d(B)$.

Now we can state the formula, which is the main result of this subsection.

Theorem 4.3. The q-hit number satisfies

$$H_d(B,x+1) \equiv C_d(B)x + h_d(B) \pmod{x^2},$$

where $C_d(B)$ is given by one of the following formulas. The two formulas agree in all terms except those involving WC; when $m<n$, these are the terms carrying the factor $(n-m)^{-1}$.

For boards $B \subseteq [m]\times[n]$ with $m<n$,

$$\begin{aligned} C_d(B) &= h_{\mathsf{Z},d-2}(B) - h_{\mathsf{S},d-2}(B) + (n-m+1)h_{\mathsf{WR},d-1}(B) \\ &\quad + \frac{n-d}{n-m}h_{\mathsf{WC},d-1}(B) - 2h_{\mathsf{Z},d-1}(B) + 2h_{\mathsf{S},d-1}(B) - (n-m+1)h_{\mathsf{WR},d}(B) \\ &\quad + \frac{2d-n-1}{n-m}h_{\mathsf{WC},d}(B) + h_{\mathsf{Z},d}(B) - h_{\mathsf{S},d}(B) + \frac{d-1}{n-m}h_{\mathsf{WC},d+1}(B) \\ &\quad + \frac{1}{4}\Big(h_d(B)(m-d)(m+2n+d-3) \\ &\quad + h_{d+1}(B)(2d+2)(n-1) + h_{d+2}(B)(d+2)(d+1)\Big). \end{aligned}$$

For boards $B \subseteq [n]\times[n]$,

$$\begin{aligned} C_d(B) &= h_{\mathsf{Z},d-2}(B) - h_{\mathsf{S},d-2}(B) + h_{\mathsf{WR},d-1}(B) + h_{\mathsf{WC},d-1}(B) \\ &\quad - 2h_{\mathsf{Z},d-1}(B) + 2h_{\mathsf{S},d-1}(B) - h_{\mathsf{WR},d}(B) - h_{\mathsf{WC},d}(B) \\ &\quad + h_{\mathsf{Z},d}(B) - h_{\mathsf{S},d}(B) + \frac{1}{4}\big(h_d(B)(n-d)(3n+d-3) \\ &\quad + h_{d+1}(B)(2d+2)(n-1) + h_{d+2}(B)(d+2)(d+1)\big). \end{aligned}$$

By Corollary 3.13, the quantity $C_d(B)$ is an integer; in Section 4.2, we will show that it is in fact non-negative. We start with an example to make the notation clear.

Example 4.4. Let $B = [2] \times [2]$ be the full 2×2 board. Since $m = n = d = 2$, the only surviving contribution in the square-board formula is $h_{\mathsf{Z},0}(B) - h_{\mathsf{S},0}(B)$. A direct computation from Equation (2.1) gives

$$H_2(B, x+1) = x^2 + 3x + 2 \equiv 3x + 2 \pmod{x^2}.$$

Here $h_{\mathsf{Z},0}(B) = \mathcal{B}(B, \mathsf{Z}) = 4$ and $h_{\mathsf{S},0}(B) = \mathcal{B}(B, \mathsf{S}) = 1$, so

$$C_2(B) = h_{\mathsf{Z},0}(B) - h_{\mathsf{S},0}(B) = 3.$$

Before we begin the proof, we present the key ingredients of the computation. The following results about q-analogues have standard proofs, which we omit.

Lemma 4.5. For all integers $a, n \geq 1$, we have the following congruences of polynomials:

$$\begin{aligned}
q^n &\equiv n(q-1) + 1 \pmod{(q-1)^2}, \\
[n]_q &\equiv \binom{n}{2}(q-1) + n \pmod{(q-1)^2}, \\
[n]!_q &\equiv \frac{n!}{2}\binom{n}{2}(q-1) + n! \pmod{(q-1)^2}, \\
\begin{bmatrix} n \\ i \end{bmatrix}_q &\equiv \binom{n}{i}\frac{i(n-i)}{2}(q-1) + \binom{n}{i} \pmod{(q-1)^2}, \\
(a;q)_n &\equiv \left(-\binom{n}{2}(1-a)^{n-1}a\right)(q-1) + (1-a)^n \pmod{(q-1)^2}.
\end{aligned}$$

We use the following relation between the classical rook and hit numbers.

Lemma 4.6. For a board B and fixed k, we have

$$\sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!} i(i-1)\cdots(i-k+1)(t-1)^i = \sum_{i=0}^{m} i(i-1)\cdots(i-k+1)(t-1)^k t^{i-k} h_i(B).$$

Proof. Take k derivatives of Equation (1.1) and then multiply by $(t-1)^k$. ☐

Finally, we have generalized rook-hit relations, analogous to the classical rook-hit relation from Equation (1.1).

Proposition 4.7. Let $B \subseteq [m] \times [n]$ be a board. Let G be a bi-colored graph with x row vertices and y column vertices. Assume that no connected component of G is isomorphic to E, and write $M = \min(m-x, n-y)$ and $N = \max(m-x, n-y)$. Then the generalized rook and hit numbers satisfy

$$\sum_{i=0}^{M} r_{G,i}(B)\frac{(N-i)!}{(N-M)!}(t-1)^i = \sum_{i=0}^{M} h_{G,i}(B)t^i.$$

Proof. We compute the same generating function in two ways. A *hit pair* of weight i is a pair of boards $\omega \subseteq \sigma \subseteq [m] \times [n]$ such that $\omega \subseteq B$, $G(\omega) \cong G \sqcup \mathsf{E}^i$, and $G(\sigma) \cong G \sqcup \mathsf{E}^M$. Let

$$Q_G(B, t) := \sum_{(\omega,\sigma)} (t-1)^{\mathrm{weight}(\omega)},$$

where the sum ranges over all hit pairs and $\mathrm{weight}(\omega) = i$ when $G(\omega) \cong G \sqcup \mathsf{E}^i$.

First fix ω of weight i. To extend ω to σ, we must place $M - i$ additional disjoint edges in the complementary $(m-x-i) \times (n-y-i)$ rectangle. The number of choices is therefore

$$r_{M-i}([m-x-i] \times [n-y-i]) = \frac{(N-i)!}{(N-M)!},$$

since the smaller side length is $M - i$ and the larger is $N - i$. As there are $r_{G,i}(B)$ choices for ω, we obtain

$$Q_G(B, t) = \sum_{i=0}^{M} r_{G,i}(B)\frac{(N-i)!}{(N-M)!}(t-1)^i.$$

Now fix σ such that $G(\sigma \cap B) \cong G \sqcup \mathsf{E}^j$. Because G has no connected component isomorphic to E, the distinguished copy of G inside $G(\sigma \cap B)$ is uniquely determined: it is the union of all connected components of $G(\sigma \cap B)$ that are not isolated edges. Thus the remaining j isolated edges are unambiguously the ones from which ω chooses its extra i edge-components, so there are exactly $\binom{j}{i}$ choices for $\omega \subseteq \sigma$ of weight i. Summing over i and σ gives

$$Q_G(B,t) = \sum_{j=0}^{M} h_{G,j}(B) \sum_{i=0}^{j} \binom{j}{i} (t-1)^i = \sum_{j=0}^{M} h_{G,j}(B) t^j.$$

Comparing the two expressions for $Q_G(B,t)$ proves the proposition. ☐

For a graph-rank pair (G, D), define

$$f_{G,D}(B,t) := \sum_{i=D}^{M+D} r_{G,i-D}(B) \frac{(n-i)!}{(n-m)!} (t-1)^i.$$

This is the form in which the generalized rook numbers appear in the proof of Theorem 4.3.

Lemma 4.8. Let $K := n - D - N$, which we call the defect. Then

$$\begin{aligned} f_{G,D}(B,t) = (t-1)^D \frac{(N-M)!}{(n-m)!} (-1)^K \sum_{i=0}^{M} \Bigg[&\sum_{\ell=0}^{K} h_{G,i+\ell}(B)(-1)^\ell \\ &\times \sum_{j=\ell}^{\ell+i} \binom{K}{j} \binom{j}{\ell} (-N-1)_{K-j} (i+\ell)_j \Bigg] t^i. \end{aligned} \tag{4.1}$$

Proof. Write $s = i - D$. Then

$$f_{G,D}(B,t) = (t-1)^D \sum_{s=0}^{M} r_{G,s}(B) \frac{(n-D-s)!}{(n-m)!} (t-1)^s.$$

Since $n - D = N + K$, we have

$$\frac{(n-D-s)!}{(n-m)!} = \frac{(N-M)!}{(n-m)!} \frac{(N+K-s)!}{(N-M)!}.$$

Using

$$(N+K-s)! = (N-s)!(-1)^K (s-N-1)_K,$$

and Vandermonde's identity

$$(s-N-1)_K = \sum_{j=0}^{K} \binom{K}{j} (-N-1)_{K-j} (s)_j,$$

for falling factorials, we obtain

$$\begin{aligned} f_{G,D}(B,t) = (t-1)^D \frac{(N-M)!}{(n-m)!} (-1)^K &\sum_{j=0}^{K} \binom{K}{j} (-N-1)_{K-j} \\ \times &\sum_{s=0}^{M} r_{G,s}(B) \frac{(N-s)!}{(N-M)!} (s)_j (t-1)^s. \end{aligned}$$

Now take j derivatives of the two sides of the equation in Proposition 4.7 and multiply by $(t-1)^j$. This gives

$$\sum_{s=0}^{M} r_{G,s}(B) \frac{(N-s)!}{(N-M)!} (s)_j (t-1)^s = (t-1)^j \sum_{u=0}^{M} (u)_j h_{G,u}(B) t^{u-j}.$$

Expanding $(t-1)^j = \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell t^{j-\ell}$ and then reindexing by $u = i + \ell$ yields

$$(t-1)^j \sum_{u=0}^{M} (u)_j h_{G,u}(B) t^{u-j} = \sum_{i=0}^{M} \left[\sum_{\ell=0}^{K} h_{G,i+\ell}(B) (-1)^\ell \binom{j}{\ell} (i+\ell)_j \right] t^i.$$

Here the index j comes from the Vandermonde expansion of the factorial ratio, while ℓ records the choice of a term from $(t-1)^j$; we may extend the upper limit from j to K because $\binom{j}{\ell} = 0$ for $\ell > j$. Substituting this into the previous display and interchanging the finite j- and ℓ-sums gives a sum with $\ell \leq j$. Since $(i+\ell)_j = 0$ for $j > i + \ell$, we may restrict the remaining j-sum to $\ell \leq j \leq i + \ell$, which is exactly Equation (4.1). ☐

Proof of Theorem 4.3. Consider Equation (2.1):

$$\sum_{i=0}^{m} H_i(B,q)t^i = q^{\binom{m}{2}} \sum_{i=0}^{m} M_i(B,q)\frac{[n-i]!_q}{[n-m]!_q}(-1)^i(t;q^{-1})_i.$$

We expand the right-hand side modulo $(q-1)^2$ and then extract the coefficient of t^d.

Write

$$\Lambda_i(B) := r_{\mathsf{Z},i-2}(B) - r_{\mathsf{S},i-2}(B) + r_{\mathsf{WR},i-1}(B) + r_{\mathsf{WC},i-1}(B).$$

By Theorem 3.9,

$$M_i(B,q) \equiv r_i(B) + (q-1)\Lambda_i(B) \pmod{(q-1)^2}.$$

From the third equation of Lemma 4.5, we have

$$[a]!_q \equiv a!\left(1 + \frac{\binom{a}{2}}{2}(q-1)\right) \pmod{(q-1)^2},$$

and therefore

$$[n-i]!_q \equiv (n-i)!\left(1 + \frac{\binom{n-i}{2}}{2}(q-1)\right) \pmod{(q-1)^2}$$

and

$$[n-m]!_q \equiv (n-m)!\left(1 + \frac{\binom{n-m}{2}}{2}(q-1)\right) \pmod{(q-1)^2}.$$

Since

$$\frac{1+\alpha(q-1)}{1+\beta(q-1)} \equiv 1 + (\alpha-\beta)(q-1) \pmod{(q-1)^2},$$

and $[n-m]!_q$ has nonzero constant term, it follows that

$$\frac{[n-i]!_q}{[n-m]!_q} \equiv \frac{(n-i)!}{(n-m)!}\left(1 + \frac{\binom{n-i}{2} - \binom{n-m}{2}}{2}(q-1)\right) \pmod{(q-1)^2}.$$

Since

$$\binom{n-i}{2} - \binom{n-m}{2} = \frac{(m-i)(2n-m-i-1)}{2},$$

this becomes

$$\frac{[n-i]!_q}{[n-m]!_q} \equiv \frac{(n-i)!}{(n-m)!} + \frac{(n-i)!}{(n-m)!}\frac{(m-i)(2n-m-i-1)}{4}(q-1) \pmod{(q-1)^2}.$$

Finally, since

$$(-1)^i(t;q^{-1})_i = (tq^{-i+1};q)_i,$$

applying the last congruence of Lemma 4.5 to $(tq^{-i+1};q)_i$ gives

$$(-1)^i(t;q^{-1})_i \equiv (t-1)^i - \binom{i}{2}t(t-1)^{i-1}(q-1) \pmod{(q-1)^2}.$$

Combining these three expansions gives

$$\begin{aligned}\sum_{i=0}^{m} H_i(B,q)t^i \equiv q^{\binom{m}{2}} \Bigg[&\sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}(t-1)^i \\ &+(q-1)\sum_{i=0}^{m} \Lambda_i(B)\frac{(n-i)!}{(n-m)!}(t-1)^i \\ &+(q-1)\sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}\frac{(m-i)(2n-m-i-1)}{4}(t-1)^i \\ &+(q-1)\sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}\left(-\binom{i}{2}t(t-1)^{i-1}\right)\Bigg] \pmod{(q-1)^2}. \end{aligned} \tag{4.2}$$

Let $A_0(t)$ denote the constant term (with respect to $x = q - 1$) inside the brackets, and let $A_1(t)$ denote the coefficient of $q - 1$ inside the brackets. Since

$$q^{\binom{m}{2}} \equiv 1 + \binom{m}{2}(q-1) \pmod{(q-1)^2},$$

Equation (4.2) becomes

$$\sum_{i=0}^{m} H_i(B,q)t^i \equiv A_0(t) + (q-1)\big(A_1(t) + \binom{m}{2}A_0(t)\big) \pmod{(q-1)^2}.$$

Thus the constant term of the whole right-hand side is $A_0(t)$, while the linear term is $A_1(t) + \binom{m}{2}A_0(t)$. By Equation (1.1),

$$A_0(t) = \sum_{i=0}^{m} h_i(B)t^i.$$

In particular, the constant term of the whole right-hand side is $\sum_i h_i(B)t^i$.

We now rewrite the generalized-rook contribution in $A_1(t)$. It is

$$\sum_{i=0}^{m} \Lambda_i(B)\frac{(n-i)!}{(n-m)!}(t-1)^i = f_{\mathsf{Z},2}(B,t) - f_{\mathsf{S},2}(B,t) + f_{\mathsf{WR},1}(B,t) + f_{\mathsf{WC},1}(B,t).$$

For $(\mathsf{Z},2)$, $(\mathsf{S},2)$, and $(\mathsf{WR},1)$, the defect is 0, so Lemma 4.8 immediately gives

$$f_{\mathsf{Z},2}(B,t) = (t-1)^2\sum_{i=0}^{m-2} h_{\mathsf{Z},i}(B)t^i, \quad f_{\mathsf{S},2}(B,t) = (t-1)^2\sum_{i=0}^{m-2} h_{\mathsf{S},i}(B)t^i,$$

and

$$f_{\mathsf{WR},1}(B,t) = (t-1)(n-m+1)\sum_{i=0}^{m-2} h_{\mathsf{WR},i}(B)t^i.$$

For $(\mathsf{WC},1)$, the defect is 0 when $m = n$ and 1 when $m < n$. The case $m = n$ again gives

$$f_{\mathsf{WC},1}(B,t) = (t-1)\sum_{i=0}^{m-2} h_{\mathsf{WC},i}(B)t^i.$$

When $m < n$, Lemma 4.8 with $K = 1$ yields

$$f_{\mathsf{WC},1}(B,t) = -(t-1)\sum_{i=0}^{m-1}\big(ih_{\mathsf{WC},i}(B) - (i+1)h_{\mathsf{WC},i+1}(B)\big)t^i.$$

Therefore, when $m < n$,

$$\begin{aligned}
&\sum_{i=0}^{m}\big(r_{\mathsf{Z},i-2}(B) - r_{\mathsf{S},i-2}(B) + r_{\mathsf{WR},i-1}(B) + r_{\mathsf{WC},i-1}(B)\big)\frac{(n-i)!}{(n-m)!}(t-1)^i \\
&= \sum_{i=0}^{m} t^{i-2}(t-1)\Big[(t-1)\big(h_{\mathsf{Z},i-2}(B) - h_{\mathsf{S},i-2}(B) - \tfrac{i-1}{n-m}h_{\mathsf{WC},i-1}(B)\big) \\
&\qquad + t\big((n-m+1)h_{\mathsf{WR},i-1}(B) + \tfrac{n-1}{n-m}h_{\mathsf{WC},i-1}(B)\big)\Big],
\end{aligned}$$

while for $m = n$,

$$\begin{aligned}
&\sum_{i=0}^{n}\big(r_{\mathsf{Z},i-2}(B) - r_{\mathsf{S},i-2}(B) + r_{\mathsf{WR},i-1}(B) + r_{\mathsf{WC},i-1}(B)\big)(n-i)!(t-1)^i \\
&= \sum_{i=0}^{n} t^{i-2}(t-1)\Big[(t-1)\big(h_{\mathsf{Z},i-2}(B) - h_{\mathsf{S},i-2}(B)\big) + t\big(h_{\mathsf{WR},i-1}(B) + h_{\mathsf{WC},i-1}(B)\big)\Big].
\end{aligned}$$

For the remaining two sums in $A_1(t)$, define

$$S_k(t) := \sum_{i=0}^{m} r_i(B)\frac{(n-i)!}{(n-m)!}(i)_k(t-1)^i.$$

Equation (1.1) together with Lemma 4.6 gives

$$S_k(t) = \sum_{i=0}^{m} (i)_k (t-1)^k t^{i-k} h_i(B).$$

Since

$$(m-i)(2n-m-i-1) = m(2n-m-1) - 2(n-1)i + i(i-1),$$

the first remaining sum is

$$\frac{m(2n-m-1)}{4} S_0(t) - \frac{n-1}{2} S_1(t) + \frac{1}{4} S_2(t),$$

and therefore

$$\begin{aligned}\sum_{i=0}^{m} r_i(B) \frac{(n-i)!}{(n-m)!} \frac{(m-i)(2n-m-i-1)}{4}(t-1)^i = \frac{m(2n-m-1)}{4} &\sum_{i=0}^{m} h_i(B) t^i \\ - \frac{n-1}{2} &\sum_{i=0}^{m} i\, h_i(B)(t-1)t^{i-1} \\ + \frac{1}{2} &\sum_{i=0}^{m} \binom{i}{2} h_i(B)(t-1)^2 t^{i-2},\end{aligned}$$

and

$$\sum_{i=0}^{m} r_i(B) \frac{(n-i)!}{(n-m)!} \left(-\binom{i}{2} t (t-1)^{i-1} \right) = -\frac{t}{2(t-1)} S_2(t) = -\sum_{i=0}^{m} h_i(B) \binom{i}{2} t^{i-1}(t-1).$$

Finally, expand each occurrence of $(t-1)$ and $(t-1)^2$, collect the coefficient of t^d, and include the additional contribution $\binom{m}{2} h_d(B)$ coming from $q^{\binom{m}{2}}$. This yields exactly the two formulas stated in the theorem: when $m < n$,

$$\begin{aligned} C_d(B) = {} & h_{\mathsf{Z},d-2}(B) - h_{\mathsf{S},d-2}(B) + (n-m+1) h_{\mathsf{WR},d-1}(B) \\ & + \frac{n-d}{n-m} h_{\mathsf{WC},d-1}(B) - 2h_{\mathsf{Z},d-1}(B) + 2h_{\mathsf{S},d-1}(B) - (n-m+1) h_{\mathsf{WR},d}(B) \\ & + \frac{2d-n-1}{n-m} h_{\mathsf{WC},d}(B) + h_{\mathsf{Z},d}(B) - h_{\mathsf{S},d}(B) + \frac{d-1}{n-m} h_{\mathsf{WC},d+1}(B) \\ & + \frac{1}{4}\Big(h_d(B)(m-d)(m+2n+d-3) \\ & + h_{d+1}(B)(2d+2)(n-1) + h_{d+2}(B)(d+2)(d+1) \Big), \end{aligned}$$

and when $m = n$,

$$\begin{aligned} C_d(B) = {} & h_{\mathsf{Z},d-2}(B) - h_{\mathsf{S},d-2}(B) + h_{\mathsf{WR},d-1}(B) + h_{\mathsf{WC},d-1}(B) \\ & - 2h_{\mathsf{Z},d-1}(B) + 2h_{\mathsf{S},d-1}(B) - h_{\mathsf{WR},d}(B) - h_{\mathsf{WC},d}(B) \\ & + h_{\mathsf{Z},d}(B) - h_{\mathsf{S},d}(B) + \frac{1}{4}\big(h_d(B)(n-d)(3n+d-3) \\ & + h_{d+1}(B)(2d+2)(n-1) + h_{d+2}(B)(d+2)(d+1) \big). \end{aligned}$$

□

4.2 Positivity in q-hit number coefficients

In Theorem 4.3, we gave an alternating formula for the coefficient $C_i(B)$ of $q-1$ in the expansion

$$H_i(B,q) \equiv h_i(B) + C_i(B)(q-1) \pmod{(q-1)^2}.$$

In this section, we show that $C_i(B) \geq 0$, thus verifying Conjecture 4.1 in the case $k = 2$.

Example 4.9. Let $B = \{(1,1), \ldots, (n,n)\}$ be the support of the $n \times n$ identity matrix. The q-rook numbers of its complement $\overline{B}$ were studied in [9,12] as a q-analogue of derangements. By the reciprocity relation [10, Prop. 3.9] and the explicit formula of [10, Prop. 3.5], a direct computation gives

$$H_{n-d}(\overline{B}, q) = q^{n(d-1)-\binom{d}{2}+\binom{n}{2}} \sum_{k=0}^{n-d} (-1)^k \binom{n}{k+d} [n-d-k]!_q \begin{bmatrix} d+k \\ d \end{bmatrix}_q.$$

Applying Lemma 4.5 to the factors in this sum shows that

$$C_{n-d}(\overline{B}) = \sum_{k=0}^{n-d} \frac{n!}{k!d!}(-1)^k \left(\binom{n}{2} - \binom{d}{2} + n(d-1) + \frac{1}{2}\binom{n-d-k}{2} + \frac{dk}{2}\right).$$

This is nonnegative, since for $k = 2j$ the summand is positive and larger in magnitude than the succeeding summand for $k = 2j+1$.

Theorem 4.10. For a board $B \subseteq [n] \times [n]$, the linear q-hit coefficient $C_i(B)$ is a non-negative integer.

We restrict to square boards here because the square-chain argument below is formulated for boards $B \subseteq [n] \times [n]$; we do not prove the corresponding statement when $m < n$.

We prove that $C_i(B)$ is non-negative using a series of inequalities relating our generalized hit numbers to the usual ones.

Lemma 4.11. For boards $B \subseteq [n] \times [n]$, we have

$$h_{\mathsf{WR},i-1}(B) + h_{\mathsf{WC},i-1}(B) - 2h_{\mathsf{Z},i-1}(B) + 2h_{\mathsf{S},i-1}(B) + \frac{1}{4}(2i+2)ih_{i+1}(B) \geq 0 \tag{4.3}$$

and

$$h_{\mathsf{Z},i}(B) - h_{\mathsf{S},i}(B) - h_{\mathsf{WR},i}(B) - h_{\mathsf{WC},i}(B) + \frac{1}{4}(2i+2)(n-i-1)h_{i+1}(B) + \frac{1}{4}(i+1)(i+2)h_{i+2}(B) \geq 0. \tag{4.4}$$

Proof. Define a square-chain as a set of cells $t \subset [n] \times [n]$ whose associated graph $G(t)$ consists of the union of $n-2$ disjoint edges and a $K_{2,2}$. For a square-chain t, let $K(t)$ be the set of cells in t corresponding to the $K_{2,2}$. For a board $B \subseteq [n] \times [n]$, define $S_i(B)$ as the set of square-chains t such that $\#((t \backslash K(t)) \cap B) = i-2$.

For any $G \in \{\mathsf{Z}, \mathsf{S}, \mathsf{WR}, \mathsf{WC}\}$, and for each board $\omega \subseteq [n] \times [n]$ with $G(\omega) \cong G \bigsqcup \mathsf{E}^{n-2}$ and $G(\omega \cap B) \cong G \bigsqcup \mathsf{E}^{i-2}$, there is exactly one $t \in S_i(B)$ such that $\omega \subseteq t$, and there are no $t \in S_k(B)$ for $k \neq i$ such that $\omega \subseteq t$.

For a board $\omega \subseteq [n] \times [n]$ with $G(\omega) \cong \mathsf{E}^n$ and $G(\omega \cap B) \cong \mathsf{E}^i$, there are exactly $\binom{i}{2}$ different $t \in S_i(B)$ such that $\omega \subseteq t$. This is because we choose two cells $c_1, c_2 \in \omega \cap B$ that are part of $K(t)$ in $\binom{i}{2}$ ways, and such a choice fixes the other two cells of $K(t)$. Similarly, there are exactly $i(n-i)$ such $t \in S_{i+1}(B)$ such that $\omega \subseteq t$.

For a square-chain t and $G \in \{\mathsf{Z}, \mathsf{S}, \mathsf{WR}, \mathsf{WC}\}$, let $C_{t,G,i}(B)$ be the number of boards $\omega \subseteq [n] \times [n]$ with $G(\omega) \cong G \bigsqcup \mathsf{E}^{n-2}$, $G(\omega \cap B) \cong G \bigsqcup \mathsf{E}^i$, and $\omega \subseteq t$. Also, let $C_{t,i}(B)$ be the number of boards $\omega \subseteq [n] \times [n]$ with $G(\omega) \cong \mathsf{E}^n$, $\omega \subseteq t$, and $G(\omega \cap B) \cong \mathsf{E}^i$.

A choice of $\sigma \subseteq [n] \times [n]$ such that $G(\sigma) \cong \mathsf{Z} \bigsqcup \mathsf{E}^{n-2}$ and $G(\sigma \cap B) \cong \mathsf{Z} \bigsqcup \mathsf{E}^{i-2}$ fixes a unique choice of t in $S_i(B)$, and so

$$\sum_{t \in S_i(B)} C_{t,\mathsf{Z},i-2}(B) = h_{\mathsf{Z},i-2}(B).$$

The same is true mutatis mutandis for S, WR, and WC.

Similarly, for the usual hit numbers, we have the relations

$$\sum_{t \in S_i(B)} C_{t,i}(B) = \binom{i}{2} h_i(B)$$

and

$$\sum_{t \in S_{i+1}(B)} C_{t,i}(B) = i(n-i)h_i(B).$$

Observe that if $t \in S_i(B)$, then the values $C_{t,\mathsf{Z},i-2}(B)$, $C_{t,\mathsf{S},i-2}(B)$, $C_{t,\mathsf{WR},i-2}(B)$, $C_{t,\mathsf{WC},i-2}(B)$, $C_{t,i}(B)$, $C_{t,i-1}(B)$, and $C_{t,i-2}(B)$ are determined by the intersection $K(t) \cap B$. Since $K(t)$ has four cells, there are only 2^4 possible configurations of $K(t) \cap B$ to check.

First, for $t \in S_{i+1}(B)$, we have the inequality

$$C_{t,\mathsf{WR},i-1}(B) + C_{t,\mathsf{WC},i-1}(B) - 2C_{t,\mathsf{Z},i-1}(B) + 2C_{t,\mathsf{S},i-1}(B) + C_{t,i+1}(B) \geq 0.$$

Checking the 2^4 possible configurations of $K(t) \cap B$ shows that this inequality holds for all $t \in S_{i+1}(B)$. Summing over $t \in S_{i+1}(B)$ gives

$$\sum_{t \in S_{i+1}(B)} C_{t,\mathsf{WR},i-1}(B) + C_{t,\mathsf{WC},i-1}(B) - 2C_{t,\mathsf{Z},i-1}(B) + 2C_{t,\mathsf{S},i-1}(B) + C_{t,i+1}(B) \geq 0,$$

and therefore, by the identities above,

$$h_{\mathsf{WR},i-1}(B)+h_{\mathsf{WC},i-1}(B)-2h_{\mathsf{Z},i-1}(B)+2h_{\mathsf{S},i-1}(B)+\frac{1}{4}(2i+2)ih_{i+1}(B)\geq 0.$$

Next, for $t\in S_{i+2}(B)$, we have the inequality

$$C_{t,\mathsf{Z},i}(B)-C_{t,\mathsf{S},i}(B)-C_{t,\mathsf{WR},i}(B)-C_{t,\mathsf{WC},i}(B)+\frac{1}{2}C_{t,i+1}(B)+\frac{1}{2}C_{t,i+2}(B)\geq 0.$$

Again, checking the 2^4 possible configurations of $K(t)\cap B$ shows that this inequality holds for all $t\in S_{i+2}(B)$. Summing over $t\in S_{i+2}(B)$ gives

$$\sum_{t\in S_{i+2}(B)} C_{t,\mathsf{Z},i}(B)-C_{t,\mathsf{S},i}(B)-C_{t,\mathsf{WR},i}(B)-C_{t,\mathsf{WC},i}(B)+\frac{1}{2}C_{t,i+1}(B)+\frac{1}{2}C_{t,i+2}(B)\geq 0,$$

and therefore, by the identities above,

$$h_{\mathsf{Z},i}(B)-h_{\mathsf{S},i}(B)-h_{\mathsf{WR},i}(B)-h_{\mathsf{WC},i}(B)+\frac{1}{4}(2i+2)(n-i-1)h_{i+1}(B)+\frac{1}{4}(i+1)(i+2)h_{i+2}(B)\geq 0.$$

□

Now we are ready to complete the proof of Theorem 4.10.

Proof of Theorem 4.10. Adding inequality (4.3) and inequality (4.4) to the two inequalities $\frac{1}{4}(n-i)(3n+i-3)h_i(B)\geq 0$ and $h_{\mathsf{Z},i-2}(B)-h_{\mathsf{S},i-2}(B)\geq 0$, we get

$$\begin{aligned}C_i(B)=h_{\mathsf{Z},i-2}(B)&-h_{\mathsf{S},i-2}(B)+\frac{1}{4}(n-i)(3n+i-3)h_i(B)\\&+h_{\mathsf{WR},i-1}(B)+h_{\mathsf{WC},i-1}(B)-2h_{\mathsf{Z},i-1}(B)+2h_{\mathsf{S},i-1}(B)\\&+h_{\mathsf{Z},i}(B)-h_{\mathsf{S},i}(B)-h_{\mathsf{WR},i}(B)-h_{\mathsf{WC},i}(B)\\&+\frac{1}{4}(2i+2)(n-1)h_{i+1}(B)+\frac{1}{4}(i+1)(i+2)h_{i+2}(B)\geq 0.\end{aligned}$$

□

5 Final Remarks

We are able to obtain alternating formulas for the coefficient of x through the orbit-counting arguments of Section 3, and in principle this approach extends further. Dividing the sum in Equation (3.2) into parts corresponding to the finite set of bi-colored graphs and ranks which can appear, and analyzing the orbit counts $\mathcal{O}_D(G,q)$ will yield an alternating formula even for higher coefficients, but there are two obstructions. The first is that the number of terms becomes extremely large, even for the quadratic coefficient. Particularly for q-hit numbers, our expansion in Lemma 4.8 simplified considerably because the defect can only be 0 or 1. The corresponding term in the quadratic coefficient of $H_d(B,x+1)$ has many more cases. The second, more important obstruction is that $\mathcal{O}_D(G,q)$ may become non-polynomial as soon as G contains at least 7 row and 7 column vertices, and so one will need to analyze all polynomials agreeing with the orbit count on unbounded subsets of $\mathbb{Z}$. Understanding these orbit counts appears to be very hard for general boards. On the other hand, by Theorem 3.12, we know $M_d(B,q)$ modulo $(q-1)^6$ is an integer polynomial in q, and therefore there must be a polynomial formula for $M_d(B,x+1) \pmod{x^6}$ in terms of generalized rook numbers $\mathcal{B}(B,G)$ for a finite set of graphs G, even if it is too large to write down. The same is true of $H_d(B,x+1)$ modulo x^6. Regarding positivity, the inequalities (4.3) and (4.4) have analogues for generalized hit numbers of other graphs, and we are curious to see if those analogues could show non-negativity of higher coefficients than in Theorem 4.10. Of course, showing the coefficients are non-negative leads us to wonder if there is some nice combinatorial object that the coefficients count. In Example 4.9, we see an infinite family of cases where the q-hit number and its first coefficient both have nice counting interpretations. For instance, $H_0(\overline{B},q)$ counts arbitrary matrices on the board $\overline{B}$, and $C_0(\overline{B})$ counts cells of the board $\overline{B}$. We have so far been unable to find this interpretation in our alternating formula.

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