

On the Asymptotics and the Non-Holonomic Character of First Returns in the Standard Euclidean Lattice

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ABSTRACT: We give precise asymptotics to the number of first-time returning random walks in the standard orthogonal lattice in \mathbb{R}^d , and we prove that these numbers do not form a P -recursive sequence. In the process, the known asymptotics of the number of closed walks are obtained in an elementary way, by using a combinatorial and geometric multiplication principle together with the classical theory of Legendre polynomials. An important observation is the connection that we make between closed walks and a sequence that has been studied when analyzing both staircase polygons and abelian squares. By showing that the relevant generating functions are G -functions, we use a form of the Hadamard convolution to find their singularities in all dimensions. We use general properties to prove that the generating function of closed walks is not invertible as a G -function, which immediately implies that the generating function of the first time returning walks is not holonomic.

Keywords: Δ -analyticity; Asymptotics; G -functions; Hadamard convolution; Legendre polynomials; Picard-Fuchs equation; Random walk; P -recurrence; Return probability; Singularity analysis; Tauberian theory.

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1. Introduction

Random walks have been studied extensively for more than a century in connection to Brownian motion, diffusion, behavior of financial time series, and that of many other stochastic processes. In this paper, we study the classical random walks that occur in an Euclidean space \mathbb{R}^d , along the directions of a fixed orthogonal system of coordinates, with equally likely probability of motion in the direction of the axes. We present some interesting connections that random walks afford between combinatorics, theory of Legendre polynomials, G -functions, and singularity analysis.

The first part of our paper has combinatorial and geometric themes and presents two multiplication principles satisfied by the number of walks ending at various points in \mathbb{R}^d . Theorem 2.1 describes an inductive, on d , procedure to calculate the number of walks of length n that end up at an arbitrary point in \mathbb{R}^d . We denote by $A_{2n}^{(d)}$ the number of walks of length $2n$ in \mathbb{R}^d that return to the origin at time $t = 2n$. These are known in the literature as “closed walks” or “excursions.” Similarly, we denote by $B_{2n}^{(d)}$ the number of walks of length $2n$ in \mathbb{R}^d that return *for the first time* at the origin at time $t = 2n$. As part of our results in Section 2, we obtain in Corollary 2.1 the following recurrence relation:

$$A_{2n}^{(d+1)} = \binom{2n}{n} \cdot \sum_{k=0}^n \binom{n}{k}^2 \frac{A_{2k}^{(d)}}{\binom{2k}{k}}.$$

This leads to the natural definition of the sequence $x_n^{(d)} = A_{2n}^{(d)} / \binom{2n}{n}$, which has a combinatorial interpretation in the context of staircase polygons [15] and abelian squares [27], and shares similar combinatorial and analytic properties as $(A_{2n}^{(d)})_n$.

The main results of the paper are contained in Section 3. In Subsection 3.2, Theorem 3.1, we obtain the asymptotics of $x_n^{(d)}$ in the form

$$x_n^{(d)} = a_d \frac{d^{2n}}{(\pi n)^{(d-1)/2}} \left(1 + \frac{r_1(d)}{n} + \frac{r_2(d)}{n^2} + \cdots + \frac{r_m(d)}{n^m} + O_{d,m} \left(\frac{1}{n^{m+1}} \right) \right),$$

where $a_d = d^{d/2}/2^{d-1}$, and $r_k(d)$ are polynomials in d divisible by $d - 1$. From these, we obtain the asymptotics given in [21]:

$$A_{2n}^{(d)} = a_d \frac{(2d)^{2n}}{(\pi n)^{d/2}} \left(1 + \frac{a_1(d)}{n} + \frac{a_2(d)}{n^2} + \dots + \frac{a_m(d)}{n^m} + O_{d,m} \left(\frac{1}{n^{m+1}} \right) \right).$$

These results are proved in a relatively elementary way using the theory of Legendre polynomials and standard results about integrals and Taylor series of basic functions like exp and log, while using from Section 2 only the above recurrence and the initial conditions for $d = 1$.

In Subsection 3.3, we use the theory of holonomic functions and the Tauberian theory of singularity analysis [12] to obtain asymptotics for the first return paths in the form

$$B_{2n}^{(d)} = b_d \frac{(2d)^{2n}}{(\pi n)^{d/2}} \left(1 + \frac{b_1(d)}{n} + \frac{b_2(d)}{n^2} + \dots + \frac{b_m(d)}{n^m} + O_{d,m} \left(\frac{1}{n^{m+1}} \right) \right), \text{ for } d \geq 3 \text{ odd.}$$

Here, $b_d = a_d/m_d^2$, where, for all $d \geq 3$, m_d is the expected number of returns to the origin, plus one: $m_d = 1 + \sum_{n=1}^{\infty} A_{2n}^{(d)}/(2d)^{2n} < \infty$. The case d even additionally involves polynomials in $\log n$. See Theorem 3.2 for full details. Applying the theory of Hadamard convolution of analytic functions, we show that the generating functions F_d and A_d , of $(x_n)_n$ and $(A_{2n})_n$, satisfy Fuchsian ODEs. We determine the singularities of these generating functions and their type. For example, A_d has finite order singularities at $1/(2k)^2$, for all $1 \leq k \leq d$ and k having the same parity as d , and at ∞ . However, the generating function B_d is not holonomic for $d \geq 2$, but it is Δ -analytic, and the theory of singularity analysis still applies. See Theorems 3.3, 3.4, and 3.5. We also obtain, in Theorem 3.6, asymptotic expansions for the (normalized) generating functions in terms of the standard functions $T(w) = 1 - w$ and $L(w) = -\log(1 - w)$, $|w| < 1$.

In Subsection 3.4 we present the necessary proofs, which are based on the identity (see equation (17))

$$F_{d+1}(w) = \frac{1}{2\pi i} \int_{|t|=1/s_d} F_d(t)P(t,w)^{-1/2} dt, \text{ for } P(t,w) = w^2 - wt(2w + 2) + t^2(w - 1)^2.$$

(Here $s_d > d^2$.) In order to adapt to our case, the determination of singularities of the Hadamard convolution of two functions, we need to carefully analyze

$$t_1(w) = \frac{w}{(\sqrt{w} + 1)^2} \text{ and } t_2(w) = \frac{w}{(\sqrt{w} - 1)^2},$$

which are the zeros of $P(t, w)$. The proof of the above identity is based on properties of the Legendre polynomials in the complex domain, and especially on the relation

$$\sum_{m=0}^{\infty} \binom{n+m}{m}^2 w^m = (1-w)^{-n-1} P_n \left(\frac{1+w}{1-w} \right), \text{ for } |w| < 1,$$

which seems to have a long history, appearing as early as 1303 (in the form of the equality between the coefficients of w^k on both sides) in the works of the Chinese mathematician Shih-Chieh Chu ([34]).

In Section 4, we look at the generating functions F_d and A_d in the context of the theory of holonomic power series and, in particular, the theory of G -functions (which are holonomic power series with coefficients algebraic numbers that satisfy some growth conditions, that automatically happen when they are integers). We prove that F_d , for $d \geq 3$, and A_d , for $d \geq 2$, are not algebraic functions, with the latter implying that $1/A_d$ and B_d are not holonomic. Hence the sequence $(B_{2n}^{(d)})_n$ is not P -recursive, in contrast with the sequences $(x_n^{(d)})_n$ and $(A_{2n}^{(d)})_n$. While most of these follow using the asymptotics proved in Section 3 and general considerations, the proof of the fact that the sequence $(B_{2n}^{(d)})_n$ is not P -recursive requires the general theory of G -functions [1]. The paper ends with an enumeration of the P -recurrences and the ODEs satisfied by the sequences $(x_n^{(d)})_n$ and $(A_{2n}^{(d)})_n$, for $n = 1$ to 5.

This article is a drastically modified version of our initial preprint [8]. We thank the anonymous referee for making us aware of some already published results that our preprint rediscovered, or which went beyond our initial approach. We refer here especially to [20] and to the concept of holonomy [12, Appendix B.4]. We also thank Alin Bostan for a careful reading of our paper, many suggestions and simplifications of various arguments (see Theorem 4.1, part (iii)), as well as for showing us how the results in the paper connect to recent literature.

2. Simple random walks in Euclidean lattices

To set up the discussion, consider the Euclidean space \mathbb{R}^d , with a fixed system of coordinates. We will use the lattice with vertices the points with integer coefficients, and with each vertex being joined to the nearest

neighbors by an undirected segment. By a (*simple*) *random walk of length n* we understand the motion of a point which starts at the origin P_0 and “walks” along n segments of the lattice, at each vertex choosing with equal probability $1/(2d)$ which segment to follow next. Sometimes we refer to the number of traveled segments as *time*, because this set-up corresponds to a discrete-time and discrete-space walk, one step of the walk occurring with each increment of the time t . In this section, we obtain explicit formulas for the *number* of random walks of length n that end up at a particular point in \mathbb{R}^d , when the starting point is fixed. In other words, we determine the geometric distribution of these numbers. See Figure 1 below, as an exemplification in \mathbb{R}^2 . Such formulas have been obtained for some time by Domb [7], using probability distribution functions. Our approach is more geometric and leads to the recurrence formula (1), which allowed us to make connections with Legendre polynomials.

For simplicity, we assume that the walker starts at the origin of the coordinate system. In the case $d = 1$, it is easy to see that the number of walks of length n is given by the binomial coefficients. When $d = 2$, one obtains a diamond-shaped distribution. Figure 1 shows this distribution in the case $n = 3$ and $n = 4$. Each green (darker) dot represents the end of a path. The number next to a dot represents the number of paths that end at that green dot, among all possible walks of length n . The gray (lighter) dots are the ends of paths from the previous time step.

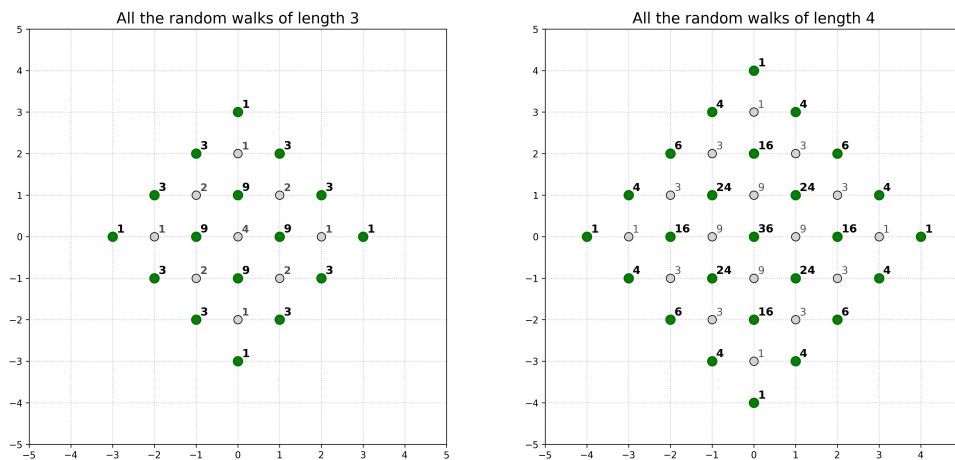


Figure 1: The diamond shaped distribution of random walks starring at $P_0 = (0, 0)$.

We refer to this pattern as the *first multiplication principle*:

Lemma 2.1. *The number of random walks of length n in the plane that start at the origin $(0, 0)$ and end at point (k, l) is given by $\binom{n}{\frac{n+k-l}{2}} \binom{n}{\frac{n+k+l}{2}}$. Here, $-n \leq k \leq n$, $-n \leq l \leq n$, $|k| + |l| \leq n$, and $n \equiv k + l \pmod{2}$.*

The proof is by induction on n . In its essence, Lemma 2.1 is due to the fact that a random walk in \mathbb{R}^2 can be viewed as the product of two independent one-dimensional random walks. This is clear if one projects the walk on the orthogonal system consisting of the diagonals of the standard system of coordinates. It is mentioned in the original paper of Pólya [24], in [9, Example 12], [12, Example VI.14], and, in the disguised form of generating functions, in [10, page 323]. See also [16].

The three-dimensional case is slightly more complicated. There is no equivalent of Lemma 2.1, perhaps not surprisingly, because the l^1 unit ball is not a cube; however, a different multiplication principle still holds. The end points of all possible random walks of a fixed length form octahedra, and a multiplication principle which holds on the *faces* of such octahedra turns out to hold for “layers” too (see below). Moreover, the result that we were able to first observe for \mathbb{R}^3 turns out to hold for any \mathbb{R}^d , with $d \geq 3$. See Theorem 2.1.

Before analyzing walks, we discuss a sequence of triangles of numbers $(\tau_n)_{n \geq 0}$ which might be of interest in its own right. This sequence is built inductively using the following procedure. At stage $n = 0$, τ_0 consists just of the number 1. At any subsequent stage $n + 1$, the numbers from τ_n duplicate one slot above, to the left, and to the right, and then they are removed from the triangle. The new triangle τ_{n+1} consists of the sums of the numbers that result at each location. One could interpret τ_n as the distributions of the number of random walks of length n in the plane, where the moves happen with equal probability equal to $1/3$ to the left, right,

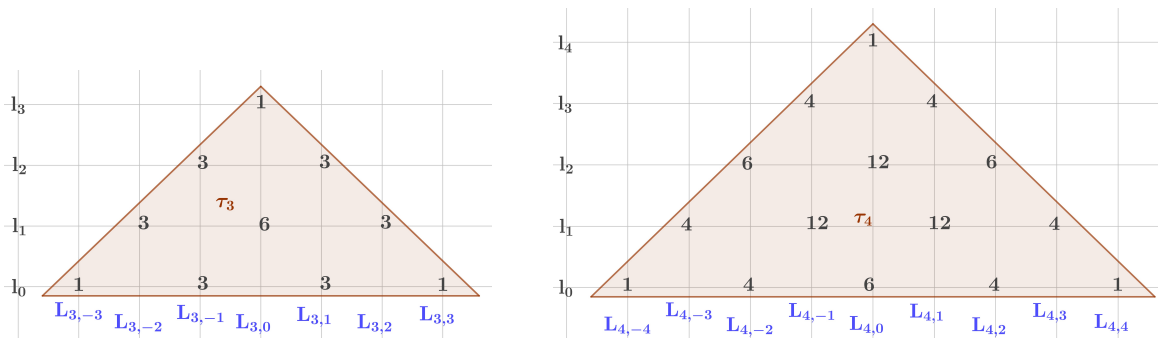


Figure 3: Obtaining the $L_{n,h}$'s as linear combinations of the 2D-distributions with coefficients from τ_n , in the case when $n = 3$ and $n = 4$.

Theorem 2.1. *With the above notations, for all $d \geq 1$, $n \geq 1$, and $0 \leq h \leq n$,*

$$L_{n,h}^{(d+1)} = \sum_{j=0}^{\lfloor \frac{n-h}{2} \rfloor} \binom{n}{n-h-2j} \binom{h+2j}{j} l_{n-h-2j}^{(d)} = \sum_{i=0 \text{ or } 1}^{n-h} \binom{n}{i} \binom{n-i}{\frac{n-h-i}{2}} l_i^{(d)}.$$

In the second sum, $i \equiv n - h \pmod{2}$.

Proof. Fix d . We will drop the superscript and keep in mind that any lower l is the entire distribution of walks in \mathbb{R}^d and any capital L is just a layer in \mathbb{R}^{d+1} . The proof is by induction on n . The case $n = 1$ is clear: $L_{1,0} = l_1 = \binom{1}{1} \binom{0}{0} l_1$ and $L_{1,1} = l_0 = \binom{1}{0} \binom{1}{0} l_0$.

The key observation is that the random walks of length $n + 1$ are obtained from the layers of the walks of length n , where each layer $L_{n,h}$ does only two “moves:” either it *expands* horizontally in the hyperplane $x_{d+1} = h$, as if the walks were d -dimensional, or it *duplicates* above and below, in the hyperplanes $x_{d+1} = h + 1$ and $x_{d+1} = h - 1$, respectively.

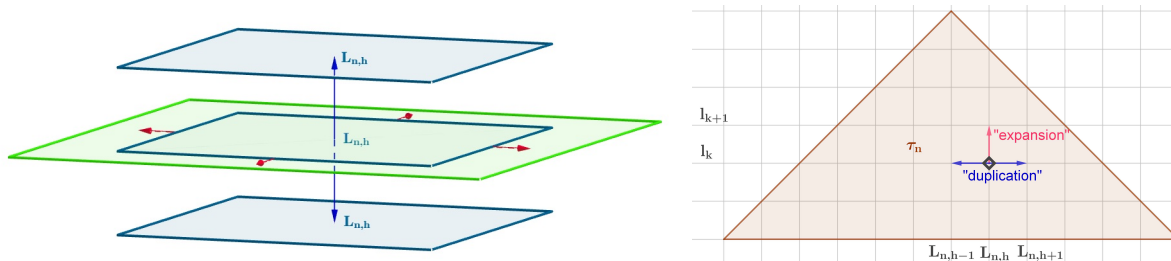


Figure 4: Expansion and duplication of layers, as the length of paths increases (left). Understanding the change in layers via τ_n (right).

If one interprets the j th column of τ_n as keeping track of what is going on in the hyperplane $x_{d+1} = j - (n + 1)$, then the observation that we have just made about the propagation of layers has the following interpretations. The horizontal expansion of a layer can be tracked by an upward addition (l_k becomes l_{k+1}). The vertical duplications translate into left and right duplications (the horizontal hyperplanes are changed). But this is exactly how τ_{n+1} is generated from τ_n . In other words, the sequence $(\tau_n)_n$ keeps track of the propagation of random walks! (See Figure 4.)

Finally, the formula given in the statement of the theorem simply uses the numbers from τ_n to compose $L_{n,h}$ out of the corresponding d -distributions l_i . □

For $m \geq 0$, we denote by $A_{2m}^{(d)}$ the number of walks of length $2m$ in \mathbb{R}^d that return to the origin at time $t = 2m$. In the literature, these are known as *closed walks* or *excursions*. We will use the first term in our paper. By convention $A_0^{(d)} = 1$.

Corollary 2.1. *For any $d \geq 1$,*

$$A_{2m}^{(d+1)} = \binom{2m}{m} \cdot \sum_{k=0}^m \binom{m}{k}^2 \frac{A_{2k}^{(d)}}{\binom{2k}{k}}. \tag{1}$$

Proof. Use Theorem 2.1 for $n = 2m$ to evaluate the layer corresponding to $h = 0$ at the origin:

$$\begin{aligned} A_{2m}^{(d+1)} &= \sum_{k=0}^m \binom{2m}{2k} \binom{2m-2k}{m-k} A_{2k}^{(d)} = \sum_{k=0}^m \frac{(2m)! \cdot (2m-2k)!}{(2k)! \cdot (2m-2k)! \cdot (m-k)!^2} A_{2k}^{(d)} \\ &= \binom{2m}{m} \cdot \sum_{k=0}^m \frac{m!^2}{k!^2 \cdot (m-k)!^2} \frac{A_{2k}^{(d)}}{\binom{2k}{k}} = \binom{2m}{m} \cdot \sum_{k=0}^m \binom{m}{k}^2 \frac{A_{2k}^{(d)}}{\binom{2k}{k}}. \end{aligned}$$

□

3. Asymptotics of the number of closed walks

In this section, we study the asymptotic behaviour of the number of closed walks (with the starting point assumed for simplicity to be the origin). For any $d \geq 1$, recall that, before Corollary 2.1, we denoted by $A_{2n}^{(d)}$ the number of closed walks of length $2n$ in \mathbb{R}^d . Similarly, we denote by $B_{2n}^{(d)}$ the number of walks of length $2n$ in \mathbb{R}^d that return for the first time at the origin at time $t = 2n$ (that is, which avoided the origin till the very end).

In the one-dimensional case, we have, of course, $A_{2n}^{(1)} = \binom{2n}{n}$ (A000984) and $B_{2n}^{(1)}$ is the sequence A284016 in [30]. The just cited reference from the OEIS gives the formula $B_{2n}^{(1)} = A_{2n}^{(1)} / (2n - 1)$, which we will justify further below (see Remark 3.1). In the two-dimensional case, we have, based on Lemma 2.1, $A_{2n}^{(2)} = \binom{2n}{n}^2$ (A002894) and $B_{2n}^{(2)}$ is the sequence A054474. In the three-dimensional case, Corollary 2.1 gives us formulas for the $A_{2n}^{(3)}$'s (A002896) and $B_{2n}^{(3)}$ is the sequence A049037. In the four-dimensional case, Corollary 2.1 gives us formulas for the $A_{2n}^{(4)}$'s (A039699). At the time our preprint [8] was posted, the sequence of $B_{2n}^{(4)}$'s was not found in [30]. Now it is A359801.

The first eight terms of these sequences are provided in the table below. It is interesting to note that when $d = 3$ and $d = 4$ the $B_{2n}^{(d)}$'s are fairly comparable with the $A_{2n}^{(d)}$'s, as opposed to the 1D- and 2D-case, where the first returns are much smaller than the returns. We will explain this behaviour and more in this section.

n	1	2	3	4	5	6	7	8
$A_{2n}^{(1)}$	2	6	20	70	252	924	3,432	12,870
$B_{2n}^{(1)}$	2	2	4	10	28	84	264	858
$A_{2n}^{(2)}$	4	36	400	4,900	63,504	853,776	11,778,624	165,636,900
$B_{2n}^{(2)}$	4	20	176	1,876	22,064	275,568	3,584,064	47,995,476
$A_{2n}^{(3)}$	6	90	1,860	44,730	1,172,556	32,496,156	936,369,720	27,770,358,330
$B_{2n}^{(3)}$	6	54	996	22,734	577,692	15,680,628	445,162,392	13,055,851,998
$A_{2n}^{(4)}$	8	168	5,120	190,120	7,939,008	357,713,664	16,993,726,464	839,358,285,480
$B_{2n}^{(4)}$	8	104	2,944	108,136	4,525,888	204,981,888	9,792,786,432	486,323,201,640

3.1 Recurrence relation for the first time returns

We show first how to generate recursively the sequences $(B_{2n}^{(d)})_n$ from the corresponding sequences $(A_{2n}^{(d)})_n$. Because in this recurrence relation the dimension does not play a role, we simplify the notation temporarily and write $\alpha_n = A_{2n}^{(d)}$ and $\beta_n = B_{2n}^{(d)}$. It is clear that we have $\beta_1 = \alpha_1$, as the walks that return in two steps at the origin return there for the first time as well. One then observes that $\beta_2 = \alpha_2 - \beta_1 \alpha_1$, because from the walks that return to the origin after 4 steps (namely, α_2) we have to subtract those generated after 2 steps by the β_1 walks that have already returned (and there are $\beta_1 \alpha_1$ such walks in total). This argument generalizes to give, for all $n \geq 2$,

$$\beta_n = \alpha_n - \sum_{k=1}^{n-1} \beta_k \alpha_{n-k}, \text{ or } B_{2n}^{(d)} = A_{2n}^{(d)} - \sum_{k=1}^{n-1} B_{2k}^{(d)} A_{2n-2k}^{(d)}. \tag{2}$$

The identity (2) can be written as the following identity of series:

$$(1 - \beta_1 x - \beta_2 x^2 - \dots - \beta_n x^n - \dots) \cdot (1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n + \dots) = 1. \tag{3}$$

The identity (3) has two main consequences. First, once we establish the asymptotics of the $A_{2n}^{(d)}$'s, it will be used to obtain the asymptotics of the $B_{2n}^{(d)}$'s. See Subsection 3.3. Second, it provides a proof of György Pólya's celebrated theorem about the recurrence and transience of random walks ([24]). We recall that an infinite simple

random walk is called *recurrent* if it is certain to return to its the starting point; if not, the random walk is called *transient*. Pólya's Theorem asserts that an infinite simple random walk on a d -dimensional lattice is recurrent for $d = 1$ and $d = 2$, and it is transient for $d \geq 3$. Theorem 3.4 below implies that, when $d \geq 3$, the series of $A_d(w)$ converges absolutely for $|w| \leq \frac{1}{(2d)^2}$ and so does the series of $B_d(w)$, since $0 < B_{2n}^{(d)} \leq A_{2n}^{(d)}$. Hence the relation $(1 - B_d(w))A_d(w) = 1$ extends to the circle of convergence by continuity, so $1 - B_d(1/(2d)^2) = 1/m_d$, with $0 < 1/m_d < 1$. It follows that $0 < p_d = B_d(1/(2d)^2) < 1$.

The proof of the recurrence of the random walk when $d = 1$ and $d = 2$ follows directly. When $d = 1$, recalling that $\alpha_n = A_{2n}^{(1)} = \binom{2n}{n}$, the series with the α_n 's is identified using Newton's generalized binomial formula for $r = -1/2$:

$$1 + \sum_{n=1}^{\infty} \alpha_n x^n = (1 - 4x)^{-1/2}.$$

This implies that

$$1 - \beta_1 x - \beta_2 x^2 - \dots - \beta_n x^n - \dots = (1 - 4x)^{1/2}, \tag{4}$$

with interval of convergence $[-1/4, 1/4]$. The probability of return to the origin, namely $\sum_{n=1}^{\infty} \beta_n/2^{2n}$, equals 1 by evaluating (4) at $x = 1/4$.

Remark 3.1. From Equation (4), it is now clear that $\beta_n = B_{2n}^{(1)} = A_{2n}^{(1)}/(2n - 1)$.

The two-dimensional case is less elementary. Recalling that $\alpha_n = A_{2n}^{(2)} = \binom{2n}{n}^2$, these coefficients are recognized in the expansion of the complete elliptic integral of the first kind:

$$\frac{K(k)}{\pi/2} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \sum_{i=0}^{\infty} \binom{2i}{i}^2 \frac{1}{16^i} \cdot k^{2i},$$

which holds true for $|k| < 1$. This implies that

$$1 - \beta_1 x - \beta_2 x^2 - \dots - \beta_n x^n - \dots = \frac{1}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 16x\right)},$$

with interval of convergence $(-1/16, 1/16)$. The limit when $x \rightarrow 1/16^-$ of ${}_2F_1$ is infinite and it follows again that the probability of return to the origin equals 1, since the β_n 's are positive, so we can move the limit inside.

3.2 Asymptotics of the numbers $A_{2n}^{(d)}$'s of closed walks

For $d \geq 1$, we denote $x_n^{(d)} = \frac{A_{2n}^{(d)}}{\binom{2n}{n}}$, with $x_0^{(d)} = 1$.* The generating formula for the $A_{2n}^{(d)}$'s contained in Equation (1) of Corollary 2.1, gives the following fundamental recurrence relation:

$$x_n^{(d+1)} = \sum_{k=0}^n \binom{n}{k}^2 x_k^{(d)} = 1 + \sum_{k=1}^n \binom{n}{k}^2 x_k^{(d)}, \text{ for } n \geq 1, d \geq 1. \tag{5}$$

To prove our next theorem, we use (5) and the asymptotics of Legendre polynomials to obtain asymptotics of higher order for the sequence $(x_n^{(d)})_n$. This is essentially different from the proofs done in [26], [27], and [23], for the first term, because the use of Legendre polynomials linearizes the problem. We also mention that [7, eqn.(29)] gives without a proof the first three terms of the expansion. The asymptotics of $(x_n^{(d)})_n$ allow us to recover the asymptotics for the $(A_{2n}^{(d)})_n$ sequence, which appear already in the literature (see [7, eqn.(27)] and [21, Appendix]).

We remark that if one uses the probability-generating function $\phi(t_1, \dots, t_d) = (t_1 + t_1^{-1} + \dots + t_d + t_d^{-1})^{2n}$ to calculate $A_{2n}^{(d)}$, then one can express ([7, eqn.(28)]) the number of closed walks as sum of squares of binomial coefficients

$$A_{2n}^{(d)} = \sum_{n_1+n_2+\dots+n_d=n} \frac{(2n)!}{(n_1!)^2(n_2!)^2 \dots (n_d!)^2}.$$

This in turn gives the formula:

$$x_n^{(d)} = \sum_{n_1+n_2+\dots+n_d=n} \frac{(n!)^2}{(n_1!n_2! \dots n_d!)^2}. \tag{5'}$$

Once known, identity (5') is actually easy to prove from (5). In [23], with a different motivation than ours, Louchard and Ward used (5') as starting point of their calculation of first-order asymptotics of the sequence $(x_n^{(d)})_n$. However, in our proof, the inductive definition (5) is used in an essential way.

*In [30], $(x_n^{(3)})_n$ is A002893 and $(x_n^{(4)})_n$ is A002895.

Theorem 3.1. *The following holds:*

$$x_n^{(d)} = a_d \frac{d^{2n}}{(\pi n)^{(d-1)/2}} \left(1 + \frac{r_1(d)}{n} + \frac{r_2(d)}{n^2} + \dots + \frac{r_m(d)}{n^m} + O_{d,m} \left(\frac{1}{n^{m+1}} \right) \right),$$

where $a_d = \frac{d^{d/2}}{2^{d-1}}$, $r_1(d) = \frac{1-d}{8}$, $r_2(d) = \frac{(d^2-1)(2d-3)}{384}$, while $r_3(d), r_4(d)$ are polynomials in d of degree 4 and 6 (see below), and it follows from [21] that $r_m(d)$ are polynomials in d of degree $\lceil \frac{3m}{2} \rceil$ and divisible by $d-1$.

Remark 3.2. *The polynomials mentioned in the theorem are:*

$$r_3(d) = \frac{1}{3072} (d-1) (6d^3 - 19d^2 + 14d + 15),$$

$$r_4(d) = \frac{1}{1474560} (d-1) (20d^5 + 2504d^4 - 10241d^3 + 9679d^2 + 309d + 945).$$

Remark 3.3. *From Theorem 3.1 and the relationship $A_{2n}^{(d)} = \binom{2n}{n} x_n^{(d)}$ we obtain*

$$A_{2n}^{(d)} = a_d \frac{(2d)^{2n}}{(\pi n)^{d/2}} \left(1 + \frac{a_1(d)}{n} + \frac{a_2(d)}{n^2} + \dots + \frac{a_m(d)}{n^m} + O_{d,m} \left(\frac{1}{n^{m+1}} \right) \right),$$

which is the formula (3.11) in [21] with $a_i(d) = Q_i(d)/2^i$. We also note that the formula for the leading coefficient appears in the original paper [24, eqn.(12)] and the first two terms $a_1(d)$ and $a_2(d)$ were obtained in [7] using Pólya's original method, which is completely different than ours.

Remark 3.4. *Using the well-known asymptotic expansion $\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}} (1 + \frac{g_1}{n} + \dots + \frac{g_k}{n^k} + \dots)$, where $g_1 = -\frac{1}{8}, g_2 = \frac{1}{128}, g_3 = \frac{5}{1024}, g_4 = -\frac{21}{32768}$, and so on, then clearly $a_m(d) = r_m(d) + g_1 r_{m-1}(d) + \dots + g_m$, so if $r_k(d)$, $k = 1, \dots, m$, are polynomials, so are $a_k(d)$, $k = 1, \dots, m$, and viceversa. Because $x_n^{(1)} = 1$ it follows that $r_m(1) = 0$, $m \geq 1$, hence $a_m(1) = g_m$, verifying (3.15) in [Joyce]. Similarly $r_m(2) = g_m$, $m \geq 1$, hence we obtain a formula for $a_m(2)$ matching (3.14) in Joyce.*

The proof of Theorem 3.1 by induction on $d \geq 2$ will be elementary and self-contained, using only basic calculus and classical results about Legendre polynomials, which we recall first.

We recall ([37, 15.1-9], [33, 4.8]) that the Legendre polynomials P_n of degree $n \geq 0$, which form an orthogonal basis in the space of real polynomials on $[-1, 1]$ with the usual L^2 -norm, are given by Rodrigues' formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$. The first ones are $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3x^2-1}{2}$, and so on. By the Leibniz rule, the above leads to the formula:

$$P_n(x) = \left(\frac{x-1}{2} \right)^n \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{x+1}{x-1} \right)^k. \tag{6}$$

There is also an integral formula

$$P_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x + \sqrt{x^2 - 1} \cos \theta)^n d\theta,$$

valid for all complex x , as it is clear that the choice of the square root is irrelevant, as all odd powers in the binomial expansion vanish.

For any (complex number) $x \notin [-1, 1]$ there is a unique complex number z satisfying $|z| > 1$ and $x = \frac{1}{2}(z + \frac{1}{z})$. This is because the quadratic in z solving for x is symmetric under $z \rightarrow 1/z$, and if $z = e^{i\theta}$ then $x = \cos \theta \in [-1, 1]$. Note that then $\Re(1 - z^{-2}) > 0$, so one has natural definition of $(1 - z^{-2})^a$ for any $a \in \mathbb{R}$ which is positive for $z > 1$. In particular, for $x > 1, z = x + \sqrt{x^2 - 1} > 1$ and the integral representation above gives

$$|P_n(x)| \leq |z|^n. \tag{7}$$

The fundamental asymptotic theorem (Laplace-Heine) for the Legendre polynomials (of argument not in $[-1, 1]$) states the following ([33, 8.21]). Given a fixed Jordan curve Γ enclosing the interval $[-1, 1]$ one has uniformly for x in the exterior of Γ :

$$P_n(x) = q_n z^n \sum_{j=0}^{m-1} q_j \frac{1 \cdot 3 \cdots (2j-1)}{(2n-1)(2n-3) \cdots (2n-2j+1)} z^{-2j} (1 - z^{-2})^{-j-1/2} + O_{\Gamma,m} \left(\frac{|z|^n}{n^{1/2+m}} \right),$$

for all $m \geq 1$, where $q_0 = 1$ and $q_k = \binom{2k}{k} / 2^{2k}$, for $k \geq 1$. In the above $x = \frac{1}{2}(z + \frac{1}{z}), |z| > 1$, so $(1 - z^{-2})^{-1/2-k}$ are naturally well defined as noted.

Let now $x, y > 1$ satisfy $x = \frac{y+1}{y-1}$, so that $z = \frac{(\sqrt{y}+1)^2}{y-1}$ and $(1 - z^{-2})^{-1/2} = \frac{\sqrt{y}+1}{2y^{1/4}}$. Formula (6) becomes

$$(y - 1)^n P_n(x) = \sum_{k=0}^n \binom{n}{k}^2 y^k,$$

while the asymptotic above becomes

$$\sum_{k=0}^n \binom{n}{k}^2 y^k = q_n \frac{(\sqrt{y} + 1)^{2n+1}}{2y^{1/4}} \sum_{j=0}^{m-1} q_{jn} \frac{(\sqrt{y} - 1)^{2j}}{4^j y^{j/2}} + O_{\Gamma, m} \left(\frac{(\sqrt{y} + 1)^{2n}}{n^{1/2+m}} \right), \tag{8}$$

where $q_{0n} = q_0 = 1$ and $q_{jn} = q_j \frac{1 \cdot 3 \cdots (2j-1)}{(2n-1)(2n-3)\cdots(2n-2j+1)}$. The growth of the coefficients appearing in equation (8) is worth noticing: $q_n = \binom{2n}{n} / 2^{2n} \sim 1/\sqrt{\pi n}$ and $q_{jn} \sim c_j n^{-j}$. We also remark that on the left-hand side of the sum of this equation, we can ignore the first term, for $k = 0$, which is 1, because it is negligible with respect to the right-hand side error.

With the same notation, the inequality in (7) translates to:

$$\sum_{k=0}^n \binom{n}{k}^2 y^k \leq (\sqrt{y} + 1)^{2n}. \tag{9}$$

We start the proof of Theorem 3.1 by first showing that in the induction step, the asymptotic carryover goes through relation (5). Using a simple version of the elimination method in asymptotic analysis, we reduce our induction step to developing asymptotics for integrals where it is straightforward to apply the Laplace method as described in [12, Appendix B.6]. We perform the standard steps of localization to the smaller range $\frac{\log^2 n}{n}$ (“Neglect the tails”), approximate the integrand by forcing out the exponential main term $(d + 1)^{2n+1}$ and using Taylor series on the resulting function (“Centrally approximate”), and finally introduce back the tails (“Complete the tails”). First, in Lemma 3.1 we deal with the asymptotic error carryover. In its statement and proof $C_{n,m,d}$, $C_{m,d}$, and so on are constants with the implied dependencies, but can change between uses.

Lemma 3.1. *Assume that for $d \geq 2, n \geq 1, m \geq 0$ we prove that*

$$x_n^{(d)} = a_d \frac{d^{2n}}{(\pi n)^{(d-1)/2}} \left(1 + \frac{r_1(d)}{n} + \frac{r_2(d)}{n^2} + \cdots + \frac{r_m(d)}{n^m} + \frac{C_{n,m,d}}{n^{m+1}} \right),$$

with $|C_{n,m,d}| \leq C_{m,d}$. Then it follows that

$$x_n^{(d+1)} = a_d \sum_{k=1}^n \binom{n}{k}^2 \frac{d^{2k}}{(\pi k)^{(d-1)/2}} \left(1 + \frac{r_1(d)}{k} + \frac{r_2(d)}{k^2} + \cdots + \frac{r_m(d)}{k^m} \right) + C_{n,m,d+1} \frac{(d+1)^{2n}}{n^{d/2+m+1}},$$

with $|C_{n,m,d+1}| \leq C_{m,d+1}$.

Proof. We substitute the above expression for $x_k^{(d)}$, $k \geq 1$, in formula (5) and note that the first term 1 is clearly subsumed in the final error. For $0 \leq k \leq n/4$, using that $\binom{n}{k} \leq 4^n$, we have $\binom{n}{k}^2 d^{2k} \leq (4\sqrt{d})^n$ so

$$\left| \sum_{k \leq n/4} \binom{n}{k}^2 \frac{d^{2k}}{(\pi k)^{(d-1)/2}} \frac{C_{k,m,d}}{k^{m+1}} \right| \leq C_{m,d} \cdot n \cdot (4\sqrt{d})^n = O_{m,d} \left(\frac{(d+1)^{2n}}{n^{d/2+m+1}} \right),$$

since $\frac{4\sqrt{d}}{(d+1)^2} < 1$.

For $k > n/4$ we have $|C_{k,m,d}| k^{(1-d)/2-m-1} \leq 4^{(d-1)/2+m+1} C_{m,d} n^{(1-d)/2-m-1}$, so the absolute value of this other part of the sum that gives the error terms is at most (with a different $C_{m,d}$)

$$C_{m,d} \frac{1}{n^{(d-1)/2+m+1}} \cdot \sum_{k > n/4} \binom{n}{k}^2 d^{2k} \leq C_{m,d+1} \frac{(d+1)^{2n}}{n^{d/2+m+1}}.$$

The last inequality comes from (8) with $y = d^2$, remembering that in formula (8) $q_n = O(1/\sqrt{n})$ and the inner sum is $O_{d,m}(1)$. □

The following two lemmas are used in justifying the steps in the Laplace method indicated above.

Lemma 3.2. *If $a, b > 0, k \geq 1$ then*

$$\int_b^\infty e^{-sk} s^{a-1} ds \leq \frac{(b+1)^a (a+1)^{a+1}}{\min(b, 1)} e^{-bk}.$$

Proof. Use induction on the integer part of a and integration by parts. □

Lemma 3.3. *If $0 \leq q \leq 1/2$ then $(e^{-q}d + 1)^{2n} \leq (d + 1)^{2n}e^{-qnd/(d+1)}$, for $d \geq 2$ and $n \geq 1$.*

Proof. We use, for $0 \leq x < 1$, that $\log(1 - x) \leq -x$ and $1 - e^{-x} \geq x/2$. Then $(e^{-q}d + 1)^{2n} = (d + 1)^{2n}(1 - \frac{(1-e^{-q})d}{d+1})^{2n}$, hence taking logarithms $2n \log(1 - \frac{(1-e^{-q})d}{d+1}) \leq -2n \frac{(1-e^{-q})d}{d+1} \leq -\frac{qnd}{d+1}$. □

Now, we use a simple version of the elimination method from [5, 3.9]. From

$$k^{(1-d)/2-j} = \frac{1}{\Gamma((d-1)/2+j)} \int_0^\infty e^{-sk} \cdot s^{(d-1)/2+j-1} ds, \text{ for } k \geq 1, d \geq 2, j \geq 0,$$

we can write (with $r_0(d) = 1$ and noting that since $d \geq 2$ we have integrability near 0)

$$\sum_{k=1}^n \binom{n}{k}^2 \frac{d^{2k} \cdot r_j(d)}{(\pi k)^{(d-1)/2} \cdot k^j} = \frac{r_j(d)}{\pi^{(d-1)/2} \cdot \Gamma((d-1)/2+j)} \int_0^\infty \sum_{k=1}^n \binom{n}{k}^2 (e^{-s}d^2)^k \cdot s^{(d-1)/2+j-1} ds.$$

So, under the assumptions of Lemma 3.1, we obtain

$$x_n^{(d+1)} = a_d \sum_{j=0}^m \frac{r_j(d)}{\pi^{(d-1)/2} \cdot \Gamma((d-1)/2+j)} \int_0^\infty \sum_{k=1}^n \binom{n}{k}^2 (e^{-s}d^2)^k \cdot s^{(d-1)/2+j-1} ds + O_{m,d} \left(\frac{(d+1)^{2n}}{n^{d/2+m+1}} \right). \quad (10)$$

Applying Lemma 3.2 with $b_n = \frac{\log^2 n}{n}$, $n \geq 2$, $b_n > 0$, $k \geq 1$, inequality (9), and Lemma 3.3, we see that we can neglect the tails in (10) up to allowable error, so we get

$$x_n^{(d+1)} = a_d \sum_{j=0}^m \frac{r_j(d)}{\pi^{(d-1)/2} \Gamma((d-1)/2+j)} \int_0^{b_n} \sum_{k=1}^n \binom{n}{k}^2 (e^{-s}d^2)^k \cdot s^{(d-1)/2+j-1} ds + O_{m,d} \left(\frac{(d+1)^{2n}}{n^{d/2+m+1}} \right).$$

Since $e^{-b_n}d^2 \sim d^2 \geq 4$, for $d \geq 2$, we can substitute (8) with $y = e^{-s}d^2$, $0 \leq s \leq b_n$, in the above. Putting $t = ns$ and using Lemma 3.3 for $j = 0, \dots, m$ we estimate the error coming from (8) as follows:

$$\int_0^{b_n} (e^{-s/2}d + 1)^{2n} s^{(d-1)/2+j-1} ds \leq \frac{(d+1)^{2n}}{n^{(d-1)/2+j}} \int_0^{\log^2 n} e^{-td/(2d+2)} t^{(d-1)/2+j-1} dt \ll_{d,m} \frac{(d+1)^{2n}}{n^{(d-1)/2+j}}.$$

So the error term coming from (8) is $\ll_{d,m} \frac{(d+1)^{2n}}{n^{(d-1)/2+j+1/2+m}} \ll_{d,m} \frac{(d+1)^{2n}}{n^{d/2+m+1}}$ as required. Note that the above considerations and the fact that $q_{kn} \sim c_k n^{-k}$ show that, if we fix $j = 0, \dots, m$ when we apply the asymptotic (8), we actually need to go only up to $k = m - j$ terms as then the next term is $\ll_{d,m} \frac{(d+1)^{2n}}{n^{d/2+m+1}}$ so is included in the required error.

Hence up to allowable error $O_{m,d} \left(\frac{(d+1)^{2n}}{n^{d/2+m+1}} \right)$, we have

$$x_n^{(d+1)} = \frac{a_d \cdot q_n}{2\sqrt{d} \cdot \pi^{(d-1)/2}} \sum_{j=0}^m \sum_{k=0}^{m-j} \frac{r_j(d) \cdot q_{kn}}{(4d)^k \cdot \Gamma((d-1)/2+j)} I_{dj k},$$

where

$$I_{dj k} = \int_0^{b_n} (e^{-s/2}d + 1)^{2n+1} \cdot (e^{-s/2}d - 1)^{2k} \cdot e^{s/4+sk/2} \cdot s^{(d-1)/2+j-1} ds.$$

Putting $t = ns$ and forcing $(d + 1)^{2n+1}$ out of the integral gives us

$$x_n^{(d+1)} = a_{n,d} \sum_{j=0}^m \sum_{k=0}^{m-j} \tilde{q}_{jk} \tilde{I}_{dj k}, \quad (11)$$

where

$$\tilde{I}_{dj k} = \int_0^{\log^2 n} \left(1 - (1 - e^{-t/(2n)}) \frac{d}{d+1} \right)^{2n+1} \cdot (e^{-t/(2n)}d - 1)^{2k} \cdot e^{t/(4n)+tk/(2n)} \cdot t^{(d-1)/2+j-1} dt, \quad (12)$$

and

$$a_{n,d} = \frac{a_d \cdot q_n \cdot (d+1)^{2n+1}}{2\sqrt{d} \cdot (\pi n)^{(d-1)/2}} \text{ and } \tilde{q}_{jk} = \frac{r_j(d) \cdot q_{kn}}{(4d)^k \cdot \Gamma((d-1)/2+j) \cdot n^j} \sim c_{d,j,k} \cdot n^{-j-k}.$$

(In the equation above, \tilde{q}_{jk} has an asymptotic in inverse powers of n starting at n^{-j-k} since q_{kn} has such asymptotic starting at n^{-k} , as already remarked right after we obtained (8).)

But if n large enough, depending only on m , so that $t \leq \log^2 n$ also makes $t/2n$ small enough, we have uniformly in $t \in [0, \log^2 n]$

$$(2n + 1) \log \left(1 - (1 - e^{-t/(2n)})^{\frac{d}{d+1}} \right) = -\frac{d}{d+1}t + \frac{p_1(t)}{n} + \dots + \frac{p_m(t)}{n^m} + O_{d,m}(n^{-m-1}),$$

with fixed polynomials $p_k(t)$, where the d dependence in each is of the type $(\frac{d}{d+1})^r$ for various r and we notice that in each term $(\frac{d}{d+1})^r t^{r_1}$ we actually have $r \leq r_1$. For example $4p_1(t) = -(\frac{d}{d+1})^2 t^2 + \frac{d}{d+1} t^2 - 2\frac{d}{d+1} t$.

Hence, the central approximation of the main term of each integrand in (12) is of the form

$$e^{-dt/(d+1)} \left(1 + \frac{\tilde{p}_1(t)}{n} + \dots + \frac{\tilde{p}_m(t)}{n^m} + O_{d,m}(n^{-m-1}) \right),$$

where $\tilde{p}_k(t)$ has the same properties as $p_k(t)$ and clearly $p_1 = \tilde{p}_1$.

On the other hand, the other exponential terms in (12), $(e^{-t/(2n)}d - 1)^{2k} \cdot e^{t/(4n)+tk/(2n)}$, have expansions of the type $(d-1)^{2k} + s_1(t)/n + \dots + s_m(t)/n^m + O_{d,m}(n^{-m-1})$, but now the dependence in d of the polynomials $s_k(t)$ is also polynomial only, so overall the integrals in (12) are a sum of integrals of the type $e^{-dt/(d+1)} \cdot t^{(d-1)/2+j-1} t^r$ with coefficients depending on d as discussed above.

Now when we complete the tails, since the main factor is $e^{-dt/(d+1)}$ we need to make the change of variable $u = \frac{d}{d+1}t$, which means that every power of t in the above integrands generates a factor of $(\frac{d+1}{d})^{(d-1)/2+j+r}$, as well as a $\Gamma((d-1)/2+j+r)$. But because the respective outer coefficient contains the term $1/\Gamma((d-1)/2+j)$, the quotient of the respective two Gammas is a polynomial in d .

If we look at the main asymptotic term, when $k = j = 0$, there is only one corresponding integrand $e^{-dt/(d+1)} \cdot t^{(d-1)/2-1}$. Keeping in mind that $q_n \sim \frac{1}{\sqrt{\pi n}}$, as well as that the asymptotic requires only $(d+1)^{2n}$, so we have an extra $d+1$ factor from (12), and that the Gamma factor cancels out completely, we obtain $a_{d+1} = a_d \frac{(d+1)^{(d+1)/2}}{2^{d/2}}$. Given that $a_2 = 1$, we immediately obtain the formula claimed in Theorem 3.1:

$$a_d = \frac{d^{d/2}}{2^{d-1}}.$$

Putting all of these observations together, Theorem 3.1 follows.

We note that for any m , the only corresponding asymptotic term that appears in (11) for $j = m$ is when $k = 0$, so as before we have only one corresponding integrand, $e^{-dt/(d+1)} t^{(d-1)/2-1+m}$, which gives an extra $(\frac{d+1}{d})^m$, so giving the term $r_m(d) (\frac{d+1}{d})^m$ in the recurrence for $r_m(d+1)$ that follows from (12). More generally, the discussion above about the shape of the coefficients of the powers of t involved in each asymptotic term, especially the fact that any coefficient $d/(d+1)$ appears to a power at most the power of the corresponding t , so in the final result is canceled by the $(d+1)/d$ power coming from completing the tails, gives the recurrence:

$$r_m(d+1) = (\frac{d+1}{d})^m r_m(d) + (\frac{d+1}{d})^{m-1} Q_{1,m}(d) r_{m-1}(d) + \dots + \frac{d+1}{d} Q_{m-1,m}(d) r_1(d) + Q_{m,m}(d),$$

where $Q_{k,m}$ are rational functions in d with denominator a power of d only. A more careful look at all the expansions involved in (11) actually shows that the degree of the denominators of $Q_{k,m}(d)$ are at most k , since they all come only from the terms of degree k in $1/n$ in the expansion of (12), and the Legendre recurrence terms are actually the only ones giving denominators powers of d , highest possible being k for the $1/n^k$ error term there. This is because the carryover term from $r_{m-k}(d)$ corresponding to the extra t^{m-k} in the starting integral (15) is precisely reflected in the coefficient $(\frac{d+1}{d})^k$, while simplification of the corresponding Gamma factors gives only polynomials in d as noted above. Also all $Q_{k,m}(d)$ have rational coefficients since $q_n = \binom{2n}{n}/2^{2n}$ and q_{jn} do when expanded in powers of $1/n$ (as of course have the Taylor series of the elementary functions involved in (12)).

For example $Q_{1,1}(d) = -\frac{1}{8d}$ so $r_1(d+1) = \frac{d+1}{d} r_1(d) - \frac{1}{8d}$, with $r_1(1) = 0$, $r_1(2) = -1/8$. This results in $r_1(d) = \frac{1-d}{8}$. Similarly $Q_{1,2}(d) = \frac{3}{8d}$ and $Q_{2,2}(d) = \frac{d^2}{192} + \frac{d}{96} + \frac{11}{192} - \frac{1}{96d} - \frac{7}{128d^2}$, giving $r_2(d+1) = (\frac{d+1}{d})^2 r_2(d) + \frac{d+1}{d} \frac{3}{8d} \frac{1-d}{8} + \frac{d^2}{192} + \frac{d}{96} + \frac{11}{192} - \frac{1}{96d} - \frac{7}{128d^2}$, with $r_2(1) = 0$, $r_2(2) = \frac{1}{128}$. This results in $r_2(d) = \frac{(d^2-1)(2d-3)}{384}$, as claimed.

3.3 Asymptotics of the first returns $B_{2n}^{(d)}$

Now we continue with the deeper discussion of the asymptotic for $B_{2n}^{(d)}$'s and related matters, which involve the singularity analysis method from [12, ch.VI] as well as the theory of holonomic (D -finite) functions and P -recursive sequences ([31,32]), with forays in the theory of Fuchsian ODEs ([36], [18]) and in particular Heun

equations ([28]). In the process, we will show that the A and B sequences are fundamentally different from a combinatorial viewpoint, as the former are P -recursive but the latter are not, except in dimension 1. We will first state the main theorems, and then we will indicate how to prove them.

Theorem 3.2. For $d \geq 3$, let $m_d = \sum_{n \geq 0} \frac{A_{2n}^{(d)}}{(2d)^{2n}} < \infty$ denote the expected number of returns to the origin, plus one, while for $d \geq 5$, let $\tilde{m}_d = \sum_{n \geq 0} \frac{nA_{2n}^{(d)}}{(2d)^{2n}} < \infty$.

(i) For $d \geq 3$ odd, we have

$$B_{2n}^{(d)} = b_d \frac{(2d)^{2n}}{(\pi n)^{d/2}} \left(1 + \frac{b_1(d)}{n} + \frac{b_2(d)}{n^2} + \dots + \frac{b_m(d)}{n^m} + O_{d,m} \left(\frac{1}{n^{m+1}} \right) \right),$$

where $b_d = \frac{a_d}{m_d^2} = \frac{d^{d/2}}{m_d^2 \cdot 2^{d-1}}$, for all $d \geq 3$. Here $b_1(3) = -\frac{3}{16} + \frac{9}{32m_3} - \frac{81}{16\pi^2 m_3^3}$ and $b_1(d) = -\frac{d}{8} - \frac{d\tilde{m}_d}{m_d}$, for $d \geq 5$.

(ii) For $d \geq 4$ even, we have

$$B_{2n}^{(d)} = b_d \frac{(2d)^{2n}}{(\pi n)^{d/2}} \left(1 + \frac{b_1(d, \log n)}{n} + \frac{b_2(d, \log n)}{n^2} + \dots + \frac{b_m(d, \log n)}{n^m} + O_{d,m} \left(\frac{b_{m+1}(d, \log n)}{n^{m+1}} \right) \right)$$

where $b_k(d, \log n)$ are now (possibly constant) polynomials in $\log n$, with the first non-constant polynomial $b_m(d, \log n)$ for $m = d/2 - 1$ with leading term $-2 \binom{d-2}{d/2-1} \frac{a_d}{\pi^{d/2} m_d} \log n$.

In particular, for $d = 4$ we have

$$B_{2n}^{(4)} = b_4 \frac{8^{2n}}{(\pi n)^2} \left(1 - \frac{8}{\pi^2 m_4} \frac{\log n}{n} + O\left(\frac{1}{n}\right) \right),$$

while for $d \geq 6$ we have $b_1(d, \log n) = b_1(d) = -\frac{d}{8} + \frac{d\tilde{m}_d}{m_d}$.

(iii) For $d = 2$, we have ([12, p.426], with γ the Euler constant and $K \doteq 0.8825424$)

$$B_{2n}^{(2)} = \pi \frac{4^{2n}}{n \log^2 n} \left(1 - 2 \frac{\gamma + \pi K}{\log n} + O\left(\frac{1}{\log^2 n}\right) \right).$$

(iv) For $d = 1$, we have

$$B_{2n}^{(1)} = \frac{A_{2n}^{(1)}}{2n-1} = \frac{2^{2n}}{2n\sqrt{\pi n}} \left(1 - \frac{3}{8n} + \frac{25}{128n^2} + O\left(\frac{1}{n^3}\right) \right).$$

After recalling a few definitions and making some observations, we collect all the information about the generating functions of $x_n^{(d)}$, $A_{2n}^{(d)}$, and $B_{2n}^{(d)}$ in Theorems 3.3, 3.4, and 3.5, respectively. Their proofs will be given in the next subsection, but here we show how they imply Theorems 3.6 and 3.2, using singularity analysis theory results like the zig-zag algorithm from [12, Ch.VI].

If $f = \sum a_n w^n$ and $g = \sum b_n w^n$ are Taylor series with (nonzero) radii of convergence R_1 and R_2 , respectively, we define their *Hadamard convolution* by $h(w) = f * g(w) = \sum a_n b_n w^n$. From the integral representation

$$h(w) = \frac{1}{2\pi i} \int_{|\zeta|=r} f(\zeta)g(w/\zeta) \frac{d\zeta}{\zeta}, \text{ for } |w| < R_1 R_2, |w|/R_2 < r < R_1,$$

h has radius of convergence at least $R_1 R_2$.

Following [12, App.B.4], a formal power series over some field K (for us $K = \mathbb{C}$, $K = \mathbb{Q}$ or $K = \mathbb{Z}/p$, p prime) $f(z) = \sum_{n \geq 1} f_n z^n$ is called *holonomic* or *D-finite* (short for differentially finite) if the vector space over $K(z)$ (the field of rational functions with coefficients in K) spanned by the set of all its formal derivatives $(\partial^j f(z))_{j=0}^\infty$ is finite dimensional. This is equivalent to the existence of a (non-trivial) finite order linear ODE with coefficients in $K(z)$ (hence by clearing denominators in $K[z]$) formally satisfied by f .

Similarly, a sequence $(f_n)_n \in K$ will be called *P-recursive* (short for polynomially recursive) if there are polynomials $P_0, \dots, P_r \in K[z]$ such that $\sum_{k=0}^r P_k(n) f_{n+k} = 0$, for $n \geq n_0$. It is a fairly simple matter to note that the holonomy of a generating formal power series f is equivalent to the P -recursiveness of its coefficients f_n .

If f is algebraic over K , so there are polynomials $Q_0, \dots, Q_r \in K[z]$, not all zero, such that $\sum_{k=0}^r Q_k(z) f^k(z) = 0$, then f is holonomic (all the formal derivatives of f belong to $K(z, f)$ which is finitely dimensional over $K(z)$ since f is algebraic). If f is not algebraic, we call it transcendental. Rational functions like $\frac{1}{1-z} = \sum_{n \geq 0} z^n$

are algebraic over any field K , while $e^z = \sum z^n/n!$ is transcendental over \mathbb{C} but holonomic since $(e^z)' - e^z = 0$, or, equivalently, $(n + 1) \cdot \frac{1}{(n+1)!} - \frac{1}{n!} = 0$ being the P -recurrence for the coefficients. By extension, we will call the sequence of coefficients $(f_n)_n$ algebraic or transcendental over K if the corresponding formal series is, though of course the definition is useful only for P -recursive coefficients, as non- P -recursive coefficients cannot be algebraic anyway. In particular, if the f_n 's are integers, it makes sense to talk about algebraicity modulo a prime p in addition to algebraicity over \mathbb{C} . By Eisenstein's Lemma ([25, part VIII, no.150]), if f is an algebraic power series over \mathbb{C} with rational coefficients, so $P_0(z) + P_1(z)f(z) + \dots + P_n(z)f^n(z) = 0$, $n \geq 1$, $P_n \neq 0$, $P_k \in \mathbb{C}[z]$, then we can actually take $P_k \in \mathbb{Z}[z]$.

The fundamental difference between \mathbb{C} and \mathbb{Z}/p for integral sequences is that pointwise multiplication of coefficient sequences $(f_n)(g_n) \rightarrow (f_n g_n)$ preserves algebraicity only in finite characteristic p , though it preserves holonomy for any field (see [32, Ch.6]).[†]

If we work over \mathbb{C} and $f = \sum f_n z^n$ is holonomic with a non-zero radius of convergence, so it actually defines an analytic function and not only a formal power series, we can use all the machinery of linear ODEs with polynomial (rational) coefficients ([18], [36]). In particular, such an ODE is called *Fuchsian* if all its solutions are locally bounded by rational functions. Equivalently, the ODE is Fuchsian iff when denoting the singularities of the ODE (the poles of the coefficients R_k when the equation is written as $f^{(m)}(z) + \sum_{0 \leq k \leq m-1} R_k(z) f^{(k)}(z) = 0$, with R_k rational functions) by q_1, \dots, q_M (including ∞ if when the equation is transformed by $z \rightarrow 1/z$, we get a singularity at 0), then, for all local solutions of the ODE, there is a N such that $(z - q_k)^N f$ is bounded near q_k , with the corresponding definition at ∞ . The usual definition ([18]) involves the notion of a regular singular point but is equivalent to the above. The Fuchsian condition constrains strongly the shape of the equation ([13]). Also a linear ODE with a priori meromorphic coefficients only, which is Fuchsian in the sense above, can actually be taken to have rational coefficients, while if f is a solution of a linear ODE (again a priori with meromorphic coefficients only) and all the singularities of f are Fuchsian (hence they are finitely many), then f is the solution of a Fuchsian ODE of order at most the order of the original ODE satisfied by f . In particular, algebraic functions over \mathbb{C} (which are automatically locally analytic except at their finitely many singularities) satisfy Fuchsian ODEs, though it is a classical and quite difficult problem to determine if a given Fuchsian ODE has at least one algebraic solution ([13]).

According to [12] or [9], an analytic function $f(w)$ given by a Taylor series with radius of convergence 1 is called Δ -analytic (or Δ -regular) if it has an analytic extension to a Δ -domain of the type $|w| < 1 + \eta$, $|\arg(w - 1)| > \phi$, for some $\eta > 0$ and $0 < \phi < \pi/2$. More generally, we call any analytic f given by a Taylor series with arbitrary finite radius of convergence Δ -analytic if $f(\zeta w)$ is so in the restricted sense above for some complex $\zeta \neq 0$. The theory of linear ODEs with polynomial (rational) coefficients ([18]) shows that holonomic functions defined by convergent power series and being solutions of such ODEs are Δ -analytic, though of course the converse is far from true since Δ -analyticity depends only on the singularities on the boundary of the disc of convergence.

We define $T(w) = 1 - w$ and $L(w) = -\log(1 - w) = \sum_{n \geq 1} \frac{w^n}{n}$, for $|w| < 1$.

Δ -analyticity is a very useful property enjoyed by many elementary functions like $T^a = (1 - w)^a$ (algebraic), $(L/w)^b = (\log(1/(1 - w)))/w^b$, for $a, b \in \mathbb{R}$, and polylogarithms $\sum_{n \geq 1} \frac{w^n}{n^k}$, $k \geq 2$, while as noted holonomy is much more restrictive. For example, while polylogarithms and T^a are holonomic, only L^n , $n \geq 1$ are holonomic. While both properties are preserved under various operations ([12, VI], [32]), including differentiation, integration, Hadamard convolution $f * g$, and multiplication $f g$, Δ -analyticity is preserved under division f/g , with g having no zeroes on a Δ -domain, but holonomy in general is not preserved under division. For example, if f is holonomic ($f(0) \neq 0$), the theorem of Harris and Sibuya [17] states that $1/f$ is holonomic if and only if f'/f is algebraic.

Our next theorems state that the generating functions of the first returns are also Δ -analytic, but only the A and x sequences lead to holonomic functions and not the B sequence.

Theorem 3.3. For $F_d(w) = \sum_{n \geq 0} x_n^{(d)} w^n$, the following hold:

- (i) F_d has radius of convergence $1/d^2$ and is uniformly convergent on the circle of convergence for $d \geq 4$. It is holonomic, hence Δ -analytic, for all $d \geq 1$ and satisfies a Fuchsian linear ODE. In particular, F_d can be extended to a multivalued analytic function in the plane with finitely many singularities.
- (ii) For $d \geq 2$, F_d has finite order singularities at $1/k^2$, for all $1 \leq k \leq d$ and k having the same parity as d , and at ∞ , with principal branch extending the power series on the plane cut from $1/d^2$ to ∞ .
- (iii) All the finite singularities of F_d are of the same type and with the same kind of asymptotics (which is also a development on angular domains) as described below in Theorem 7 for the circle of convergence singularity. At infinity, the singularity will always be of a logarithmic nature (for $d \geq 3$).

[†]Over the complex field, this pointwise multiplication corresponds to Hadamard convolution, if the formal power series have non zero radius of convergence, and by a slight abuse of terminology is also called convolution or even Hadamard convolution over any field K , though we do not have an integral representation in general.

(iv) $F_1(w) = 1/(1-w)$, $F_2(w) = 1/\sqrt{1-4w} = A_1(w)$, $F_3(1/9) = \infty$.

Remark 3.5. We conjecture that, for all d , F_d satisfies an ODE of order $(d-1)$.

Theorem 3.4. For $A_d(w) = \sum_{n \geq 0} A_{2n}^{(d)} w^n$, the following hold:

- (i) $A_d(w) = F_d(w) * A_1(w) = F_d(w) * F_2(w)$, for all $d \geq 1$. $A_1(w) = 1/\sqrt{1-4w} = F_2(w)$ and $A_2(w) = A_1(w) * A_1(w) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 16w\right)$ is a hypergeometric function.
- (ii) A_d has radius of convergence $1/(2d)^2$ and is uniformly convergent on the circle of convergence for $d \geq 3$, with $A_d(1/(2d)^2) = m_d > 1$, for $d \geq 3$, and $A_1(1/4) = A_2(1/16) = \infty$. It is holonomic, hence Δ -analytic, for all $d \geq 1$, and satisfies a Fuchsian linear ODE of order d .[‡] In particular, A_d can be extended to a multivalued analytic function in the plane with finitely many singularities.
- (iii) A_d has finite order singularities at $1/(2k)^2$, for all $1 \leq k \leq d$ and k having the same parity as d , and at ∞ , with its principal branch extending the power series on the plane cut from $1/(2d)^2$ to ∞ .
- (iv) All the finite singularities of A_d are of the same type and with the same kind of asymptotics (which is also a development on angular domains) as described below in Theorem 7 for the circle of convergence singularity. For d even, the singularity at infinity will always have logarithmic and half-integral powers (e.g., it is well known that $A_2(w) \sim (c_1 + c_2 \log w)\sqrt{1/w}$ for $|w| \rightarrow \infty$ [6]). For $d \geq 3$ odd, the singularity at infinity is of square root type and contains no logarithmic terms.

Theorem 3.5. For $B_d(w) = \sum_{n \geq 1} B_{2n}^{(d)} w^n$, the following hold:

- (i) $(1 - B_d(w)) \cdot A_d(w) = 1$.
- (ii) B_d has radius of convergence $1/(2d)^2$ and is uniformly convergent on the circle of convergence. It is not holonomic for $d \geq 2$, but it is still Δ -analytic, for all $d \geq 1$.
- (iii) $B_1(w) = 1 - \sqrt{1-4w}$, $B_1(1/4) = B_2(1/16) = 1$, and, for $d \geq 3$, $B_d(1/(2d)^2) = 1 - 1/m_d = p_d$, Pólya's return probability.

Next, we consider the normalized series $\tilde{F}_d(w) = \sum_{n \geq 0} \frac{x_n^{(d)}}{d^{2n}} w^n$, with radius of convergence 1, and the analogue definitions for $\tilde{A}_d(w)$ and $\tilde{B}_d(w)$.

Theorem 3.6. The following hold:

- (i) For $d \geq 2$ even, \tilde{F}_d has an asymptotic expansion in terms of half-integer and integer powers of T , with the first asymptotic term being $\frac{a_d \Gamma(\frac{3-d}{2})}{\pi^{(d-1)/2}} T^{(d-3)/2}$. For $d \geq 5$ odd, \tilde{F}_d has an asymptotic expansion in terms of LT^k and T^m , with the first asymptotic term being $\frac{a_d}{d! \pi^{(d-1)/2}} LT^{(d-3)/2}$. Also $\tilde{F}_1 = 1/T$ and $\tilde{F}_3 = \frac{3\sqrt{3}}{4\pi} L + f_{3,1} + f_{3,2}LT + \dots$.
- (ii) For $d \geq 1$ odd, \tilde{A}_d has an asymptotic expansion in terms of half-integer and integer powers of T , with the first asymptotic term being $\frac{a_d \Gamma(\frac{2-d}{2})}{\pi^{d/2}} T^{(d-2)/2}$. For $d \geq 4$ even, \tilde{A}_d has an asymptotic expansion in terms of LT^k and T^m , with the first asymptotic term being $\frac{a_d}{d! \pi^{d/2}} LT^{(d-2)/2}$. Also $\tilde{A}_2 = L/\pi + K + a_{2,2}LT + \dots$.
- (iii) For $d \geq 1$ odd, \tilde{B}_d has an asymptotic expansion in terms of half-integer and integer powers of T , with the first asymptotic term being $\frac{a_d \Gamma(\frac{2-d}{2})}{m_d^2 \pi^{d/2}} T^{(d-2)/2}$. For $d \geq 4$ even, \tilde{B}_d has an asymptotic expansion in terms of $L^j T^k$, $k - j \geq d/2 - 2$, and T^m , $k, j, m \geq 1$, with the first asymptotic term being $\frac{a_d}{d! \pi^{d/2} m_d^2} LT^{(d-2)/2}$. For $d = 2$ the asymptotic is in terms of L^{-j} , $j \geq 1$, since they are all higher order (converging much slower to zero at $1/16$) than any $L^j T^k$, $j \geq 0, k \geq 1$.

The fundamental result is Theorem 3.3 and most notably the statements about holonomy and the principal branch, which we will prove rigorously in the next subsection, while we will give a sketch of proof for the rest of the claims. Theorems 3.4 and 3.5 follow from Theorem 3.3 and a few basic considerations, which we will also address in the next subsection. Here we prove Theorems 3.6 and 3.2, given Theorems 3.3, 3.4, and 3.5.

We have explicit computations in dimension 1, since we know the A series exactly to be $1/\sqrt{1-4w}$, so we can compute the B series and hence its coefficients from $(1-B) \cdot A = 1$. When $d = 2$, the random 2-walk is the direct product of two independent one-dimensional random walks, and the full analysis of this fairly difficult

[‡]The order claim is proved in [2, Prop.6.1]. Further, in [3], the conjecture that the order d is minimal is verified till $d = 30$.

case, as the small logarithmic error terms show, is done in [12, Example VI.14]. These remarks justify parts (iii) and (iv) of Theorem 3.2, which we included in the statement for completion, so we prove Theorems 3.2 and 3.6 only in the case $d \geq 3$.

Note that for $d \geq 3$ the random walk we consider is not a product of 1-dimensional walks and we see this reflected in the formulas for $A_{2n}^{(d)}$'s and $B_{2n}^{(d)}$'s, which involve $(2d)^{2n}$ rather than 2^{2nd} . However, due to the similar asymptotic expansion of the Legendre polynomials $(d^2 - 1)^n P_n(\frac{d^2+1}{d^2-1})$ to the one for the binomial coefficient $\binom{2n}{n}$, the asymptotic expansions of $A_{2n}^{(d)}/(2d)^{2n}$ and $B_{2n}^{(d)}/(2d)^{2n}$ are similar to the ones from the d -product of 1-dimensional walks discussed by Flajolet and Sedgewick (Pólya's drunkard problem).

Theorems 3.3 and 3.4 (and their proofs) show inductively that F and A (hence \tilde{F} and \tilde{A}) are analytic functions amenable to singularity analysis in the sense of [12, Ch.VI], so they are not only holonomic, hence Δ -analytic, but also admit an asymptotic representation in terms of the basic functions T, L and their powers which can be determined (up to non-negative powers of T) from the asymptotics of their coefficients.

The coefficients of $\tilde{A}_d(w)$ are asymptotic with

$$\sum_{k \geq 1} a_{d,k} \frac{1}{(\pi n)^{d/2+k-1}}$$

and those of $\tilde{F}_d(w)$ are asymptotic with

$$\sum_{k \geq 1} f_{d,k} \frac{1}{(\pi n)^{(d-1)/2+k-1}},$$

where $f_{d,1} = a_{d,1} = \frac{d^{(d/2)}}{2^{d-1}}$. We know that $\tilde{F}_1(w) = (1-w)^{-1} = T^{-1}$ and $\tilde{F}_2(w) = \tilde{A}_1(w) = (1-w)^{-1/2}$, hence when the coefficients are asymptotic with odd powers of $1/n$ the function is asymptotic to a sum of half integer (and integer) powers of T , and when the dominant term is $n^{q/2}$, $q \geq 3$, the series being convergent, we start with the constant $\tilde{A}_d(1) = m_d$, $d \geq 3$ odd, and $\tilde{F}_d(1) = f_d$, $d \geq 4$ even.

When the powers are even and the dominant term is $1/n$ the series are divergent at 1 and start with L/π for $\tilde{A}_2(w)$, $a_2 = 1$, and $\frac{3\sqrt{3}}{4\pi}L$ for $\tilde{F}_3(w)$, $a_3 = f_3 = \frac{3\sqrt{3}}{4}$, respectively, followed by a sum of T^j , $j \geq 0$, and LT^m , $m \geq 1$. When the dominant term is $1/n^q$, $q \geq 2$, the series are again convergent at 1. This means that they start with a constant m_d or f_d and then we have a sum of T^j , $j \geq 1$, and LT^m , $m \geq q-1 \geq 1$. These comments prove the claims from Theorem 3.6 about the first "true" asymptotic term being $LT^{(d-2)/2}$ for $\tilde{A}_d(w)$ and $LT^{(d-3)/2}$ for $\tilde{F}_d(w)$, respectively.

Since the asymptotic of $\tilde{B}_d(w)$ follows from the relation $(1 - \tilde{B}_d(w)) \cdot \tilde{A}_d(w) = 1$, essentially, by identifying coefficients, we have to do an analysis considering how fast terms converge to 0 at 1, so we have to look at the terms $O(1)$, $o(T)$, $O(T)$, $o(T^2)$, $O(T^2)$, and so on. There are two cases to be considered.

Case 1. When $d \geq 3$ odd (using that $T^{d/2}$ has main asymptotic coefficient $\frac{1}{\Gamma(\frac{2-d}{2})}$),

$$\tilde{A}_d(w) = m_d + \sum_{k \geq 1} \tilde{a}_{d,k} T^{d/2+k-2} + \sum_{j \geq 1} q_{d,j} T^j, \text{ with } \tilde{a}_{d,1} = \frac{a_d \Gamma(\frac{2-d}{2})}{\pi^{d/2}},$$

hence, by identifying coefficients, we get

$$\tilde{B}_d(w) = (1 - 1/m_d) + \sum_{k \geq 1} \tilde{b}_{d,k} T^{d/2+k-2} + \sum_{j \geq 1} \tilde{q}_{d,j} T^j, \text{ with } \tilde{b}_{d,1} = \frac{a_d \Gamma(\frac{2-d}{2})}{\pi^{d/2} m_d^2}.$$

In particular the dominant power of the coefficients of $\tilde{B}_d(w)$ has coefficient $\frac{a_d}{(\pi n)^{d/2} m_d^2}$, which also gives the main claim of Theorem 3.2 for $d \geq 3$ odd, as the error terms are the same as for the A coefficients.

Case 2. When $d \geq 4$ even (using that $LT^{d/2}$ has main asymptotic coefficient $d!$),

$$\tilde{A}_d(w) = m_d + \sum_{k \geq 1} \tilde{a}_{d,k} LT^{d/2+k-2} + \sum_{j \geq 1} q_{d,j} T^j, \text{ with } \tilde{a}_{d,1} = \frac{a_d}{\pi^{d/2} d!}.$$

Here we notice that, when we write $(1 - \tilde{B}_d(w)) \cdot \tilde{A}_d(w) = 1$, the constant and the first true asymptotic term are the same[§] as in the odd case, so

$$\tilde{B}_d(w) = (1 - 1/m_d) + \sum_{j \geq 1} \tilde{q}_{d,j} T^j + \frac{1}{\pi^{d/2} d!} \frac{a_d}{m_d^2} LT^{d/2-1} + \dots$$

[§]We also have powers of T of order less than $LT^{d/2-2}$ for $d \geq 6$ say when we start with LT^m , $m \geq 2$, so have T, \dots, T^{m-1} before that in asymptotic order at 1, but those are of course negligible as the coefficients asymptotic go since they change only the first $m-1$ coefficients.

But now we notice that when we do the term by term multiplication, we start having also terms L^2T^j , L^3T^j , and so on, first coming from the need to cancel the LT^j terms in A multiplied with a LT^k term in B and then of course we need to cancel LT^j terms in A multiplied with the previous L^2T^k in B etc. According to [12, Ch.VI] we know that these terms add logarithmic errors with the logarithmic term a polynomial of degree $m - 1$ and the dominant power of $1/n$ being as usual $k + 1$ in L^mT^k , so in particular for L^2T^j we get a dominant term $\log n/n^{j+1}$. But now since the first asymptotic term is $LT^{d/2-1}$ the first L^2T^j term is L^2T^{d-2} with dominant coefficient term $\log n/n^{d-1}$. On the other hand the second asymptotic term in A is $LT^{d/2}$, hence B also has this term $LT^{d/2}$ with “good” dominant term of order $1/n^{d/2+1}$. It follows that the only case when the L^2T^{d-2} dominates and appears as the main error is when $d = 4$, so $d - 2 = d/2$. In this case, L^2T^2 goes slower to zero at 1 than LT^2 , so it is of lower order and appears first in the asymptotic expansion, and its main term $\log n/n^3$ becomes the main error here. For $d \geq 6$, so satisfying $d/2 < d - 2$, the term $LT^{d/2}$ remains the second asymptotic term, so its dominant term $1/n^{d/2+1}$ is still the main error in the asymptotic for the coefficients B , and with this Theorem 3.6 is proved.

Theorem 3.2 just restates the functional asymptotics of \tilde{B} from Theorem 3.6 into asymptotics for the coefficients, so the shape of the coefficients is the same. The explicit computations follow from simple algebraic manipulations using our earlier result that $a_1(d) = -d/8$, noting that L^2T^k has main asymptotic coefficient $2k! \log n/n^{k+1}$, and the fact that for $d = 3$ we have the explicit computation from [19], where the coefficients of $T^{1/2}$, $T^{3/2}$ follow from our considerations above too, but the coefficient of T follows only from the method of the cited paper:

$$\tilde{A}_3 = m_3 - \frac{3\sqrt{3}}{2\pi}T^{1/2} + \frac{9}{32}\left(m_3 + \frac{6}{\pi^2m_3}\right)T - \frac{3\sqrt{3}}{4\pi}T^{3/2} + O(T^2).$$

With this, the proofs of Theorem 3.2 and Theorem 3.6 are completed.

3.4 Analytic Proofs

Lemma 3.4. *The following identities hold for $|w| < 1$:*

$$\sum_{m=0}^{\infty} \binom{n+m}{m}^2 w^m = (1-w)^{-2n-1} \sum_{k=0}^n \binom{n}{k}^2 w^k = (1-w)^{-1}(w-1)^{-n} P_n\left(\frac{w+1}{w-1}\right).$$

Proof. The first equality can be found in [34, 35]. The last equality follows from our earlier formula (6) and the fact that $P_n(-x) = (-1)^n P_n(x)$. \square

Hence when we substitute the recurrence $x_n^{(d+1)} = \sum_{k=0}^n \binom{n}{k}^2 x_k^{(d)}$ in the series for F_{d+1} , we obtain from Lemma 3.4 that:

$$F_{d+1}(w) = \sum_{n \geq 0} \left(x_n^{(d)} (1-w)^{-1} (w-1)^{-n} P_n\left(\frac{w+1}{w-1}\right) \right) w^n, \text{ for } |w| < 1/(d+1)^2. \tag{13}$$

Denoting $u = \frac{w}{w-1}$, such that $(w-1)(u-1) = 1$ and $\frac{w+1}{w-1} = 2u-1$, the equation (13) becomes:

$$F_{d+1}(w) = (1-u) \sum_{n \geq 0} (x_n^{(d)} P_n(2u-1) u^n), \text{ for } |w| < 1/(d+1)^2. \tag{14}$$

By the Legendre polynomials generating function definition, we recall that

$$G(t, u) = \sum_{n \geq 0} P_n(u) t^n = (1 - 2tu + t^2)^{-1/2}.$$

Writing $P_n(2u-1) = \sum_{m \geq 0} q_{m,n} u^m$, we consider two power series in two variables (convergent on a small neighborhood of $(0, 0)$):

$$\begin{aligned} V_1(t, u) &= \sum_{m, n \geq 0} x_n^{(d)} t^n u^m = \frac{1}{1-u} F_d(t), \\ V_2(t, u) &= \sum_{m, n \geq 0} q_{m,n} t^n u^m = \sum_{n \geq 0} P_n(2u-1) t^n = G(t, 2u-1). \end{aligned}$$

With this notation, Equation (14) can then be written as

$$(1-w)F_{d+1}(w) = (V_1 * V_2)(t, u)|_{t=u} = \left(\frac{1}{1-u} F_d(t)\right) * G(t, 2u-1)|_{t=u}, \text{ for } w = \frac{u}{u-1}. \tag{15}$$

Since the kernel G is algebraic and the two-variable convolution evaluation is at the linear relation $t = u$, we have at our disposal the powerful convolution theorems for holonomic functions ([32], [22]). In particular since $F_1(w) = \frac{1}{1-w}$ is rational, we get inductively ([22, Thm.2.7, Prop.2.3]) that $F_d(w)$ is holonomic for all $d \geq 2$. Also since $\frac{1}{1-t}$ is the convolution unit, we get that $F_2(w) = \frac{1}{1-w} G(\frac{w}{w-1}, 2\frac{w}{w-1} - 1) = (1-4w)^{-1/2}$, as expected. Similarly, since $A_1 = F_2$ and $A_d(w) = F_d(w) * A_1(w) = F_d(w) * F_2(w)$, for $d \geq 1$, we get inductively that A_d is holonomic for all $d \geq 1$.

Using again that $\frac{1}{1-u}$ is the convolution unit (now with respect to u), we can explicit the formula (15), for u sufficiently small, as:

$$(1-w)F_{d+1}(w) = \frac{1}{2\pi i} \int_{|\zeta|=s_d} F_d(\frac{1}{\zeta}) G(\zeta u, 2u-1) \frac{d\zeta}{\zeta}, \text{ for } s_d > d^2, w = \frac{u}{u-1}. \tag{16}$$

Changing variables back to w and $t = \frac{1}{\zeta}$ we can also write

$$F_{d+1}(w) = \frac{1}{2\pi i} \int_{|t|=1/s_d} F_d(t) P(t, w)^{-1/2} dt, \text{ for } P(t, w) = w^2 - wt(2w+2) + t^2(w-1)^2. \tag{17}$$

Next we will use the method of Hadamard ([4, Article 88]) to analyze inductively the singularities of F_{d+1} in terms of those of F_d and $P^{-1/2}$ by deforming the circle $|t| = 1/s_d$ in such a way that the singularities of $F_d(t)$ stay outside all the deformation curves, while the singularities of $P^{-1/2}$ stay inside.

We first notice that $P(w, t) = P(t, w)$ and P has roots

$$t_1(w) = \frac{w}{(\sqrt{w}+1)^2} \text{ and } t_2(w) = \frac{w}{(\sqrt{w}-1)^2}, \tag{18}$$

where changing the determination of the square root just switches them. Also $t_1(0) = t_2(0) = 0$, $t_1(\infty) = t_2(\infty) = 1$, $t_1(1) = \frac{1}{4}$, $t_2(1) = \infty$, in the usual limit sense in the latter cases (i.e., as $w \rightarrow 1$ the roots $t_1(w) \rightarrow \frac{1}{4}$, $t_2(w) \rightarrow \infty$). It is the case that $t_{1,2}(w) \in [0, \infty]$ if and only if $w \in [0, \infty]$.

Given a Jordan curve $t \rightarrow \Gamma(t)$ in the (extended) t plane and taking w in the (extended) w plane, the only condition for a continuous (hence analytic) determination of $t \rightarrow P^{-1/2}(t, w)$ is that the zeroes $t_1(w)$, $t_2(w)$ are not separated by Γ (and of course are not on Γ either). Then, by fixing an initial determination at some w_0 , such a determination can be chosen jointly continuous on any $\Gamma \times U$, where $U \ni w_0$ is a connected open set in the w plane with the property above. In particular, if we can also choose a continuous determination of $F_d(t)$ on Γ (where we proceed inductively to define F_d as a multivalued analytic function starting from the given power series around zero), the corresponding formula

$$F_{d+1, \Gamma}(w) = \frac{1}{2\pi i} \int_{\Gamma} F_d(t) P(t, w)^{-1/2} dt, \text{ for } P(t, w) = w^2 - wt(2w+2) + t^2(w-1)^2 \tag{19}$$

defines an analytic function for w in the open set for which $t_{1,2}$ are not separated by Γ . However, if Γ doesn't separate at least a singularity of F_d from the zeroes of P , a simple deformation argument shows that for such w the integral is zero, hence for any Γ the set of interest U_Γ is defined to contain only those w for which Γ separates the singularities of F_d from the zeroes $t_{1,2}(w)$. In general U_Γ is an open but possibly disconnected set and $0 \in U_\Gamma$ precisely when Γ separates 0 from the singularities of F_d (since $t_{1,2}(0) = 0$). In particular, this means that 0 is inside Γ by the inductive description of those singularities. Clearly any w for which $t_{1,2}(w)$ is not a singularity of F_d will belong to such a U_Γ since F_d has only finitely many singularities and we claim that for any $w \neq 1, \infty$ we can find Γ so that $0 \in U_\Gamma$ and w is in the connected component of U_Γ containing 0. This then gives us an analytic continuation of F_{d+1} from the power series at zero (given also by (17)) to w , as we can choose the circle around 0 in (17) to be small enough so is contained inside Γ , and then Cauchy theorem immediately implies that

$$\int_{|t|=1/s_d} F_d(t) P(t, w)^{-1/2} dt = \int_{\Gamma} F_d(t) P(t, w)^{-1/2} dt,$$

for $|w|$ small enough, so $F_{d+1}(w) = F_{d+1, \Gamma}(w)$ for $|w|$ small enough.

The claim about singularities follows (inductively) since for any $w_0 \notin [0, \infty]$ we have $t_{1,2}(w_0) \notin [0, \infty]$, so taking a small tubular neighborhood T_{w_0} of the segment $[0, w_0]$ that stays away from the positive half axis except near 0, we have that $\{t_{1,2}(w) | w \in T_{w_0}\}$ is a compact set that stays away from the positive half axis except for a small neighborhood of zero, while the singularities of F_d are inductively on $[0, \infty]$ and at a positive distance from zero, so we can separate the two sets by a Jordan curve Γ .

Let $r \in [0, \infty)$, $r \neq 1$, be such that $t_1(r)$, $t_2(r)$ are not among the inductively assumed (finite) singularities of F_d , so not among $\frac{1}{k^2}$, for $1 \leq k \leq d$ with k and d of same parity. We consider T_r a small tubular neighborhood of the curve formed by the segments $[0, ir \sin \epsilon]$, $[ir \sin \epsilon, ir \sin \epsilon + r \cos \epsilon]$ and the arc $[re^{i\epsilon}, r]$. For $r < 1$ we

choose $\epsilon < 0$ such that $\rho < \cos \epsilon$ for all $\rho > 0$ in the tubular neighborhood, while for $r > 1$ we can choose any $\epsilon < \pi/2$ and the tubular neighborhood so that all $\rho > 0$ near r in it also satisfy $\rho > 1$. A quick computation using (18) shows that t_1, t_2 stay on one side of the real axis when we move along this tubular neighborhood towards r , so we can still separate the compact set $t_{1,2}(T_r)$ from the singularities $\frac{1}{k^2}$, $1 \leq k \leq d$, as there is no closed loop in $t_{1,2}(T_r)$ encircling one of those. This reduces the inductive determination of the singularities of F_{d+1} to showing that $t_{1,2}(r)$ belongs to the set of singularities of F_d precisely when r is in the claimed set of singularities of F_{d+1} and dealing with the cases $w = 1$ and $w = \infty$. The first part is easy since $\frac{r}{(\sqrt{r \pm 1})^2} = \frac{1}{k^2}$ if and only if $r = \frac{1}{(k+1)^2}$ or $r = \frac{1}{(k-1)^2}$ (with the usual convention that $1/0 = \infty$), so at least outside of the cases 1 and ∞ we indeed get inductively that the singularities of F_{d+1} are included in the set $\frac{1}{k^2}$, $1 \leq k \leq d+1$, with k and $d+1$ of the same parity.

A closer analysis of the quadratic polynomial root decomposition, induction, and the analysis in [12, VI] show that indeed those are singularities of the same type as the main singularity at $1/(d+1)^2$, so in particular of finite order. This analysis also shows that if we restrict ourselves to the principal branch of F_d defined (for $d \geq 2$) on the plane minus the half line $[\frac{1}{d^2}, \infty]$, we can define inductively F_{d+1} single valued and analytic on $[\frac{1}{(d+1)^2}, \infty]$.

The only issue that remains in the inductive analysis of the singularities of F_{d+1} is what happens when $w = 1$ and $w = \infty$, since we have $t_1(\infty) = t_2(\infty) = 1$. In the $d \geq 2$ even case, when 1 is not a singularity of F_d , we can first define an analytic function around infinity by the corresponding formula (17) but on a small circle centered at 1, so

$$F_{d+1,1}(w) = \frac{1}{2\pi i} \int_{|t-1|=\epsilon} F_d(t)P(t, w)^{-1/2} dt, \text{ for } |w| > R_d.$$

Now there exists a curve Γ such that 0 and ∞ belong to U_Γ , but a simple topological argument shows that unfortunately 0 and ∞ will never be in the same connected component of U_Γ . Hence the corresponding integral (19) will define two separate analytic functions, our $F_{d+1}(w)$ continuing the power series from 0 on the component of U_Γ containing 0 and $F_{d+1,1}(w)$ continuing the power series from ∞ on the component of U_Γ containing ∞ . Taking curves Γ containing zero inside and the singularities of F_{d+1} outside and Γ_1 containing the singularities of $F_{d+1,1}$ inside, one defines F_{d+2} and $F_{d+2,1}$, respectively, at say any non real positive complex number and then by representing those as double integrals and deforming appropriately, one can show that actually the two integrals define the same function there. Since as above, all points on the extended positive real axis except $1/k^2$, $1 \leq k \leq d+2$, k even, 1 and ∞ can be reached by the first construction and all points on the extended positive real axis except $1/k^2$, $1 \leq k \leq d+2$, k even, 0 and ∞ can be reached by the second construction, we obtain that F_{d+2} can indeed be extended as a multivalued analytic function to all points except $1/k^2$, $1 \leq k \leq d+2$, k even, and ∞ , and all its singularities are Fuchsian (and of finite order). Note that the analysis before shows the logarithmic nature of the singularity at infinity for all $d \geq 3$. In particular, this implies that $1/F$ can never be holonomic for any $d \geq 3$.

Since F_d is holonomic and with singularities of finite order, it also satisfies a Fuchsian ODE as discussed in the previous section, so Theorem 3.3 is proved. We note in Section 4 that indeed F_2, F_3, F_4 , and F_5 satisfy ODEs of order $d-1$ so we conjecture that this holds for all $d \geq 6$.

Theorem 3.4 is a simple consequence of the defining relation $A_d(w) = F_d(w) * F_2(w) = F_d(w) * A_1(w)$ and of Theorem 3.3. This is because the singularities of F_d are finite in number while the singularities of $F_2(w) = A_1(w) = 1/\sqrt{1-4w}$ are $1/4$ and ∞ , so the finite singularities of the convolution are included in the products of those of F_d and $1/4$. An analysis similar to that in [12, VI] shows that all of these will be of the same type as the main singularity, since the ones of F_d are so. At infinity, in the odd case, we will get only half-integer powers, as for the finite singularities. In the even case, we also get half-integer powers, in addition to the expected logarithmic terms. This happens as in the A_2 case and with the same type of analysis at large arguments of the convolution of functions that behave like half-integral powers at their finite singularities. A_1, A_2, A_3, A_4 , and A_5 satisfy ODEs of order d (see the end of Section 4) and the fact that this holds for all d was proved in [2, Prop.6.1].

Note that by the theory of Fuchsian ODEs [18], F_{2d} and A_{2d+1} actually have Puiseux expansions in fractional (half integral) powers of $z-w_0$ at all finite singularities, that hold in angular domains near the singularity, while F_{2d+1} and A_{2d} have similar expansions with logarithmic terms instead of fractional powers. So the asymptotic expansions of \hat{F} and \hat{A} from Theorem 3.6 are actually equalities in angular domains at 1 and extend to similar expansions in the neighborhoods of all the finite singularities and infinity (with the appropriate of an angular domain there).

From the relation $(1-B_d)A_d = 1$ and the observations that $A_d(1/(2d)^2) = m_d$, with $m_d = \infty$, for $d = 1, 2$, and $m_d > 0$, for $d \geq 3$, and that B_d has positive coefficients and it converges uniformly on the circle of convergence, for all $d \geq 1$ with $|B_d(w)| \leq B_d(|w|) = p_d \leq 1$, we conclude that $A_d \neq 0$ on the circle of convergence. All these imply that $B_d = 1 - 1/A_d$ is Δ -analytic too, while the rest of the claims of Theorem 3.5 are clear from the definition of B , except for non-holonomy, which will be shown in the next subsection. That

is not surprising since A is holonomic but not algebraic for $d \geq 2$ (as will be shown also in the next subsection), so $1/A$ (hence B) is then usually non-holonomic.

4. ODEs and P -recursion

We prove that the x and A sequences are generally transcendental over \mathbb{C} , beyond the trivial low-dimensional explicit examples, so in particular also F and A are transcendental analytic functions, but they are algebraic (in the sense of generating formal power series discussed earlier) modulo any prime. This is not that surprising considering that $\binom{2n}{n}^2$ (which is $A_n^{(2)}$) satisfies this property ([32, Exercise 6.3]). On the other hand, the B sequences are not holonomic, as noted earlier. We also record at the end of this section the recursions and ODEs for a few low-dimensional cases.

In the preprint form of this paper [8], we observed that the above statements, except the non-holonomy of B_{2d+1} , $d \geq 1$, follow from earlier considerations and known results in the literature. We proved the B_{2d+1} case by exploiting the fact that the sequences x and A satisfy the Lucas property. We thank Alin Bostan for suggesting the more direct proof given below, which does not require separate treatment of the odd and even cases.

Here is the main result of this section:

Theorem 4.1. (i) For $d \geq 3$, $(x_n^{(d)})_n$ is a P -recursive sequence, and its generating function $F_d(x)$ is transcendental over $\mathbb{C}(x)$, but algebraic modulo p prime.

(ii) For $d \geq 2$, $(A_{2n}^{(d)})_n$ is a P -recursive sequence, and its generating function $A_d(x)$ is transcendental over $\mathbb{C}(x)$, but algebraic modulo p prime.

(iii) For $d \geq 2$, $(B_{2n}^{(d)})_n$ is not a P -recursive sequence.

Proof. We showed in Theorems 3.3 and 3.4, respectively, that the power series F_d and A_d generated by $(x_n^{(d)})_n$ and $(A_{2n}^{(d)})_n$ are holonomic, which is equivalent with the P -recursivity of the given sequences [32, Ch.6]. Also, the algebraicity modulo any prime p of the sequences follows immediately by induction from the convolution representations of the generating series F and A from the previous section, since the convolution kernels G and $F_2 = A_1$ are algebraic ([14, 29]).

Since the singularities of algebraic functions (over \mathbb{C}) cannot contain logarithms, dilogarithms etc. (so the normalized coefficients of the power series at the radius of convergence singularity cannot be asymptotic with negative integral powers of n), it also follows that F_{2d+1} and A_{2d} , $d \geq 1$, cannot be algebraic over \mathbb{C} from the results of the previous section. Similarly, using that holonomic functions cannot have arbitrarily high logarithmic powers or negative logarithmic powers in their asymptotic formulas, it follows from Theorem 3.6 (iii) that B_{2d} , $d \geq 1$, cannot be holonomic.

For the case of half odd integral powers asymptotics (F_{2d} , $d \geq 2$, and A_{2d+1} , $d \geq 1$) transcendence over \mathbb{C} follows from arithmetic considerations precisely as in [11], since the integral nature of the coefficients ensure that if one of the corresponding F_{2d} , $d \geq 2$, or A_{2d+1} , $d \geq 1$, were algebraic over $\mathbb{C}[x]$, it would actually be algebraic over $\mathbb{Q}[x]$ by Eisenstein Lemma, hence the Puiseux expansion at the radius of convergence singularity would have algebraic coefficients. However, the leading asymptotic coefficient is transcendental as we see in Theorem 3.6, parts (i) and (ii). Note that the coefficients $\frac{a_2 \Gamma(\frac{3-2}{2})}{\pi^{(2-1)/2}} = 1$ for F_2 and $\frac{a_1 \Gamma(\frac{2-1}{2})}{\pi^{1/2}} = 1$ are indeed algebraic as those are the only cases when the Γ term cancels the power of π in the denominator for either F or A and that of course corresponds to the fact that $F_2 = A_1$ is indeed algebraic. This completes the proof of parts (i) and (ii).

For part (iii), assume that B_d is holonomic. This implies that $1/A_d$ is holonomic. Now both A_d and B_d are G -functions. (See [1] for G -functions. The growth conditions of the coefficients of a G -function is ensured by Remark 3.5 and Theorem 3.2, respectively.) It follows that A_d is a unit in the algebra of all G -functions, which means that A_d is algebraic, based on [1, SCHOLIE, p.123]. This is a contradiction with part (i) of the theorem. It follows that the B_d 's are not holonomic for $d \geq 2$. \square

We conclude with some remarks that illustrate the use of the theorems from the paper in obtaining insight in the shape of the recurrences and of the ODEs that the sequences x and A satisfy.

Remark 4.1. We were able to derive some results that are available in the literature about the sequences $(x_n^{(d)})_n$. The case $d = 3$ is A002893, $d = 4$ is A002895, and $d = 5$ is A169714, all from [30], though here all the sequences appear in a unified context (not mentioned in Sloane). The P -recurrences and the differential equations are given without proof in [15], where staircase polygons were studied. We obtained the P -recurrences by guess-and-check using an intuition based on the singularities of F_d 's and the structure of the convolution (see equation (17)).

(i) The P -recurrences for $d = 3, 4$, and 5 , respectively,[¶] are:

$$\begin{aligned} (n+2)^2 x_{n+2}^{(3)} - (10n^2 + 30n + 23)x_{n+1}^{(3)} + 9(n+1)^2 x_n^{(3)} &= 0; \\ (n+2)^3 x_{n+2}^{(4)} - 2(2n+3)(5n^2 + 15n + 12)x_{n+1}^{(4)} + 64(n+1)^3 x_n^{(4)} &= 0; \\ (n+3)^4 x_{n+3}^{(5)} - (35n^4 + 350n^3 + 1323n^2 + 2240n + 1433)x_{n+2}^{(5)} \\ + (n+2)^2(259n^2 + 1036n + 1062)x_{n+1}^{(5)} - 3^2 \cdot 5^2 \cdot (n+1)^2(n+2)^2 x_n^{(5)} &= 0. \end{aligned}$$

(ii) The ODEs satisfied by F_3, F_4 and F_5 , respectively, are given below^{||} and their P -symbols are consistent with the asymptotics obtained in the previous subsections. We obtained the ODEs using the knowledge about the singularities (Theorem 4), the structure of Fuchsian differential equations, and the P -recurrences above.

$$\begin{aligned} z(z-1)(9z-1)F''(z) + (27z^2 - 20z + 1)F'(z) + 3(3z-1)F(z) &= 0; \\ z^2(4z-1)(16z-1)F'''(z) + 3z(128z^2 - 30z + 1)F''(z) \\ + (448z^2 - 68z + 1)F'(z) + 4(16z-1)F(z) &= 0. \\ z^3(z-1)(9z-1)(25z-1)F^{(4)}(z) + z^2(2700z^3 - 2590z^2 + 280z - 6)F'''(z) \\ + z(8550z^3 - 6501z^2 + 518z - 7)F''(z) + (7200z^3 - 3963z^2 + 196z - 1)F'(z) \\ + 5(180z^2 - 57z + 1)F(z) &= 0. \end{aligned}$$

Remark 4.2. These recurrences and ODEs can also be proved algorithmically using the fact that $x_n^{(d)}$ is equal to the constant term in the expansion of the Laurent polynomial $\phi(t_1, \dots, t_d) = (t_1 + \dots + t_d)^n \cdot (t_1^{-1} + \dots + t_d^{-1})^n$, together with computer algebra algorithms (via “creative telescoping”).

Remark 4.3. We were also able to derive some results about the sequences $(A_n^{(d)})_n$. The case $d = 3$ is A002896, $d = 4$ is A039699, and $d = 5$ is A287317, all from [30]. The P symbols are consistent with the asymptotics obtained in the previous subsections. After a change of variables, the ODEs for $d = 4$ and $d = 5$ become exactly the respective explicit L operators from [3, page 5].

(i) The P -recurrences for $d = 3, 4$, and 5 , respectively,^{**} are:

$$\begin{aligned} (n+2)^3 A_{n+2}^{(3)} - 2(2n+3)(10n^2 + 30n + 23)A_{n+1}^{(3)} + 36(2n+3)(2n+1)(n+1)A_n^{(3)} &= 0; \\ (n+2)^4 A_{n+2}^{(4)} - 4(2n+3)^2(5n^2 + 15n + 12)A_{n+1}^{(4)} + 256(2n+3)(2n+1)(n+1)^2 A_n^{(4)} &= 0; \\ (n+3)^5 A_{n+3}^{(5)} - 2(2n+5)(35n^4 + 350n^3 + 1323n^2 + 2240n + 1433)A_{n+2}^{(5)} \\ + 4(2n+5)(2n+3)(259n^3 + 1554n^2 + 3134n + 2124)A_{n+1}^{(5)} \\ - 1800(2n+5)(2n+3)(2n+1)(n+1)(n+2)A_n^{(5)} &= 0. \end{aligned}$$

(ii) The ODEs satisfied by A_3, A_4 and A_5 , respectively,^{††} are:

$$\begin{aligned} z^2(4z-1)(36z-1)A'''(z) + 3z(288z^2 - 60z + 1)A''(z) \\ + (972z^2 - 132z + 1)A'(z) + 6(18z-1)A(z) &= 0. \\ z^3(16z-1)(64z-1)A^{(4)}(z) + 2z^2(5120z^2 - 320z + 3)A'''(z) \\ + z(25344z^2 - 1172z + 7)A''(z) + (14592z^2 - 424z + 1)A'(z) + 8(96z-1)A(z) &= 0. \\ z^4(4z-1)(36z-1)(100z-1)A^{(5)}(z) \\ + 10z^3(1800 \cdot 14z^3 - 6216z^2 + 175z - 1)A^{(4)}(z) \\ + z^2(1800 \cdot 730z^3 - 268740z^2 + 5992z - 25)A'''(z) \\ + z(1800 \cdot 1275z^3 - 369240z^2 + 5964z - 15)A''(z) \\ + (1800 \cdot 600z^3 - 124020z^2 + 1196z - 1)A'(z) \\ + 10(1800 \cdot 3z^2 - 342z + 1)A(z) &= 0. \end{aligned}$$

[¶]When $d = 1$, the recurrence $x_{n+1}^{(1)} - x_n^{(1)} = 0$ is reducible and the true “recurrence” is $x_n^{(1)} = 1$. When $d = 2$, the recurrence is $(n+1)x_{n+1}^{(2)} - 2(2n+1)x_n^{(2)} = 0$.

^{||}When $d = 1$, the ODE $(z-1)F' + F = 0$ is reducible and the true “ODE” is $(z-1)F + 1 = 0$. When $d = 2$, it is $(4z-1)F' + 2F = 0$.

^{**}When $d = 1$, $(n+1)A_{n+1}^{(1)} - 2(2n+1)A_n^{(1)} = 0$. When $d = 2$, $(n+1)^2 A_{n+1}^{(2)} - 4(2n+1)^2 A_n^{(2)} = 0$.

^{††}When $d = 1$, the ODE is $(4z-1)A' + 2A = 0$. When $d = 2$, it is $z(16z-1)A'' + (32z-1)A' + 4A = 0$.

Remark 4.4. *These recurrences and ODEs can also be proved algorithmically using the fact that $A_{2n}^{(d)}$ is equal to the constant term in the expansion of the Laurent polynomial $\phi(t_1, \dots, t_d) = (t_1 + t_1^{-1} + \dots + t_d + t_d^{-1})^{2n}$, together with computer algebra algorithms (via “creative telescoping”).*

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