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Explicit Formulas and Combinatorial Interpretation of Triangular Arrays via Weighted Paths

Aubert Voalaza Mahavily Romuald and Randrianirina Benjamin

Department of Mathematics, Faculty of Science, Laboratoire de Mathématiques et Applications de l'Université de Fianarantsoa (LaMAF), University of Fianarantsoa, Postal Address 601, Fianarantsoa, Madagascar
Email: aubert@aims.ac.za, benjamin.randrianirina@univ-fianarantsoa.mg

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ABSTRACT: Using the lattice paths in $\mathbb{N}\times\mathbb{N}$, we derive a general formula for sequences $\big(T(n,k)\big)$ satisfying the recurrence relation of the form:

$$T(n,k)=a_{n,k}T(n-1,k)+b_{n,k}T(n-1,k-1).$$

We apply this result to the case where $a_{n,k}=a_0+a_1k+a_2n$ and $b_{n,k}=b_0+b_1k+b_2n$. This leads to explicit expressions for $T(n,k)$, with simpler formulas arising in the case $b_2=0$, as well as in the fully general case, using Faà di Bruno's type expression. In particular, we analyze the case $b_{n,k}=1$, which frequently occurs in enumerative combinatorics. Applications include explicit formulas for the r-Eulerian numbers. We also express the case $b_{n,k}=1$, using a transition matrix. We apply our results to several sequences.

Keywords: Combinatorial interpretation; Triangular recurrence; r-Eulerian numbers; Weighted paths
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1. Introduction

In this paper, we investigate a certain sequence of numbers that satisfies a triangular recurrence of the form

$$T(n,k)=(a_0+a_1k+a_2n)T(n-1,k)+(b_0+b_1k+b_2n)T(n-1,k-1). \tag{1}$$

In their book [4], Graham et al. study binomial coefficients, Stirling numbers, and Eulerian numbers, and propose a generalization problem of the form (1). We call the resulting sequences GKP numbers. In combinatorics, several approaches have been developed to study these numbers. One of the earliest results is due to Neuwirth [13] (see also Spivey [17]), who obtained an explicit formula for the case $b_2=0$ using the Galton triangle. Spivey [17] further investigated several cases using finite differences. Analytical approaches have also appeared in various works, including those of Théorêt [18], Wilf [19], and Barbero [1].

Recall that a weighted set is a pair (E,v), where E is a finite nonempty set and v is a map from E to a ring of formal power series with coefficients in $\mathbb{C}$. We are generally interested in $|E|_v=\sum_{x\in E}v(x)$. Using weighted paths in the $\mathbb{N}\times\mathbb{N}$, we derive a general formula for the sequence $\big(T(n,k)\big)$ of the form:

$$T(n,k)=a_{n,k}T(n-1,k)+b_{n,k}T(n-1,k-1).$$

We apply this result to the special case of GKP numbers. Thus, the aim of this work is to study such numbers as the total weights of certain paths, with initial conditions usually taken as

$$T(0,0)=1,\quad T(n,k)=0 \text{ if } n<\max(k,0). \tag{2}$$

We remain in the setting where $a_{n,k}=a_0+a_1k+a_2n$ and $b_{n,k}=b_0+b_1k+b_2n$, with $a_{n,k}\neq 0$ and $b_{n,k}\neq 0$. In particular, we study the case $b_{n,k}=1$, which we denote mostly by $\big(F(n,k)\big)$. In this way, we obtain a unified approach to generalized Stirling numbers (see Hsu and Shiue [7], Maier [8]), the r-Whitney numbers (see [11]), the r-Lah numbers (see Niúl and Rácz [14]) and many other sequences in combinatorics.

However, generalizations of Eulerian numbers where $b_{n,k}\neq 1$ are also important examples when discussing sequences satisfying (1). We provide an explicit formula for $T(n,k)$ with $a_{n,k}=a_0+a_1k+a_2n$ and $b_{n,k}=b_0+b_1k$, and apply it to the r-Eulerian numbers.

Our Contributions and Paper Outline

We prove the main results, which provide explicit formulas for $T(n,k)$ of Faà di Bruno's type expressions:

Theorem (See Theorem 2.3)**.** *Let* $\big(T(n,k)\big)$ *be a sequence satisfying* (1) *with* $a_{n,k} = a_1k + a_0$*,* $b_{n,k} = b_2n + b_1k + b_0$*, and the usual initial conditions* (2)*. Then*

$$T(n,k) = \sum_{\{c_0+c_1+\cdots+c_k=n-k, 0\leq c_i\leq n\}} a_0^{c_0}\prod_{i=1}^{k}\big(a_0+a_1i\big)^{c_i}\Big(b_0+(b_1+b_2)i+b_2(\sum_{j=0}^{j=i-1}c_j)\Big).$$

This sum ranges over all sequences $(c_0, c_1, \cdots, c_k)$ such that, for all i, c_i is non-negative integer, and $c_0 + c_1 + \cdots + c_k = n-k$.

More generally, we can express $T(n,k)$ as a sum over all increasing $(n-k)$-tuples $p = (p_i)_{i=1}^{n-k}$ with entries in $[n]$, i.e., $1 \leq p_1 < \cdots < p_{n-k} \leq n$. It is given by the following theorem.

Theorem (See Theorem 2.5)**.** *Let* $\big(T(n,k)\big)$ *be a sequence satisfying* (1) *and the usual initial conditions* (2)*. Then*

$$T(n,k) = \sum_{\{1\leq p_1<p_2<\cdots<p_{n-k}\leq n\}}\prod_{i=1}^{n-k}\Big((a_2+a_1)p_i - a_1i + a_0\Big)\prod_{i=1}^{k}\Big(b_0+(b_1+b_2)i+b_2i+b_2\sharp\{l : p_l - l < i\}\Big).$$

In section 2, we will interpret $T(n,k)$ as the total weight of a set of paths and derive an explicit formula. It includes Theorem 2.1, Corollary 2.2 and the two above theorems. We also apply these results to sequences defined by triangular recurrence.

Section 3 contains brief remarks that we can also represent the sequences as transition matrices. We prove that the case where $b_{n,k} = 1$, it is a transition matrix in the vector space $\mathbb{R}[x]$.

In section 4, we present a simpler form of the expression obtained in Theorem 4.5 for the case $b_2 = 0$, that is, when $a_{n,k} = a_2n + a_1k + a_0$ and $b_{n,k} = b_1k + b_0$.

2. Weighted paths in $\mathbb{N}\times\mathbb{N}$ associated with sequences of numbers satisfy a triangular recurrence

We define $\mathcal{R}_{n,k}$ as the set of paths starting from $(0,0)$ to (n,k) in the lattice $\mathbb{N}\times\mathbb{N}$, using only East steps **E** $\big((i-1,j)\to(i,j)\big)$ and North-East steps **NE** $\big((i-1,j-1)\to(i,j)\big)$. Any element $\pi\in\mathcal{R}_{n,k}$ is a path of length n and height k.
Let $a = (a_{i,j})_{i,j\in\mathbb{N}}$ and $b = (b_{i,j})_{i,j\in\mathbb{N}}$ be two sequences of real numbers. We assign a weight to each East step **E** $(i-1,j)\to(i,j)$ by $a_{i,j}$, and to each North-East step **NE** $(i-1,j-1)\to(i,j)$ by $b_{i,j}$ so one can define a valuation (weighting) $\omega_{a,b} : \mathcal{R}_{n,k}\to\mathbb{R}$, such that the weight of a path $\pi = (p_1, p_2, \cdots, p_n)\in\mathcal{R}_{n,k}$ is $\omega_{a,b}(\pi) :=$ the product of the weights of each p_i. We denote by $T_{a,b}(n,k) = |\mathcal{R}_{n,k}|_{\omega_{a,b}} = \sum_{\pi\in\mathcal{R}_{n,k}}\omega_{\mathrm{a,b}}(\pi)$ the total weight of $\mathcal{R}_{n,k}$, and by $\mathbb{T}_{a,b} = (T_{a,b}(n,k))_{n,k\in\mathbb{N}}$ the associated infinite matrix.

Lemma 2.1. *The matrix* $\mathbb{T}_{a,b}$ *satisfies the following recurrence relation.*

$$T_{a,b}(n,k) = a_{n,k}\mathrm{T}_{\mathrm{a,b}}(n-1,k) + b_{n,k}\mathrm{T}_{\mathrm{a,b}}(n-1,k-1).$$

It also satisfies the usual initial condition (2)*. Conversely, any relation of this form, for* $a_{n,k}, b_{n,k}\in\mathbb{R}$*, may be interpreted as a weighting of* $\mathcal{R}_{n,k}$*.*

Proof. Indeed, any path from $(0,0)$ to (n,k) is either a path from $(0,0)$ to $(n-1,k)$ followed by an East step, or a path from $(0,0)$ to $(n-1,k-1)$ followed by an East step. ☐

Example 2.1. Figure 1 represents a path from $(0,0)$ to $(6,3)$ whose weight is $a_{1,0}b_{2,1}b_{3,2}a_{4,2}b_{5,3}a_{6,3}$.

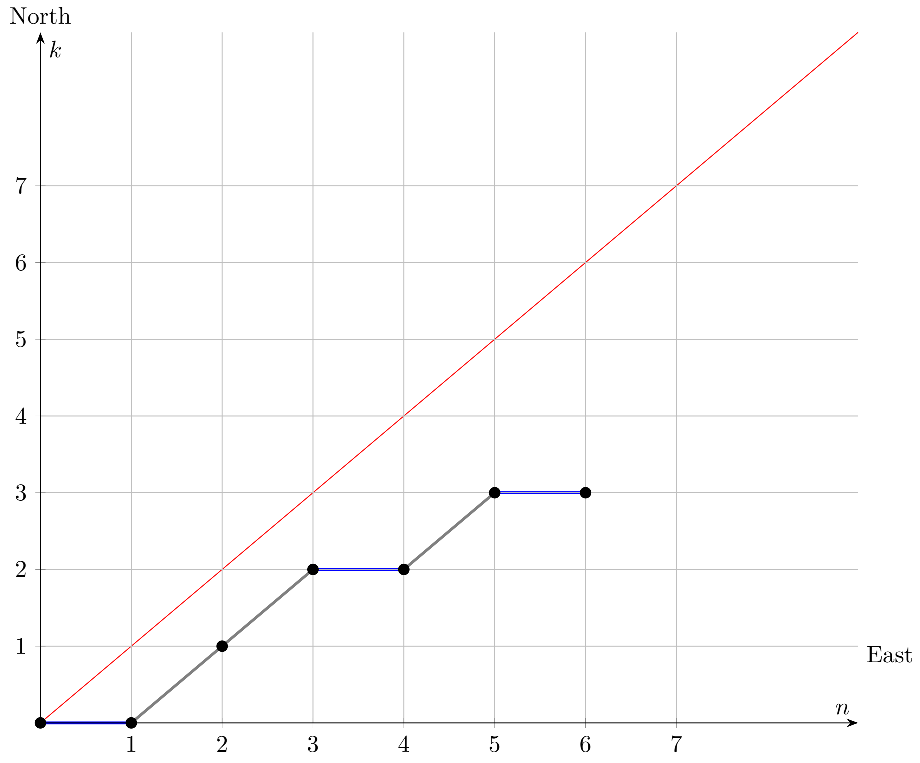

Figure 1: A path from $(0,0)$ to $(6,3)$

Let $\mathcal{C}^{\uparrow}(k,n)$ be the set of strictly increasing functions from $[k]$ to $[n]$. We identify $\pi \in \mathcal{R}_{n,k}$ with $\sigma \in \mathcal{C}^{\uparrow}(k,n)$ by defining $\sigma(i) = j$ if the i-th **NE** step is $(j-1, l-1) \to (j,l)$. Given $\sigma \in \mathcal{C}^{\uparrow}(k,n)$, one can associate the increasing list $\tilde{\sigma} = (\tilde{\sigma}_i)_{1\le i\le n-k}$ of the elements of $[n] \setminus \sigma([k])$, and $\sharp\mathcal{C}^{\uparrow}(k,n) = \binom{n}{k}$.

Example 2.2. The path in Figure 1 is identified with the map σ such that $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 5$, and $\tilde{\sigma} = (1,4,6)$.

Theorem 2.1. *Let $a = (a_{n,k})_{n,k\in\mathbb{N}}$, $b = (b_{n,k})_{n,k\in\mathbb{N}}$ and $\big(T(n,k)\big)$ be three sequences such that T satisfies the recurrence relation $T(n,k) = a_{n,k}T(n-1,k) + b_{n,k}T(n-1,k-1)$, with the initial conditions* (2). *Then*

$$T(n,k) = \sum_{\sigma\in\mathcal{C}^{\uparrow}(k,n)} \prod_{i=1}^{n-k} \left(a_{\tilde{\sigma}_i,\tilde{\sigma}_i-i}\right) \prod_{i=1}^{k} \left(b_{\sigma(i),i}\right). \tag{3}$$

Proof. Recall our notation: $\omega_{a,b} : \mathcal{R}_{n,k} \to \mathbb{R}$, the weighting, assigns the weight $a_{i,j}$ to each East step **E** $(i-1,j) \to (i,j)$ and the weight $b_{i,j}$ to each North-East step **NE** $(i-1,j-1) \to (i,j)$. By Lemma 2.1, $T(n,k)$ is the total weight of $\mathcal{R}_{n,k}$. So $T(n,k) = \sum_{\pi\in\mathcal{R}_{n,k}} \omega_{\mathrm{a,b}}(\pi)$.

It is straightforward to see that the $i-$th East step is weighted by $a_{\tilde{\sigma}_i,\tilde{\sigma}_i-i}$ and the $i-$th North-East step is weighted by $b_{\sigma(i),i}$. Thus, the result follows. ☐

The following case is very common in combinatorics.

Corollary 2.1. *Let $\big(F(n,k)\big)$ be a sequence satisfying* (1) *with $a_{n,k} = a_2n + a_1k + a_0$, $b_{n,k} = 1$, that is,*

$$F(n,k) = F(n-1,k-1) + (a_2n + a_1k + a_0)F(n-1,k),$$

and the usual initial conditions (2). *Then*

$$F(n,k) = \sum_{\{1\le p_1<p_2<\cdots<p_{n-k}\le n\}} \prod_{i=1}^{n-k} \Big((a_2+a_1)p_i - a_1i + a_0\Big).$$

Proof. Since $b_{n,k} = 1$, eq. (3) becomes

$$F(n,k) = \sum_{\sigma\in\mathcal{C}^{\uparrow}(k,n)} \prod_{i=1}^{n-k} \left(a_{\tilde{\sigma}_i,\tilde{\sigma}_i-i}\right).$$

But here $a_{n,k} = a_2 n + a_1 k + a_0$, so

$$a_{\tilde{\sigma}_i, \tilde{\sigma}_i - i} = a_2 \tilde{\sigma}_i + a_1(\tilde{\sigma}_i - i) + a_0 = (a_2 + a_1)\tilde{\sigma}_i - a_1 i + a_0.$$

Using the definition of $\mathcal{C}^{\uparrow}(k, n)$, we conclude the proof. $\square$

Lemma 2.2. *If* $b_{n,k} = b_0 + b_1 k$, *then*

$$T(n, k) = (b_0 + b_1 |\, b_1\,)^{(\overline{k})} F(n, k),$$

where $(b_0 + b_1 |\, b_1\,)^{(\overline{k})} = (b_0 + b_1)(b_0 + 2b_1) \cdots (b_0 + kb_1)$.
In particular, if $b_1 = 0$ *then:*

$$T(n, k) = b_0^k F(n, k).$$

And if $b_0 = 0$, *then:*

$$T(n, k) = b_1^k k! F(n, k).$$

Proof. Suppose that $b_{n,k} = b_0 + b_1 k$. Since each element $\pi \in \mathcal{R}_{n,k}$ has k **NE** steps, and the weight of each of these steps is independent of n, the step at height 1 is weighted by $b_0 + b_1$, the step at height 2 by $b_0 + 2b_1$, $\cdots$, and the step at height k by $b_0 + kb_1$. Hence, the total weight of the **NE** steps is

$$(b_0 + b_1)(b_0 + 2b_1) \cdots (b_0 + kb_1).$$

This means that, for the same $a_{n,k}$, if the weight of $\pi \in \mathcal{R}_{n,k}$ is $\omega(\pi)$ in the case $b_{n,k} = 1$, and $\omega'(\pi)$ in the case $b_{n,k} = b_0 + b_1 k$, then

$$\omega'(\pi) = (b_0 + b_1)(b_0 + 2b_1) \cdots (b_0 + kb_1)\, \omega(\pi).$$

Summing over all $\pi \in \mathcal{R}_{n,k}$, we obtain the desired equality. $\square$

Combining Corollary 2.1 and Lemma 2.2, we obtain

Corollary 2.2. *Let* $\big(T(n, k)\big)$ *be a sequence satisfying* (1) *with* $a_{n,k} = a_2 n + a_1 k + a_0$, $b_{n,k} = b_1 k + b_0$, *and the usual initial conditions* (2). *Then*

$$T(n, k) = (b_0 + b_1 |\, b_1\,)^{(\overline{k})} \sum_{\{1 \le p_1 < p_2 < \cdots < p_{n-k} \le n\}} \prod_{i=1}^{n-k} \Big((a_2 + a_1) p_i - a_1 i + a_0 \Big).$$

Applications 2.2. We now present some applications of Corollary 2.1, including the generalized Stirling numbers and a generalization of the r-Lah numbers. First, setting $a_2 = -\alpha$, $a_1 = \beta$, $a_0 = \alpha + r$, we obtain the generalized Stirling Numbers $S_{\alpha,\beta,r}$ defined by Hsu and Shiue in [7], satisfying the recurrence relation ([7] equation (7))

$$S_{\alpha,\beta,r}(n, k) = S_{\alpha,\beta,r}(n-1, k-1) + (-\alpha(n-1) + \beta k + r) S_{\alpha,\beta,r}(n-1, k). \tag{4}$$

Applying this corollary, the generalized Stirling Numbers can be rewritten as

$$S_{\alpha,\beta,r}(n, k) = \sum_{\{1 \le p_1 < p_2 < \cdots < p_{n-k} \le n\}} \prod_{i=1}^{n-k} \Big((\beta - \alpha) p_i - \beta i + \alpha + r \Big).$$

Note that $S_{m,0,r}(n; k)$ is the r-Whitney number of first kind, defined by Mezö in [11], and $S_{0,m,r}(n; k)$ is the r-Whitney number of second kind. Also, in [15], Randrianirina introduces several generalizations of the Lah numbers. One of them is the sequence defined by the following recurrence relation

$$\lambda_{m,r}(n, k) = \lambda_{m,r}(n, k-1) + (m(n-1) + k + r) \lambda_{m,r}(n, k). \tag{5}$$

In this case $a_0 = -m + r$; $a_1 = 1$ and $a_2 = m$. Then

$$\lambda_{m,r}(n, k) = \sum_{\{1 \le p_1 < p_2 < \cdots < p_{n-k} \le n\}} \prod_{i=1}^{n-k} \Big((m+1) p_i - i - m + r \Big).$$

$\big(\lambda_{m,r}(n, k)\big)$ is the matrix associated with the $\mathbb{L}$-species (see [16]) $z\mathbb{E}_\beta \circ \mathcal{Q}_m^+) \cdot \mathcal{Q}_m^r = T_{Z=z, X=x} \mathcal{G}(ZX^r)$, $\mathcal{G}$ being the combinatorial differential operator associated with the grammar $G = \{z \to \mu z x^{m+1}, x \to x^{m+1}\}$, $\mathbb{E}_\beta$ is the species of sets weithed by $w(A) = \mu^{|A|}$, and $\mathcal{Q}_m$ is the $\mathbb{L}$-species of m-Stirling permutation, weighted by $w'(\sigma) = x^{mn+1}$ if $\sigma \in \mathcal{Q}[n]$. Another one is the number $\Lambda_{m,r,s}(n, k)$, satisfying the relation

$$\Lambda_{m,r,s}(n, k) = \Lambda_{m,r,s}(n-1, k-1) + (m(n+k) + r + s) \Lambda_{m,r,s}(n-1, k). \tag{6}$$

As $a_2 = a_1 = m$, $a_0 = r + s$, we have

$$\Lambda_{m,r,s}(n,k) = \sum_{\{1\le p_1<p_2<\cdots<p_{n-k}\le n\}} \prod_{i=1}^{n-k} \Big(m(2p_i - i) + r + s \Big).$$

$\big(\Lambda_{m,r,s}(n,k)\big)$ is the matrix associated with the $\mathbb{L}$-species $\mathbb{F} = T_{V=v,U=u,X=x}\mathcal{G}(VU^sX^r)$, where $\mathcal{G}$ is the combinatorial differential operator associated with the grammar $G = \{v \to \mu v u^m x^m, u \to u x^m, x \to x^{m+1}\}$. Note that $\Lambda_{m,r,r}(n,k)$ is the r-Whitney-Lah numbers defined by Gyimesi and Nyúl in [5], and $\Lambda_{1,r,r}(n,k)$ is the r-Lah numbers defined by Nyúl and Rácz [14]). We can consider also the case where $a_1 = a_2 = m$ and $a_0 = mr$ corresponding to $\big(\tau_{m,r}(n,k)\big)$ the sequence of recurrence

$$\tau_{m,r}(n,k) = \tau_{m,r}(n-1,k-1) + m(n+k+r)\tau_{m,r}(n-1,k), \tag{7}$$

which Randrianirina mention in [15] as associated with $(z\mathbb{E}_\beta \circ (\mathbb{L}_\gamma)^+) \cdot (\mathbb{L}_{\gamma'})^r = T_{Z=z,X=x}\mathcal{G}(ZX^r)$, where $\mathcal{G}$ is the combinatorial differential operator associated with the grammar $G = \{v \to \mu z x^r, x \to m x^2\}$. Again from Corollary 2.1, we have

$$\tau_{m,r}(n,k) = \sum_{\{1\le p_1<p_2<\cdots<p_{n-k}\le n\}} \prod_{i=1}^{n-k} \Big(m(2p_i - i + r) \Big).$$

Now, we try to make the formula (3) less heavy, namely, to reduce the simultaneous use of σ and $\tilde{\sigma}$. To do that, let us introduce a useful tool. We aim to write down the precise identification between $\sigma \in \mathcal{C}^{\uparrow}(k,n)$ and $\tilde{\sigma} \in \mathcal{C}^{\uparrow}(n-k,n)$ used in Theorem 2.1. The aim is to transfer (identify) both to weak compositions of $n-k$ into $k+1$ parts, given by the set $\mathcal{WC}(n-k,k+1)$. We often identify any function σ from $[k]$ to $[n]$ by a sequence $(\sigma_i)_{i=1}^k \subseteq [n]$, so the following forms.

$$\begin{aligned}
\mathcal{C}^{\uparrow}(k,n) =& \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k) : 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le n\};\\
\mathcal{C}^{\uparrow}(n-k,n) =& \{\beta = (\beta_1, \beta_2, \cdots, \beta_{n-k}) : 1 \le \beta_1 < \beta_2 < \cdots < \beta_{n-k} \le n\} \text{ and}\\
\mathcal{WC}(n-k,k+1) =& \{\underline{c} = (c_0, c_1, \cdots, c_k) : \sum_{i=0}^{k} c_i = n-k, \quad \forall i, c_i \in \{0,1,\cdots,n\}\}.
\end{aligned}$$

We equip $\mathcal{C}^{\uparrow}(n-k,n)$ and $\mathcal{WC}(n-k,k+1)$ with the weights ε and δ defined by

$$\varepsilon(\beta) := \prod_{j=1}^{n-k} \big(a_0 + (\beta_j - j)a_1\big) \text{ and } \eta(\underline{c}) := \prod_{i=0}^{k} \big(a_0 + i a_1\big)^{c_i}.$$

Lemma 2.3. *Let us define the correspondences*

- $\psi : \mathcal{WC}(n-k,k+1) \to \mathcal{C}^{\uparrow}(k,n)$, $\underline{c} \mapsto \alpha$ *(with* $\alpha_i = i + \sum_{j=0}^{i-1} c_j$*) and*
- $\phi : \mathcal{C}^{\uparrow}(n-k,n) \to \mathcal{WC}(n-k,k+1)$, $\beta \mapsto \underline{c}$ *(with* $c_i := \sharp\{j : \beta_j - j = i\}$*).*

Then, the following statements hold.

1. *ψ and ϕ are well-defined bijections. Moreover $\big(\mathcal{C}^{\uparrow}(n-k,n), \varepsilon\big)$ and $\big(\mathcal{WC}(n-k,k+1), \eta\big)$ are isomorphic as weighted sets via ϕ.*
2. *The map $\psi \circ \phi : \mathcal{C}^{\uparrow}(n-k,n) \to \mathcal{C}^{\uparrow}(k,n)$ coincides with the identification between $\mathcal{C}^{\uparrow}(n-k,n)$ and $\mathcal{C}^{\uparrow}(k,n)$ used in formula (3) of Theorem 2.1. More precisely,*

$$\psi \circ \phi(\tilde{\sigma}) = \sigma.$$

Proof. First, let us prove Item 1. For ψ, well-definedness and injectivity are clear. Since the sets are finite and have the same cardinality, injectivity implies bijectivity. They have the same cardinality because we have noticed it earlier $\sharp\mathcal{C}^{\uparrow}(k,n) = \binom{n}{k}$ and the known fact about weak composition of $n-k$ in $k+1$ parts.

For ϕ, the map is well defined because each c_i lies in $\{0,1,\cdots,n\}$ and the sum $\sum_{i=0}^{k} c_i$ is precisely the length of β, namely $|\beta| = n-k$. Its inverse is defined recursively as follows:

- for $i = 0$: for all l with $1 \le l \le c_0$, set $\beta_l = l$;
- for $i = 1, \ldots, k$: for all l with

  $$\sum_{j=0}^{i-1} c_j < l \le \sum_{j=0}^{i} c_j,$$

  set $\beta_l = l + i$.

By construction, $\beta_1 \geq 1$ and $\beta_l < \beta_{l+1} \leq (n-k)+k = n$, so $\beta \in \mathcal{C}^{\uparrow}(n-k,n)$. It is also clear that this is the inverse of our correspondence.

To show that it is a morphism (preserves weights), it suffices to use the definition of c_i, which counts the indices j such that $\beta_j - j = i$. Now, we have done with the first part and we move to Item 2.

We recall that σ is the unique increasing sequence in $[n]$ complementary to $\widetilde{\sigma}$ in the sense that for any $i \in [k]$ and $m \in [n-k]$, $\sigma_i \neq \widetilde{\sigma}_m$. So we need to prove that for any $\beta \in \mathcal{C}^{\uparrow}(n-k,n)$, $\alpha = \psi \circ \phi(\beta)$ is the unique element in $\mathcal{C}^{\uparrow}(k,n)$ that is complementary to β. By definition of ψ and ϕ,

$$(\psi \circ \phi(\beta))_i = \alpha_i = i + \sharp C i, \qquad C_i := \{l \in [n-k] : \beta_l - l < i\}.$$

Clearly $C_i \subseteq C_{i+1}$, hence α is strictly increasing. Moreover, since $\sharp C_i \leq n-k$ (as $C_i \subseteq [n-k]$), we recover that $\alpha \in A$. What we need to prove here is then

$$\forall i \in [k],\ m \in [n-k] : \quad \alpha_i \neq \beta_m.$$

Assume that there exist $i \in [k]$ and $m \in [n-k]$ such that $\alpha_i = \beta_m$. Case one: $C_i = \emptyset$. Then $\beta_m = \alpha_i = i$. So $\beta_m - m < i$, and $m \in C_i$, a contradiction with $C_i = \emptyset$. Case two: $C_i \neq \emptyset$. In this case, we observe that if $l+1 \in C_i$, then $\beta_l - l \leq \beta_{l+1} - (l+1) < i, \quad l \in C_i$. This implies that $C_i = [\sharp C_i]$. As a subcase, if $m \leq \sharp C_i$, then $m \in C_i$. Hence

$$i + \sharp C_i - m = \alpha_i - m = \beta_m - m < i \implies \sharp C_i < m,$$

which is a contradiction. In the other subcase, if $m > \sharp C_i$, then $m \notin C_i$, so

$$i + \sharp C_i - m = \alpha_i - m = \beta_m - m \geq i \implies m \leq \sharp C_i,$$

which is again a contradiction. □

This lemma yields one of the main results of this paper. We obtain the following theorem.

Theorem 2.3. *Let* $\big(T(n,k)\big)$ *be a sequence satisfying* (1) *with* $a_{n,k} = a_1 k + a_0$, $b_{n,k} = b_2 n + b_1 k + b_0$, *and the usual initial conditions* (2). *Then*

$$T(n,k) = \sum_{\{c_0+c_1+\cdots+c_k = n-k, 0 \leq c_i \leq n\}} a_0^{c_0} \prod_{i=1}^{k} \big(a_0 + a_1 i\big)^{c_i} \Big(b_0 + (b_1+b_2)i + b_2(\sum_{j=0}^{j=i-1} c_j)\Big).$$

Proof. First, with $a_{n,k} = a_1 k + a_0$ and $b_{n,k} = b_2 n + b_1 k + b_0$, Theorem 2.1 becomes

$$T(n,k) = \sum_{\tilde{\sigma} \in \mathcal{C}^{\uparrow}(n-k,n)} \prod_{i=1}^{n-k} \big(a_0 + a_1(\tilde{\sigma}_i - i)\big) \prod_{i=1}^{k} \big(b_0 + i b_1 + b_2 \sigma_i\big).$$

Next, we use Item 2 of Lemma 2.3, together with definition of $\varepsilon(\tilde{\sigma})$, to obtain

$$T(n,k) = \sum_{\tilde{\sigma} \in \mathcal{C}^{\uparrow}(n-k,n)} \varepsilon(\tilde{\sigma}) \prod_{i=1}^{k} \big(b_0 + i b_1 + b_2(\psi \circ \phi(\tilde{\sigma}))_i\big).$$

Applying Lemma 2.3, Item 1 to the last equality, we obtain

$$\begin{aligned} T(n,k) &= \sum_{\tilde{\sigma} \in \mathcal{C}^{\uparrow}(n-k,n)} \eta\big(\phi(\tilde{\sigma})\big) \prod_{i=1}^{k} \big(b_0 + i b_1 + b_2(\psi \circ \phi(\tilde{\sigma}))_i\big) \\ &= \sum_{\underline{c} \in C} \eta(\underline{c}) \prod_{i=1}^{k} \big(b_0 + i b_1 + b_2(\psi(\underline{c}))_i\big) \\ &= \sum_{\underline{c} \in C} \prod_{i=0}^{k} \big(a_0 + i a_1\big)^{c_i} \prod_{i=1}^{k} \big(b_0 + i b_1 + b_2\big(i + \sum_{j=0}^{i-1} c_j\big)\big). \end{aligned}$$

So we obtain the desired conclusion. □

Applications 2.4. We can use this theorem to derive Faà di Bruno-type expressions for the mth-order Eulerian numbers and the r-Whitney–Eulerian numbers. First, recall that an m-Stirling permutation is a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{mn}$ of the multiset $1^m 2^m \cdots n^m$ such that for all i, j, k, if $i < j < k$ and $\sigma_i = \sigma_k$ then $\sigma_j \geq \sigma_i$.

In [6], Tian-Xiao He defines the mth order Eulerian numbers of order as the number $T^{(r)}(n,k)$ of m-Stirling permutations having exactly k descents, and proves that these numbers satisfy the following recurrence relation

$$T^{(r)}(n,k) = \big(rn-k+(1-r)\big)T^{(r)}(n-1,k-1)+(k+1)T^{(r)}(n-1,k),$$

with $T^{(r)}(n,0)=1$ for n≥ 1 and $T^{(r)}(n,k)=0$ if $n\leq k$ or $k<0$. We apply Theorem 2.3 to get

$$T^{(r)}(n,k) = \sum_{\{c_0+c_1+\cdots+c_k=n-k, 0\leq c_i\leq n\}} \prod_{i=1}^{k} (1+i)^{c_i}\Big((r-1)(i-1)+r\sum_{j=0}^{j=i-1} c_j\Big).$$

Now, let us consider the r-Whitney-eulerian numbers defined by Mezö and Ramirez in [12] (see also Toufik Mansour and al. in [10]), satisfying the recurrence relation

$$A_{m,r}(n,k) = (mk+r)A_{m,r}(n-1,k)+(mn-mk+m-r)A_{m,r}(n-1,k-1).$$

From Theorem 2.3, the r-Whitney-Eulerian numbers can rewritten as

$$A_{m,r}(n,k) = \sum_{\{c_0+c_1+\cdots+c_k=n-k, 0\leq c_i\leq n\}} r^{c_0}\prod_{i=1}^{k} (r+mi)^{c_i}\Big(m-r+m(\sum_{j=0}^{j=i-1} c_j)\Big).$$

In a similar way, one can obtain the general case, as follows.

Theorem 2.5. *Let* $\big(T(n,k)\big)$ *be a sequence satisfying* (1) *and the usual initial conditions* (2). *Then*

$$T(n,k) = \sum_{\{1\leq p_1<p_2<\cdots<p_{n-k}\leq n\}} \prod_{i=1}^{n-k}\Big((a_2+a_1)p_i-a_1 i+a_0\Big)\prod_{i=1}^{k}\Big(b_0+(b_1+b_2)i+b_2\big(\sum_{j=0}^{i-1}\sharp\{l:p_l-l=j\}\big)\Big).$$

That is

$$T(n,k) = \sum_{\{1\leq p_1<p_2<\cdots<p_{n-k}\leq n\}} \prod_{i=1}^{n-k}\Big((a_2+a_1)p_i-a_1 i+a_0\Big)\prod_{i=1}^{k}\Big(b_0+(b_1+b_2)i+b_2\sharp\{l:p_l-l<i\}\ \Big).$$

Proof. From Lemma 2.3, item 2, the bijection that identifies $\tilde{\sigma}=(\tilde{\sigma}_i)_{1\leq i\leq n-k}\in\mathcal{C}^{\uparrow}(n-k,n)$ with $\sigma=(\sigma(i))_{1\leq i\leq k}\in\mathcal{C}^{\uparrow}(k,n)$ can be described as follows

$$\sigma(i) = i+\sharp\{l\in[n-k]:\beta_l-l<i\} = i+\sum_{j=0}^{i-1}\sharp\{l:\tilde{\sigma}_l-l=j\}.$$

Now, we can rewrite Theorem 2.1 for $a_{n,k}=a_2n+a_1k+a_0$ and $b_{n,k}=b_2n+b_1k+b_0$ as

$$T(n,k) = \sum_{\tilde{\sigma}\in\mathcal{C}^{\uparrow}(n-k,n)}\prod_{i=1}^{n-k}\big(a_2\tilde{\sigma}_i+a_1(\tilde{\sigma}_i-i)+a_0\big)\prod_{i=1}^{k}\Big(b_2\big(i+\sum_{j=0}^{i-1}\sharp\{l:\tilde{\sigma}_l-l=j\}\big)+b_1 i+b_0\Big).$$

Hence the result. ☐

In a similar way, we obtain another formula for the generalized r-Whitney numbers of the second kind. These numbers $W_{p,q}(n,k)=W_{p,q}(n,k;r,m)$, introduced in [9], satisfy the recurrence relation

$$W_{p,q}(n,k) = W_{p,q}(n-1,k-1)+\big([r]_p+m[k]_q\big)W_{p,q}(n-1,k),$$

where $[a]_b = 1+b+b^2+\cdots+b^{a-1}$.

Proposition 2.1. *If* $q\neq 1$, *then*

$$W_{p,q}(n,k) = \sum_{\{c_0+c_1+\cdots+c_k=n-k, 0\leq c_i\leq n\}}\prod_{i=0}^{k}\left(A+\frac{m}{q-1}q^i\right)^{c_i},$$

where $A=[r]_p+\frac{m}{1-q}$.

If $q=1$, *then*

$$W_{p,1}(n,k) = \sum_{\{c_0+c_1+\cdots+c_k=n-k, 0\leq c_i\leq n\}}\prod_{i=0}^{k}\left([r]_p+mi\right)^{c_i}.$$

Proof. In this case, $a_{n,k} = a_k = [r]_p + m[k]_q$ and $b_{n,k} = 1$, so Theorem 2.1 implies

$$W_{p,q}(n,k) = \sum_{\sigma\in\mathcal{C}^{\uparrow}(k,n)} \prod_{i=1}^{n-k} \big(a_{\tilde{\sigma}_i,\tilde{\sigma}_i - i}\big) = \sum_{\beta\in\mathcal{C}^{\uparrow}(n-k,n)} \prod_{i=1}^{n-k} \big([r]_p + m[\beta_i - i]_q\big).$$

We now generalize the definitions of the weights ε and η on $\mathcal{C}^{\uparrow}(n-k,n)$ and $\mathcal{WC}^{\uparrow}(n-k,k+1)$, respectively, by

$$\varepsilon(\beta) := \prod_{j=1}^{n-k} f(\beta_j - j) \qquad \text{and} \qquad \eta(\underline{c}) := \prod_{i=0}^{k} f(i)^{c_i}.$$

Since $c_i := \sharp\{j : \beta_j - j = i\}$ under the transformation ϕ, the map ϕ is still an isomorphism of weighted sets. Then, by taking $f(i) = a_i = [r]_p + m[i]_q$, we obtain

$$W_{p,q}(n,k) = \sum_{\{c_0+c_1+\cdots+c_k=n-k, 0\le c_i\le n\}} \prod_{i=0}^{k} \big([r]_p + m[i]_q\big)^{c_i}.$$

For $q = 1$, the result is now clear and for $q \neq 1$, we use $[i]_q = \frac{1-q^i}{1-q}$ to get the conclusion. ☐

Now, let's consider the generating series $T_n(x) = \sum_{k=0}^{n} T(n,k)x^k$ and $\overline{T}_k(y) = \sum_{n\ge k} T(n,k)y^n$.

Theorem 2.6. *The generating series* $T_n(y)$ *and* $\overline{T}_k(y)$ *satisfy*

$$T_n(x) = \big((b_2n + b_1 + b_0)x + a_2n + a_0\big)T_{n-1}(x) + (b_1x^2 + a_1x)T'_{n-1}(x)$$

and

$$(1 - (a_2 + a_1k + a_0)y)\overline{T}_k(y) = a_2y^2\overline{T}'_k(y) + b_2y^2\overline{T}'_{k-1}(y) + (b_2 + b_1k + b_0)y\overline{T}_{k-1}(y).$$

Proof. We set $\mathcal{R}_{n,*} := \bigcup_{k\ge0} \mathcal{R}_{n,k}$, $\mathcal{R}_{*,k} := \bigcup_{n\ge k} \mathcal{R}_{n,k}$ and if $\pi \in \mathcal{R}_{n,k}$ then $l(\pi) = n$ and $h(\pi) = k$. We have

$$\begin{aligned} T_n(x) &= \sum_{\pi\in\mathcal{R}_{n,*}} \omega_{a,b}(\pi)x^{h(\pi)} = \sum_{\pi\in\mathcal{R}_{n-1,*}} (b_2n + b_1(h(\pi)+1) + b_0)\omega_{a,b}(\pi)x^{h(\pi)+1} \\ &\quad + \sum_{\pi\in\mathcal{R}_{n-1,*}} (a_2n + a_1h(\pi) + a_0)\omega_{a,b}(\pi)x^{h(\pi)} \\ &= (b_2n + b_1 + b_0)x \sum_{\pi\in\mathcal{R}_{n-1,*}} \omega_{a,b}(\pi)x^{h(\pi)} + b_1x^2 \sum_{\pi\in\mathcal{R}_{n-1,*}} h(\pi)\omega_{a,b}(\pi)x^{h(\pi)-1} \\ &\quad + (a_2n + a_0) \sum_{\pi\in\mathcal{R}_{n-1,*}} \omega_{a,b}(\pi)x^{h(\pi)} + a_1x \sum_{\pi\in\mathcal{R}_{n-1,*}} h(\pi)\omega_{a,b}(\pi)x^{h(\pi)-1} \\ &= \big((b_2n + b_1 + b_0)x + a_2n + a_0\big)T_{n-1}(x) + (b_1x^2 + a_1x)T'_{n-1}(x). \end{aligned}$$

On the other hand

$$\begin{aligned} \overline{T}_k(y) &= \sum_{n\ge k} T(n,k)y^n = \sum_{\pi\in\mathcal{R}_{*,k}} \omega_{a,b}(\pi)y^{l(\pi)} \\ &= \sum_{\pi\in\mathcal{R}_{*,k}} (a_2(l(\pi)+1) + a_1k + a_0)\omega_{a,b}(\pi)y^{l(\pi)+1} \\ &\qquad\qquad + \sum_{\pi\in\mathcal{R}_{*,k-1}} (b_2(l(\pi)+1) + b_1k + b_0)\omega_{a,b}(\pi)y^{l(\pi)+1} \\ &= a_2y^2 \sum_{\pi\in\mathcal{R}_{*,k}} l(\pi)\omega_{a,b}(\pi)y^{l(\pi)-1} + (a_2 + a_1k + a_0)y \sum_{\pi\in\mathcal{R}_{*,k}} \omega_{a,b}(\pi)y^{l(\pi)} \\ &\quad + b_2y^2 \sum_{\pi\in\mathcal{R}_{*,k-1}} l(\pi)\omega_{a,b}(\pi)y^{l(\pi)-1} + (b_2 + b_1k + b_0)y \sum_{\pi\in\mathcal{R}_{*,k-1}} \omega_{a,b}(\pi)y^{l(\pi)} \\ &= a_2y^2\overline{T}'_k(y) + (a_2 + a_1k + a_0)y\overline{T}_k(y) + b_2y^2\overline{T}'_{k-1}(y) + (b_2 + b_1k + b_0)y\overline{T}_{k-1}(y). \end{aligned}$$

☐

Noticing $T_0(x) = 1$ and if $a_2 = 0$, $\overline{T}_0(y) = \frac{1}{1-a_0y}$, we get directly the following.

Corollary 2.3. *If* $b_1 = a_1 = 0$, *then*

$$T_n(x) = \prod_{i=1}^{n} \big((b_2 i + b_1 + b_0)x + a_2 i + a_0\big).$$

And if $a_2 = b_2 = 0$, *then*

$$\overline{T}_k(y) = \frac{1}{1-a_0y} \prod_{i=1}^{k} \frac{(b_1k + b_0)y}{(1-(a_1k+a_0)y)}.$$

These identities include the following results. The r-Whitney numbers of first kind $w_{m,r}(n,k)$, corresponding to $a_2 = -m$, $a_1 = 0$, $a_0 = m - r$, $b_2 = b_1 = 0$ and $b_0 = 1$, satisfy

$$T_n(x) = \sum_{k=0}^{n} w_{m,r}(n,k)x^k = \prod_{i=0}^{n-1}(x - r - mi).$$

The r-Whitney numbers of second kind $W_{m,r}(n,k)$, corresponding to $a_2 = 0$, $a_1 = m$, $a_0 = r$, $b_2 = b_1 = 0$ and $b_0 = 1$, satisfy

$$\overline{T}_k(y) = \sum_{n\geq 0} W_{m,r}(n,k)y^n = \prod_{i=0}^{k} \frac{y}{(1-(mk+r)y)}.$$

These results are seen in several combinatorial pieces of literature ([15], [2]).

3. Defining $\big(F(n,k)\big)_{k,n}$ as a change-of-basis matrix

We denote

$$(y|a)^{(\overline{m})} = \prod_{i=0}^{m-1}(y+ia) \quad \text{and} \quad (y|a)^{(\underline{m})} = (y|-a)^{(\overline{m})} = \prod_{i=0}^{m-1}(y-ia).$$

Recall that the polynomial sequences

$$\Big((x|a_2)^{(\overline{n})}\Big)_{n\in\mathbb{N}} \quad \text{and} \quad \Big((x-a_0-a_2|a_1)^{(\underline{n})}\Big)_{n\in\mathbb{N}}$$

form bases of the vector space $\mathbb{R}[x]$. In this section, we interpret the triangular array $\big(F(n,k)\big)_{k,n\in\mathbb{N}}$ appearing in Theorem 3.1 as the associated change-of-basis matrix between these two bases. Since it is a change-of-basis matrix, it is therefore invertible, and we also compute its inverse matrix $\big(F(n,k)\big)^{-1}_{n,k\in\mathbb{N}}$.

Theorem 3.1. *For any sequences* $\big((F(n,k))\big)$, *that satisfies the usual initial conditions*

$$F(0,0) = 1, \quad F(n,k) = 0 \text{ if } n < \max(k,0),$$

the following facts are equivalent.

$$\forall n,k: \quad F(n,k) = F(n-1,k-1) + (a_2n + a_1k + a_0)F(n-1,k).$$

and

$$\forall x: \quad (x|a_2)^{(\overline{n})} = \sum_{k=0}^{n} F(n,k)(x-a_0-a_2|a_1)^{(\underline{k})}, \tag{8}$$

where $(x|a_2)^{(\overline{n})} = \prod_{i=0}^{i=n-1}(x+ia_2)$ *and* $(x-a_0-a_2|a_1)^{(\underline{k})} = \prod_{i=0}^{k-1}(x-a_0-a_2-ia_1)$.

Proof. Let $\mathcal{R}_n = \bigcup_{k\geq 0}\mathcal{R}_{n,k}$ be the set of paths of length n in $\mathbb{N}\times\mathbb{N}$. We want to interpret the right-hand side of eq. (8) as the total weight of $\mathcal{R}_n$, denoted by $|\mathcal{R}_n|_v$, for a suitable weighting v. Since $F(n,k) = \sum_{\pi\in\mathcal{R}_{n,k}}\omega(\pi)$, where $\omega = \omega_{a,b}$ is the weighting associated with the sequence $\big(F(n,k)\big)$ (see the beginning of section 2), the desired weighting is defined as

$$v(\pi) := \omega(\pi)(x-a_0-a_2|a_1)^{(\underline{h(\pi)})},$$

where $h(\pi)$ denotes the height of π. So we can write

$$|\mathcal{R}_n|_v = \sum_{\pi\in\mathcal{R}_n} v(\pi) = \sum_{k\geq 0}\sum_{\pi\in\mathcal{R}_{n,k}} \omega(\pi)(x-a_0-a_2|a_1)^{(\underline{k})} = \sum_{k\geq 0} F(n,k)(x-a_0-a_2|a_1)^{(\underline{k})}. \tag{9}$$

Since a path π of length n can be obtained from a path u of length $n-1$ by adding either an **E** step or a **NE** step, the set $\mathcal{R}_n$ is the disjoint union

$$\mathcal{R}_n = \mathcal{D}_n \sqcup \mathcal{E}_n,$$

where $\mathcal{D}_n$ consists of the paths obtained from $\mathcal{R}_{n-1,k}$ and $\mathcal{E}_n$ consists of the paths obtained from $\mathcal{R}_{n-1,k-1}$. For $\pi \in \mathcal{D}_n$, $\pi = u + p_n$ with $u \in \mathcal{R}_{n-1,k}$ and p_n an **E** step of weight $a_2 n + a_1 h(\pi) + a_0$. As $h(\pi) = h(u)$, we have $\omega(\pi) = (a_2 n + a_1 h(u) + a_0)\omega(u)$ and

$$v(\pi) = \omega(\pi)(x - a_0 - a_2|a_1)^{(\underline{h(\pi)})} = \big(a_2 n + a_1 h(u) + a_0\big)\omega(u)(x - a_0 - a_2|a_1)^{(\underline{h(u)})}.$$

For $\pi \in \mathcal{E}_n$, $\pi = u + p_n$ with $u \in \mathcal{R}_{n-1,k-1}$ and p_n a **NE** step of weight 1. So $\omega(\pi) = \omega(u)$ and

$$v(\pi) = \omega(\pi)(x - a_0 - a_2|a_1)^{(\underline{h(\pi)})} = \omega(u)\big(x - a_0 - a_2 - h(u)a_1\big)(x - a_0 - a_2|a_1)^{(\underline{h(u)})}.$$

Noticing that each of $\mathcal{D}_n$ and $\mathcal{E}_n$ can be identified with $\mathcal{R}_{n-1}$, we combine the above identities to obtain

$$\begin{aligned}
|\mathcal{R}_n|_v &= \sum_{\pi \in \mathcal{R}_n} v(\pi) = \sum_{\pi \in \mathcal{D}_n} v(\pi) + \sum_{\pi \in \mathcal{E}_n} v(\pi) \\
&= \sum_{u \in \mathcal{R}_{n-1}} (a_2 n + a_1 h(u) + a_0)\omega(u)(x - a_0 - a_2|a_1)^{(\underline{h(u)})} \\
&\quad + \sum_{u \in \mathcal{R}_{n-1}} \omega(u)\big(x - a_0 - a_2 - h(u)a_1\big)(x - a_0 - a_2|a_1)^{(\underline{h(u)})} \\
&= \sum_{u \in \mathcal{R}_{n-1}} \Big((a_2 n + a_1 h(u) + a_0) + (x - a_0 - a_2 - h(u)a_1)\Big)\omega(u)(x - a_0 - a_2|a_1)^{(\underline{h(u)})} \\
&= (x + a_2(n-1))|\mathcal{R}_{n-1}|_v.
\end{aligned}$$

Since $|\mathcal{R}_0|_v = 1$, $|\mathcal{R}_n|_v = (x|a_2)^{(\overline{n})}$. Together with eq. (9), this concludes the proof. ☐

Remark 3.1. A $\big((x|a_2)^{(\overline{n})}\big)_{n\in\mathbb{N}}$ and $\big((x - a_0 - a_2|a_1)^{(\underline{k})}\big)_{k\in\mathbb{N}}$ form a basis of $\mathbb{R}[x]$, Theorem 3.1 says that $\mathbb{F}^T = \big(F(n,k)\big)_{k,n\in\mathbb{N}}$ is a transition matrix from $\big((x|a_2)^{(\overline{n})}\big)_{n\in\mathbb{N}}$ to $\big((x - a_0 - a_2|a_1)^{(\underline{k})}\big)_{k\in\mathbb{N}}$. The matrix $\mathbb{F}^T$ is therefore invertible[*]. Its inverse is given in Theorem 3.3.

Applications 3.2. Equation (8) generalizes several classical results. We now present some representative examples. By Theorem 3.1, the generalized Stirling Numbers in eq. (4) $S_{\alpha,\beta,r}$ verify

$$(x|-\alpha)^{(\overline{n})} = \sum_{k=0}^{n} S_{\alpha,\beta,r}(n,k)(x - r|\beta)^{(\underline{k})}.$$

It is the equation (1) of [7] (seen also in [8]). In particular, the r-Whitney numbers of first kind, verify

$$\prod_{i=0}^{n-1}(x - mi) = \sum_{k=0}^{n} w_{m,r}(n,k)(x + r)^k.$$

By replacing x with mx in this equation, we obtain equation (2) of [11]. Also the r-Whitney number of second kind, verify

$$x^n = \sum_{k=0}^{n} W_{m,r}(n,k)\prod_{i=0}^{n-k}(x - r - im).$$

By replacing x with $mx + r$ in this equation, we obtain equation (1) of [11]. Now, let's consider the sequence $(\lambda_{m,r}(n,k)$ defined in Applications 2.2 by eq. (5). Theorem 3.1 implies

$$(x|m)^{(\overline{n})} = \sum_{k=0}^{n} \lambda_{m,r}(n, k(x - r|1)^{(\underline{k})}.$$

Similarly, for $\big(\Lambda_{m,r,s}(n,k)\big)$ and $\big(\tau_{m,r}(n,k)\big)$ defined by eqs. (6) and (7),

$$(x|m)^{(\overline{n})} = \sum_{k=0}^{n} \Lambda_{m,r,s}(n,k)(x - r - s - m|m)^{(\underline{k})}, \quad (x|m)^{(\overline{n})} = \sum_{k=0}^{n} \tau_{m,r}(n,k)(x - mr - m|m)^{(\underline{k})}.$$

---

[*] This can even be obtained quickly since $F(n,n) = 1$ for all n, because $b_{n,k} = 1$.

Theorem 3.3. *The inverse of* $\mathbb{F}^T$, *is the matrix* $\mathbb{H}^T = \big((H(n,k))\big)_{k,n}$ *satisfying*

$$\forall n,k: \quad H(n,k) = H(n-1,k-1) + (-a_1 n - a_2 k + a_1 - a_0 - a_2)H(n-1,k).$$

Proof. Consider the inverse $\big(H(n,k)\big)$ of $\big(F(n,k)\big)$. Using eq. (8), we have

$$\forall x: \quad (x-a_0-a_2|a_1)^{(\underline{n})} = \sum_{k=0}^{n} H(n,k)(x|a_2)^{(\overline{k})}.$$

We can change $y = x - a_0 - a_2$ and use the fact $(z|c)^{(\overline{l})} = (z|-c)^{(\underline{l})}$ in this equation to get

$$\forall y: \quad (y|-a_1)^{(\overline{n})} = \sum_{k=0}^{n} H(n,k)(y+a_0+a_2|-a_2)^{(\underline{k})}.$$

Set $a_2' := -a_1$; $\quad a_1' := -a_2$; $\quad a_0' := a_1 - a_0 - a_2$, and plug in this equation. Then

$$\forall y: \quad (y|a_2')^{(\overline{n})} = \sum_{k=0}^{n} H(n,k)(y-a_0'-a_2'|a_1')^{(\underline{k})}.$$

So applying Theorem 3.1, we get

$$\forall n,k: \quad H(n,k) = H(n-1,k-1) + (a_2' n + a_1' k + a_0')H(n-1,k).$$

By substituting $a_2' = -a_1$, $a_1' = -a_2$, and $a_0' = a_1 - a_0 - a_2$, we obtain the desired conclusion. ☐

4. Closed formula for $F(n,k)$, and for $T(n,k)$ if $b_{n,k} = b_1 k + b_0$

We now propose to prove a closed formula for the sequences $F(n,k)$, corresponding to the case where $b_{n,k} = 1$ and $a_{n,k} = a_2 n + a_1 k + a_0$. The case of the sequences $\big(T(n,k)\big)$, with $b_{n,k} = b_0 + b_1 k$, then follows as a direct consequence.

4.1 The case where $a_1 \neq 0$

For this, we need the following two lemmas.

Lemma 4.1. *Let* $P_n(x) := \prod_{l=1}^{l=n}(a_0 + a_1 x + a_2 l)$. *If* $a_1 \neq 0$, *we have*

$$F(n,k) = \frac{1}{a_1^k}[x^{(\underline{k})}]P_n(x).$$

Proof. Note that $P_n(x) = (a_0 + a_1 x + a_2 n)P_{n-1}(x)$ and $x \cdot x^{(\underline{k})} = x^{(\underline{k+1})} + k \cdot x^{(\underline{k})}$. These facts imply that $G(n,k) := [x^{(\underline{k})}]P_n(x)$ satisfies

$$G(0,0) = 1; \quad G(n,k) = (a_0 + a_1 k + a_2 n)G(n-1,k) + a_1 G(n-1,k-1).$$

Using Lemma 2.2, we have $G(n,k) = a_1^k F(n,k)$. ☐

Lemma 4.2. *For any polynomial* $P(x) = \sum_{i=0}^{i=n} \alpha_i x^i$,

$$[x^{(\underline{k})}]P(x) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}P(j).$$

Proof. We know that the Stirling numbers of the second kind $\big(S(i,k)\big)$ satisfy $x^i = \sum_{k\geq 0} S(i,k)x^{(\underline{k})}$ (see Comtet [3]). Then

$$\begin{aligned} P(x) = {} & \textstyle\sum_{i=0}^{n} \alpha_i \sum_{k=0}^{n} S(i,k)x^{(\underline{k})} = \sum_{k=0}^{n}\Big(\sum_{i=0}^{n}\alpha_i S(i,k)\Big)x^{(\underline{k})} \\ & = \textstyle\sum_{k=0}^{n}\Big(\sum_{i=0}^{n}\alpha_i \frac{1}{k!}\sum_{j=0}^{j=k}(-1)^{k-j}\binom{k}{j}j^i\Big)x^{(\underline{k})} \\ & = \textstyle\sum_{k=0}^{n}\Big(\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\sum_{i=0}^{n}\alpha_i j^i)\Big)x^{(\underline{k})}. \end{aligned}$$

So the conclusion. ☐

Applying Lemma 4.2 to the polynomial $P_n(x)$ of Lemma 4.1, we obtain:

Theorem 4.1. *Let $\big(F(n,k)\big)$ be a sequence satisfying* (1) *with $a_{n,k} = a_2 n + a_1 k + a_0$, $b_{n,k} = 1$, and the usual initial conditions* (2).

If $a_1 \neq 0$, then

$$F(n,k) = \frac{1}{a_1^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=1}^{n} (a_0 + a_1 j + l a_2). \tag{10}$$

Applications 4.2. This theorem allows us to obtain other expressions for the sequences given in Applications 2.2. Firstly, the generalized Stirling numbers verify

$$S_{\alpha,\beta,r}(n,k) = \frac{1}{\beta^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=0}^{n-1} (r + \beta j - l\alpha)$$

In particular, the r-Whitney numbers $W_{m,r}(n,k)$ of second kind satisfy

$$W_{m,r}(n,k) = \frac{1}{m^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (r + mj)^n.$$

In this case, we obtain a result of Mezö and Ramirez in [12] (corollary 8). Also, the numbers $\lambda_{m,r}(n,k)$ and $\Lambda_{m,r,s}(n,k)$ of eqs. (5) and (6) can be written explicitly as

$$\lambda_{m,r}(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=1}^{n} (r + j + ml), \quad \Lambda_{m,r,s}(n,k) = \frac{1}{m^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=j}^{n+j} (r + s + ml).$$

In particular, the r-Whitney-Lah numbers $\Lambda_{m,r,r}$ verify

$$\Lambda_{m,r,r}(n,k) = \frac{1}{m^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=j}^{n+j} (2r + ml)$$

and the r-Lah numbers $\Lambda_{1,r,r}(n,k)$ verify

$$\Lambda_{1,r,r}(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=j}^{n+j} (2r + l).$$

The numbers $\tau_{m,r}(n,k)$ of eq. (7) verify

$$\tau_{m,r}(n,k) = \frac{m^{n-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{l=j+1}^{j+n} (r + l) = \frac{m^{n-k}}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(r+j+n)!}{(r+j)!}.$$

4.2 The case where $a_1 = 0$

Let us now consider the case $a_1 = 0$ and $a_2 \neq 0$. Define $Q_n(x) = \prod_{i=1}^{n}(x + a_0 + a_2 i)$. On the one hand, we have: $Q_n(x) = \sum_{k=0}^{n} \big(\sum_{A \subset [n], |A| = n-k} \prod_{i \in A} (a_0 + i a_2) \big) x^k$. On the other hand, we have $\prod_{i=0}^{n-1}(u+i) = \sum_{j=0}^{n} c(n,j) u^j$, where $c(n,j)$ is the unsigned Stirling numbers of the first kind (see Comtet [3]). Then:

$$\begin{aligned} Q_n(x) &= \prod_{i=0}^{n-1} (x + (a_0 + a_2) + a_2 i) = a_2^n \prod_{i=0}^{n-1} \Big(\frac{x + a_0 + a_2}{a_2} + i\Big) = a_2^n \sum_{j=0}^{n} c(n,j) \Big(\frac{x + a_0 + a_2}{a_2}\Big)^j \\ &= a_2^n \sum_{j=0}^{n} c(n,j) \Big(\sum_{k=0}^{j} \binom{j}{k} a_2^{-j} x^k (a_0 + a_2)^{j-k}\Big) = \sum_{k=0}^{n} \Big(\sum_{j=k}^{n} c(n,j) \binom{j}{k} a_2^{n-j} (a_0 + a_2)^{j-k}\Big) x^k. \end{aligned}$$

So

$$[x^k] Q_n(x) = \sum_{A \subset [n], |A| = n-k} \prod_{i \in A} (a_0 + i a_2) = \sum_{j=k}^{n} c(n,j) \binom{j}{k} a_2^{n-j} (a_0 + a_2)^{j-k}.$$

Let $\big(F(n,k)\big)$ be a sequence satisfying (1) with $a_{n,k} = a_2 n + a_0$, $b_{n,k} = 1$, and the usual initial conditions (2). The weight of each East step $\mathbf{E}$ $(i-1, j) \to (i, j)$ by $a_{i,j}$ is independent of j and is equal to $a_0 + a_2 i$ when it is the i-th step. Thus, if $\pi \in \mathcal{R}_{n,k}$ and A_π denotes the set of positions of the East steps of π, then its weight is given by $\omega(\pi)=\prod_{i\in A_\pi}(a_0+a_2 i)$. Since each path $\pi\in\mathcal{R}_{n,k}$ can be identified with the set A_π and $|A_\pi|=n-k$, it follows that

$$F(n,k)=\sum_{\pi\in\mathcal{R}_{n,k}}\omega(\pi)=\sum_{A\subset[n],|A|=n-k}\prod_{i\in A}(a_0+ia_2).$$

Hence, we obtain the following theorem.

Theorem 4.3. *If $\big(F(n,k)\big)_{n,k\in\mathbb{N}}$ be a sequence satisfying* (1) *with $a_{n,k}=a_2n+a_0$, $b_{n,k}=1$, and the usual initial conditions* (2)*, then*

$$\sum_{k=0}^{n}F(n,k)x^k=\prod_{i=1}^{n}(x+a_0+a_2i).$$

That is

$$F(n,k)=\sum_{j=k}^{n}c(n,j)\binom{j}{k}a_2^{n-j}(a_0+a_2)^{j-k}.$$

Applications 4.4. Now, the r-Whitney numbers of the first kind provide one example of this case for $a_2=-m$ and $a_0=m-r$. Thus, Theorem 4.3 yields

$$w_{m,r}(n,k)=\sum_{j=k}^{n}(-1)^{n-k}m^{n-j}c(n,j)\binom{j}{k}r^{j-k}.$$

We obtain a result of Randrianirina in [15] (equation (37)). More generally, let $\big(F(n,k)\big)$ be a sequence satisfying

$$F(n,k)=F(n-1,k-1)+(a_0+a_1k)F(n-1,k).$$

Equation (10) says that

$$F(n,k)=\frac{1}{a_1^k\,k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(a_0+a_1j)^n.$$

Theorem 3.3 says that the inverse of the matrix $\mathbb{F}^T=(F(n,k))_{k,n}$, is the matrix $\mathbb{H}^T=\big(H(n,k)\big)_{k,n}$ satisfying

$$H(n,k)=H(n-1,k-1)+(-a_1n+a_1-a_0)H(n-1,k).$$

We have

$$H(n,k)=\sum_{j=k}^{n}c(n,j)\binom{j}{k}(-1)^{n-k}a_1^{n-j}a_0^{j-k}.$$

Corollary 4.1. *Let $\big(F(n,k)\big)$ be a sequence satisfying* (1) *with $a_{n,k}=a_2n+a_1k+a_0$, $b_{n,k}=1$, and the usual initial conditions* (2)*. We have*

$$F(n,k)=\sum_{j=k}^{n}\left(\sum_{i=j}^{n}c(n,i)\binom{i}{j}a_2^{n-i}(a_2+a_0)^{i-j}\right)a_1^{j-k}S(j,k).$$

That is

$$F(n,k)=\sum_{i=k}^{n}\sum_{j=k}^{i}\binom{i}{j}c(n,i)S(j,k)a_2^{n-i}(a_2+a_0)^{i-j}a_1^{j-k}.$$

Proof. Let $\big(U(n,k)\big)$ and $\big(V(n,k)\big)$ be two triangular sequences satisfying the usual initial conditions (2), with

$$U(n,k)=(a_2n+a_0)U(n-1,k)+U(n-1,k-1),$$

and

$$V(n,k)=a_1kV(n-1,k)+V(n-1,k-1).$$

We have adopted the convention that $\mathbb{F}$, $\mathbb{U}$, and $\mathbb{V}$ denote respectively the infinite matrices associated with $\big(F(n,k)\big)$, $\big(U(n,k)\big)$, and $\big(V(n,k)\big)$. It is straightforward to see that $\mathbb{F}=\mathbb{U}\mathbb{V}$. On the other hand, the weights corresponding to $\big(V(n,k)\big)$ differ from those of $S(n,k)$ only by a factor of a_1^{n-k}, since $S(n,k)$ satisfies $S(n,k)=kS(n-1,k)+S(n-1,k-1)$. Hence, we have $V(n,k)=a_1^{n-k}S(n,k)$. Applying Theorem 4.3, we obtain the desired conclusion. ☐

4.3 The case where $a_{n,k} = a_2 n + a_1 k + a_0$ and $b_{n,k} = b_1 k + b_0$

Consider the sequence $(T(n,k))$ satisfying

$$T(n,k) = (a_2 n + a_1 k + a_0)T(n-1,k) + (b_1 k + b_0)T(n-1,k-1)$$

with initial condition $T(0,0) = 1$ and $T(n,k) = 0$ if $n > k$.

We combine Lemma 2.2 with Theorems 4.1 and 4.3 to obtain the following.

Theorem 4.5. *Let* $\big(T(n,k)\big)$ *be a sequence satisfying* (1) *with* $a_{n,k} = a_2 n + a_1 k + a_0$, $b_{n,k} = b_1 k + b_0$, *and the usual initial conditions* (2).

If $a_1 \neq 0$, *then*

$$T(n,k) = \frac{(b_0 + b_1|b_1)^{(\overline{k})}}{a_1^k\, k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \prod_{r=1}^{n} (a_0 + a_1 j + r a_2),$$

where $(b_0 + b_1|\, b_1\,)^{(\overline{k})} = (b_0 + b_1) \times (b_0 + 2b_1) \times \cdots \times (b_0 + kb_1)$.

If $a_1 = 0$, *then*

$$T(n,k) = (b_0 + b_1|\, b_1\,)^{(\overline{k})} \sum_{j=k}^{n} a_2^{n-j} c(n,j) \binom{j}{k} a_0^{j-k}. \tag{11}$$

One can check that eq. (11) coincides with that of Neurwirth [13]. But in general, we can get the results of Neurwirth using these theorems. This is stated in Spivey [17] (equation (2)) as follows, with minor corrections and slight adjustments.

Theorem 4.6. *Let* $\big(T(n,k)\big)$ *be a sequence satisfying* (1) *with* $a_{n,k} = a_2 n + a_1 k + a_0$, $b_{n,k} = b_1 k + b_0$, *and the usual initial conditions* (2). *Then*

$$T(n,k) = (b_0 + b_1|\, b_1\,)^{(\overline{k})} \sum_{i=k}^{n} \sum_{j=k}^{i} \binom{i}{j} c(n,i) S(j,k) a_2^{n-i} (a_2 + a_0)^{i-j} a_1^{j-k}.$$

This now follows directly from Lemma 2.2 and Corollary 4.1.

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