ecajournal.haifa.ac.il **Enumerative Combinatorics and Applications**

ECA **5:3** (2025) Article #S2R21
https://doi.org/10.54550/ECA2025V5S3R21

Euler Characteristics and Duality in Riemann Functions and the Graph Riemann-Roch Rank

Nicolas Folinsbee and Joel Friedman

Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2, CANADA
Email: nicolasfolinsbee@gmail.com, jf@cs.ubc.ca

Received: July 25, 2024, **Accepted**: July 14, 2025, **Published**: August 15, 2025

Abstract: By a *Riemann function* we mean a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $f(\mathbf{d}) = f(d_1, \ldots, d_n)$ is equals 0 for $\deg(\mathbf{d}) = d_1 + \cdots + d_n$ sufficiently small, and equals $\deg(\mathbf{d}) + C$ for a constant, C, for $\deg(\mathbf{d})$ sufficiently large. For such an f, for any $\mathbf{K} \in \mathbb{Z}^n$ there is a unique Riemann function $f^{\wedge}_{\mathbf{K}}$ such that for all $\mathbf{d} \in \mathbb{Z}^n$ we have

$$f(\mathbf{d}) - f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C$$

which we call a *generalized Riemann-Roch formula*. Our motivation for this definition is that (1) adding 1 to the Baker-Norine rank function of any graph yields a Riemann function; and (2) for the results below, we need to consider non-negative valued functions f.

We demonstrate a class of Riemann functions $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ that are modeled by sheaves, $\mathcal{M}_{\mathbf{d}}$ with $\mathbf{d} \in \mathbb{Z}^2$ over a finite topological space, that models the associated generalized Riemann-Roch formula as expressing the Euler characteristic: the nonzero Betti numbers of $\mathcal{M}_{\mathbf{d}}$ are the zeroth and first, which respectively equal $f(\mathbf{d})$ and $f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d})$. The sheaves $\mathcal{M}_{\mathbf{d}}$ satisfy many properties akin to the sheaves that model the classical Riemann-Roch formula as expressing an Euler characteristic.

Any Riemann function $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ can be written as the difference of two functions modeled by sheaves, so that the generalized Riemann-Roch formula of f is modeled as an Euler characteristic formula of a family, $\{\mathcal{M}_{\mathbf{d}}\}_{\mathbf{d}\in\mathbb{Z}^2}$, of virtual (i.e., a formal difference of) sheaves. We do the same for any Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ with $n \geq 2$, by restricting any $n-2$ of its variables, and varying the remaining two variables. We show that the resulting family of virtual sheaves obtained, $\{\mathcal{M}_{\mathbf{d}}\}_{\mathbf{d}\in\mathbb{Z}^n}$, are—up to isomorphism—independent of all the choices made.

Keywords: Betti numbers; Euler characteristic; Cohomology; Duality; Graph Riemann-Roch theorem; Riemann function; Riemann's theorem
2020 Mathematics Subject Classification: 05C99; 55N99; 14H55

1 Introduction

The main goal of this article is to develop a way to understand a large class of what we call *generalized Riemann-Roch formulas* as formulas that express an Euler characteristic of a certain *sheaves* of vector spaces; we later show that such sheaves satisfy a property akin to Serre duality for line bundles on curves.

This article does not assume any prior knowledge of sheaf theory. In fact, we mostly speak of k-*diagrams*, where k is an arbitrary field, which is a structure of five k-vector spaces with some linear transformations between them. We mention sheaves only in the last section of this article, where we explain the connection of k-diagrams and their invariants to sheaf theory.

This article was motivated by the question of Baker-Norine [5] as to whether their "graph Riemann-Roch formula" can be viewed as such an Euler characteristic formula. However, our main results apply to any such formula that arises from a much wider and simpler class of functions that we call *Riemann functions*. Roughly speaking, our main result says is better that any such formula can be modeled as such, provided that (1) one is willing to work with "formal differences" of k-diagrams (or sheaves), and (2) one is willing to make a number of ad hoc choices in building the model (which we will prove do not change the equivalence class of the formal difference of k-diagrams). We therefore view this article as a first step in modeling Riemann-Roch formulas, that we hope will ultimately lead to better—meaning simpler and less ad hoc—models of Riemann-Roch formulas. Beyond this, the foundations we develop to construct our models have a number of interesting byproducts.

We emphasize that the main results in this article do not assume any prior knowledge beyond some basic combinatorics and linear algebra. We do not assume the reader is familiar with the Baker-Norine formula for any of our main results. However, some examples we use to illustrate our theorems—which are not essential to their statements or proofs—are chosen from the Baker-Norine formula for graphs and related formulas; hence, we briefly describe the Baker-Norine formula and similar formulas. We do not assume any familiarity with sheaf theory (either on graphs, as in [13], or in the classical setting) or with the Riemann-Roch formula; however, our techniques mimic ideas from there, and we briefly discuss these connections in Section 10.

At this point, let us summarize our main results, using notation that is common in the literature and made precise starting in the next section.

1.1 Riemann Functions

We use $\mathbb{Z}$ to denote the integers, and $\mathbb{N} = \mathbb{Z}_{\geq 1} = \{1, 2, \ldots\}$ for the natural numbers. For $n \in \mathbb{N}$ we use $[n]$ to denote $\{1, \ldots, n\}$. For $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, the *degree* of $\mathbf{d}$ is defined as $\deg(\mathbf{d}) = d_1 + \cdots + d_n$, and endow $\mathbb{Z}^n$ with its usual partial order, writing $\mathbf{d}' \leq \mathbf{d}$ to mean $d'_i \leq d_i$ for all $i \in [n]$.

By a *Riemann function* we mean a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ such that:

1. $f(\mathbf{d}) = 0$ for $\deg(\mathbf{d})$ sufficiently small, and
2. for some $C \in \mathbb{Z}$—called the *offset of* f—we have $f(\mathbf{d}) = \deg(\mathbf{d}) + C$ for $\deg(\mathbf{d})$ sufficiently large.

If so, setting $h\colon \mathbb{Z}^n \to \mathbb{Z}$ to be the function given by

$$h(\mathbf{d}) \stackrel{\text{def}}{=} \deg(\mathbf{d}) + C,$$

we have that for each $\mathbf{K} \in \mathbb{Z}^n$ the function $f^{\wedge}_{\mathbf{K}}\colon \mathbb{Z}^n \to \mathbb{Z}$ defined by

$$f^{\wedge}_{\mathbf{K}}(\mathbf{d}) \stackrel{\text{def}}{=} f(\mathbf{K} - \mathbf{d}) - h(\mathbf{K} - \mathbf{d}) \tag{1}$$

satisfies

$$f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) \stackrel{\text{def}}{=} f(\mathbf{d}) - h(\mathbf{d}),$$

and therefore

$$\forall \mathbf{d} \in \mathbb{Z}^n, \quad f(\mathbf{d}) - f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = h(\mathbf{d}) = \deg(\mathbf{d}) + C; \tag{2}$$

moreover, (1) easily implies that $f^{\wedge}_{\mathbf{K}}$ is also a Riemann function (with offset $-\deg(\mathbf{K}) - C$; see Proposition 2.1). We refer to the above formula as a *generalized Riemann-Roch formula for* f. We say that (2) or f is *self-dual* if $f^{\wedge}_{\mathbf{K}} = f$.

The point of articles such as [2, 5] is to study certain Riemann functions of interest, f, and determine if such $f = f^{\wedge}_{\mathbf{K}}$ for some $\mathbf{K}$. Our approach may seem a bit "happy-go-lucky," in that we develop combinatorics and models for any Riemann-Roch formula, whether or not self-duality holds. However, as we explain below, self-duality is not preserved under *restrictions*—which is how we build our models—and hence we will be forced to consider Riemann-Roch formulas without self-duality.

The motivating example for us is that if $G = (V, E)$ is a graph with an ordered vertex set $V = \{v_1, \ldots, v_n\}$, then Baker-Norine [5] defined the *rank*, a function $r_{\mathrm{BN},G}\colon \mathbb{Z}^n \to \mathbb{Z}$; it is a consequence of the Baker-Norine *graph Riemann-Roch formula* there that $1 + r_{\mathrm{BN},G}$ is a Riemann function. There is a large literature on these and related functions [2–5, 8], which is strongly related to *chip firing games* and the *sandpile model*; see [4, 8] and the references there for more historical context. Although we have organized this article primarily for the reader interested in the Baker-Norine rank and related functions, our results do not require any knowledge of such functions. Our motivation for the term *Riemann function* is the classical *Riemann's theorem* for curves.

1.2 Weights and Models for Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ that are Perfect Matchings

We model Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ by starting with a particularly simple case of functions, related to what we call *perfect matchings*. To describe this case, we note that for each Riemann function $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ there is a unique function $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ such that for all $\mathbf{d} \in \mathbb{Z}^2$ we have

$$f(\mathbf{d}) = \sum_{\mathbf{d}' \leq \mathbf{d}} W(\mathbf{d});$$

we call W a *weight function* or *the weight of* f (Subsection 2.5) furthermore, $W(\mathbf{d}) \geq 0$ for all $\mathbf{d}$ holds if and only if there is a bijection $\pi\colon \mathbb{Z} \to \mathbb{Z}$ such that $W(i, j) = 1$ if $j = \pi(i)$, and otherwise $W(i, j) = 0$. In this case, we call W a *perfect matching*.

If W is a perfect matching, then the formula (2) can be viewed as an *Euler characteristic* in a natural way: namely, we will define a family of k-diagrams (which are essentially sheaves of k-vector spaces on a fixed diagram) $\{\mathcal{M}_{W,\mathbf{d}}\}_{\mathbf{d} \in \mathbb{Z}}$ indexed on $\mathbf{d} \in \mathbb{Z}^2$ such that:

1. for all $\mathbf{d} \in \mathbb{Z}^2$, $\mathcal{M}_{W,\mathbf{d}}$ has *Betti numbers*, $b^i(\mathcal{M}_{W,\mathbf{d}})$, which vanish except for $i = 0, 1$;
2. for all $\mathbf{d} \in \mathbb{Z}^2$,
$$f(\mathbf{d}) = b^0(\mathcal{M}_{W,\mathbf{d}}), \tag{3}$$
and
3. for all $\mathbf{d} \in \mathbb{Z}^2$, and any $\mathbf{K} \in \mathbb{Z}^2$,
$$f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = b^1(\mathcal{M}_{W,\mathbf{d}}) \tag{4}$$
(the left-hand-side is independent of $\mathbf{K}$).

We will explain what this means, and we assume no knowledge of sheaf theory or Betti numbers. Defining the *Euler characteristic*, $\chi(\mathcal{M})$ of a k-diagram (or sheaf), $\mathcal{M}$, as usual, i.e., as the alternating sum of its Betti numbers, (2) is equivalent to
$$\chi(\mathcal{M}_{W,\mathbf{d}}) = \deg(\mathbf{d}) + C.$$

The construction of $\mathcal{M}_{W,\mathbf{d}}$ has two additional important properties: first, for $j = 1, 2$, one has a simple relationship between $\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_j}$ and $\mathcal{M}_{W,\mathbf{d}}$ (involving a *skyscraper* k-diagram) that immediately implies
$$\chi(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_j}) = \chi(\mathcal{M}_{W,\mathbf{d}}) + 1; \tag{5}$$
hence as soon as one verifies that $\chi(\mathcal{M}_{W,\mathbf{d}}) = \deg(\mathbf{d}) + C$ for some $C \in \mathbb{Z}$ and for a single $\mathbf{d} \in \mathbb{Z}^2$, it immediately follows that this holds for all $\mathbf{d} \in \mathbb{Z}^2$. Second, as part of our discussion of weights, it will turn out that for any $\mathbf{K} \in \mathbb{Z}^2$, setting $\mathbf{L} = \mathbf{K} + (1, 1)$, the weight of $f^{\wedge}_{\mathbf{K}}$ is the function $W^*_{\mathbf{L}}$ given by $W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L} - \mathbf{d})$. It will follow that one has, for $i = 0, 1$,
$$b^i(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) = b^{1-i}(\mathcal{M}_{W,\mathbf{d}}).$$
We will show that this equality of integers actually arises from an isomorphism
$$H^i(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})^* \to H^{1-i}(\mathcal{M}_{W,\mathbf{d}}), \tag{6}$$
which in turn arises from a statement akin to Serre duality, that states that for some k-diagrams, $\mathcal{F}$, and for $i = 0, 1$, there is an isomorphism
$$H^i(\mathcal{F})^* \to \mathrm{Ext}^{1-i}(\mathcal{F}, \underline{k}_{/B_1,B_2}), \tag{7}$$
where $\underline{k}_{/B_1,B_2}$ is a k-diagram that therefore plays the role of the *canonical sheaf* in Serre duality. We caution the reader that there is "bad news" here: although (7) does hold of k-diagrams, the "dualizing sheaf," $k_{/B_1,B_2}$ reflects a property of k-diagrams and nothing about the "geometry" of the sheaves $\mathcal{M}_{W,\mathbf{d}}$. In a stronger type of duality theorem one would expect that (6) would (1) involve a dualizing sheaf closer in "geometry" to the $\mathcal{M}_{W,\mathbf{d}}$, and (2) would follow by taking global sections of an expression involving "sheaf Hom." See Subsection 10.10 for more details, and [9].

1.3 Models for General Riemann Functions $\mathbb{Z}^2 \to \mathbb{Z}$

If $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ is a general Riemann function, its weight, W, may have negative values. In this case, one can model f by Euler characteristics, provided that one passes to *virtual k-diagrams* and *virtual Euler characteristics* in the following sense: by a *virtual k-diagram* or *formal difference of k-diagrams* we mean a pair of k-diagrams $(\mathcal{F}_1, \mathcal{F}_2)$, where we consider $(\mathcal{F}_1, \mathcal{F}_2)$ to be equivalent $(\mathcal{F}'_1, \mathcal{F}'_2)$ if for some k-diagram $\mathcal{G}$ we have
$$\mathcal{F}_1 \oplus \mathcal{F}'_2 \oplus \mathcal{G} \simeq \mathcal{F}_2 \oplus \mathcal{F}'_1 \oplus \mathcal{G}$$
(one often calls this the Grothendieck group arising from a commutative monoid); assuming that we work over the category of k-diagrams with finite Betti numbers, we define
$$b^i(\mathcal{F}_1, \mathcal{F}_2) = b^i(\mathcal{F}_1) - b^i(\mathcal{F}_2),$$
which is independent of the equivalence class of $(\mathcal{F}_1, \mathcal{F}_2)$. We will prove that for any weight $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ of a Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$ can be written as
$$W = W_1 + \cdots + W_k - \tilde{W}_1 - \cdots - \tilde{W}_{k-1} \tag{8}$$
for some $k \geq 1$, where each W_i and $\tilde{W}_i$ are perfect matchings; we then define the formal difference
$$\mathcal{M}_{W,\mathbf{d}} = (\mathcal{F}_{\mathbf{d}}, \widetilde{\mathcal{F}}_{\mathbf{d}}),$$
where
$$\mathcal{F}_{\mathbf{d}} = \mathcal{M}_{W_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_k,\mathbf{d}}, \quad \tilde{\mathcal{F}}_{\mathbf{d}} = \mathcal{M}_{\tilde{W}_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{\tilde{W}_{k-1},\mathbf{d}};$$
it is easy to verify that, up to equivalence, $(\mathcal{F}_{\mathbf{d}}, \widetilde{\mathcal{F}}_{\mathbf{d}})$ is independent of the way one writes W in (8). Then the formal difference $\mathcal{M}_{W,\mathbf{d}}$, or really the equivalence class $[\mathcal{M}_{W,\mathbf{d}}]$, models $f(\mathbf{d})$ in the sense that (3), (4), and (5) hold with $[\mathcal{M}_{W,\mathbf{d}}]$ replacing $\mathcal{M}_{W,\mathbf{d}}$.

1.4 Modeling Riemann functions $\mathbb{Z}^n \to \mathbb{Z}$ for $n \geq 3$

To model a Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$, we piece together various *two-variable restrictions of* f in the following sense: for any distinct $i, j \in [n]$ and $\mathbf{d} \in \mathbb{Z}^n$, we define the *two-variable restriction of* f *at* $i, j, \mathbf{d}$ to be the function $f_{i,j,\mathbf{d}}\colon \mathbb{Z}^2 \to \mathbb{Z}$ given by

$$f_{i,j,\mathbf{d}}(a_i, a_j) \stackrel{\text{def}}{=} f\bigl(\mathbf{d} + a_i\mathbf{e}_i + a_j\mathbf{e}_j\bigr)$$

(and hence $f_{i,j,\mathbf{d}}(0,0) = f(\mathbf{d})$). If W is the weight of $f_{i,j,\mathbf{d}}$, we use $[\mathcal{M}_{f;i,j,\mathbf{d}}]$ to denote the virtual k-diagram $[\mathcal{M}_{W,\mathbf{0}}]$. It follows that $[\mathcal{M}_{f;i,j,\mathbf{d}}]$ is a family of k-diagrams that satisfies

$$b^0([\mathcal{M}_{f;i,j,\mathbf{d}}]) = f(\mathbf{d}),$$

$$\chi([\mathcal{M}_{f;i,j,\mathbf{d}}]) = \deg(\mathbf{d}) + C$$

where C is the offset of f, and for any $\mathbf{K} \in \mathbb{Z}^2$ we have

$$b^1([\mathcal{M}_{f;i,j,\mathbf{d}}]) = f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}).$$

Of course, $\mathcal{M}_{f;i,j,\mathbf{d}}$ appears to depend on the choice of i, j; however, we will prove that for any $j' \in [n]$ distinct from i, j, $[\mathcal{M}_{f;i,j,\mathbf{d}}]$ is equivalent to $[\mathcal{M}_{f;i,j',\mathbf{d}}]$, and hence the equivalence class $[\mathcal{M}_{f;i,j,\mathbf{d}}]$ is independent of the choice of i, j.

Moreover, in case $\mathcal{M}_{f;i,j,\mathbf{d}}$ and $\mathcal{M}_{f;i,j',\mathbf{d}}$ are k-diagrams, not just virtual k-diagrams (i.e., the weights of $f_{i,j,\mathbf{d}}$ and $f_{i,j',\mathbf{d}}$ are perfect matchings), we will prove that $\mathcal{M}_{f;i,j,\mathbf{d}}$ and $\mathcal{M}_{f;i,j',\mathbf{d}}$ are isomorphic as k-diagrams.

We therefore use the notation $[\mathcal{M}_{f \text{ at } \mathbf{d}}]$ to denote the equivalence class of $[\mathcal{M}_{f;i,j,\mathbf{d}}]$, which is independent of i, j. We will show that (6) gives rise, for $i = 0, 1$ to an isomorphism

$$H^i\bigl([\mathcal{M}_{f \text{ at } \mathbf{d}}]\bigr)^* \to H^{1-i}\bigl([\mathcal{M}_{f^{\wedge}_{\mathbf{K}} \text{ at } \mathbf{K}-\mathbf{d}}]\bigr). \tag{9}$$

This involves the following fundamental fact: if $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is a Riemann function, and if we fix $d_3, \ldots, d_n$ and $\mathbf{K} \in \mathbb{Z}^n$, and we set $g(d_1, d_2) = f(\mathbf{d})$ viewing d_1, d_2 as variables, then the resulting generalized Riemann-Roch formula for g is

$$g(d_1, d_2) - g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2) = d_1 + d_2 + C_g$$

where C_g is the offset of g. It is not hard to see that this formula is the restriction of (2), in the sense that

$$d_1 + d_2 + C_g = \deg(\mathbf{d}) + C_f$$

where C_f is the offset of f, and

$$g(d_1, d_2) = f(\mathbf{d}), \quad g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2) = f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}). \tag{10}$$

It follows that the generalized Riemann-Roch formulas (2) restricts to two-variable generalized Riemann-Roch formulas, and that all the two-variable formulas (i.e., fixing some $n-2$ variables and varying the two remaining variables) determine the all the n-variable formulas.

The articles [2, 5] focus on proving that the Riemann functions there are self-dual. We remark that the notion of self-duality is not well-behaved under two-variable restrictions: indeed, if $f\colon \mathbb{Z}^n \to \mathbb{Z}$ satisfies $f^{\wedge}_{\mathbf{K}} = f$ for some $K \in \mathbb{Z}^n$, then (10) implies that for $d_3, \ldots, d_n$ and $\mathbf{K}$ fixed we have

$$g(d_1, d_2) = f(d_1, \ldots, d_n) \quad g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2) = f(K_1 - d_1, \ldots, K_n - d_n).$$

Hence, g is not generally self-dual. Hence, if f is self-dual, the two-variable restrictions in a single generalized Riemann-Roch formula still come in pairs, g and $g^{\wedge}_{(K_1,K_2)}$.

1.5 Additional Remarks and Future Work

The invariants of k-diagrams that we compute—such as their Betti numbers, and Euler characteristics—all arise from their cohomology groups, which to each k-diagram, $\mathcal{M}$, are computed as the kernel and cokernel of an associated linear transformation $\tau_{\mathcal{M}}$. Therefore, the reader who prefers can translate our entire discussion and use of k-diagrams into equivalent statements regarding the kernel and cokernel of the associated linear transformations.

We also remark that our duality theorems, as stated above, may seem trivial: for example, given that for a perfect matching W we have

$$b^i(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) = b^{1-i}(\mathcal{M}_{W,\mathbf{d}}),$$

it is immediate that the spaces

$$H^i(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \quad \text{and} \quad H^{1-i}(\mathcal{M}_{W,\mathbf{d}})$$

and their duals are all isomorphic, since these are all k-vector spaces of this same dimension. Hence, in our theorems and their proofs, it is also important to note the way we construct these duality isomorphisms; often, we make this explicit in the statement of the theorem (see, e.g., Theorem 9.2).

The Riemann functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$ associated with the Baker-Norine rank [5] and its generalizations studied by Amini and Manjunath [2] are *periodic* (in a sense described in Subsection 2.7). In this case, the k-diagrams $\mathcal{M}_{W,\mathbf{d}}$ associated the two-variable restrictions of f have a much stronger structure: namely, they are $\mathcal{O}$-modules, where $\mathcal{O}$ is a k-diagram of rings. In this case, we believe that the diagrams $\mathcal{M}_{W,\mathbf{d}}$ themselves act as canonical k-diagrams in a form of Serre duality, we explain this at the end of this article, and plan to address this in future work. For this reason, when $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is periodic, when writing W as a difference of a sum of perfect matchings (8), we will be interested in showing that the W_i and $\tilde{W}_i$ can be chosen with the same periodicity.

A good challenge for future work is to develop models that explain generalized Riemann-Roch formulas as a type of sheaf or diagram of k-vector spaces that does not have all the ad hoc choices we make, and that does not need to pass to virtual diagrams or virtual sheaves.

Another—perhaps independent challenge—is to use the theory of diagrams or sheaves to give proofs of self-duality, such as in the Baker-Norine formula [5] and some more general situations, such as those studied by Amini and Manjunath [2].

1.6 Organization of the Rest of this Article

In Section 2, we introduce some basic notation and state some theorems about the *weight* of a Riemann function, referring the reader to [11] for the proofs. In Section 3, we will prove some theorems regarding the weights of Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ that we will use. In Section 4, we give some conventions regarding the sheaves we build—that we call k-diagrams—and show how to use them to model a Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$ whose weight is non-negative, i.e., is a *perfect matching*. In Section 5 we discuss morphisms between k-diagrams, and a number of related ideas needed later on; in particular, to define virtual k-diagrams, we need to know some facts about direct sums and isomorphisms of k-diagrams. In Section 6 we introduce *indicator* k-diagrams that gives an alternate way to view the k-diagrams that we use to model Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$; we will need them when we prove duality theorems later on. In Section 7, we describe our conventions about virtual k-diagrams and show that any Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$ can be modeled by a single equivalence class of virtual k-diagrams. In Section 8, we model any Riemann function $\mathbb{Z}^n \to \mathbb{Z}$ by diagrams obtained by fixing any $n-2$ of its variables and modeling the resulting Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$. In Section 9, we will prove the $i = 1$ case of (7), (6), and (9). In Section 10, we prove the $i = 0$ case of (7) and explain the connection of k-diagrams to sheaf theory; we tie up a few other loose ends, including a discussion of periodic Riemann functions and a *Serre functor* computation that yields $\underline{k}_{/B_1,B_2}$.

2 Basic Terminology and Weights

In this section, we introduce some basic terminology used throughout this paper, including the definition of a *Riemann function* and its *weight function*. Then we derive some combinatorial results about the weights of Riemann functions that we will need to construct our models.

The weight function of a Riemann function is quite interesting for its own sake, and we refer to [11] for a fuller discussion of weights of Riemann functions.

2.1 Basic Notation

We use $\mathbb{Z}, \mathbb{N}$ to denote the integers and positive integers; for $a \in \mathbb{Z}$, we use $\mathbb{Z}_{\le a}$ to denote the integers less than or equal to a, and similarly for the subscript $\ge a$. For $n \in \mathbb{N}$ we use $[n]$ to denote $\{1, \ldots, n\}$. We use bold face $\mathbf{d} = (d_1, \ldots, d_n)$ to denote elements of $\mathbb{Z}^n$, using plain face for the components of $\mathbf{d}$; by the *degree* of $\mathbf{d}$, denoted $\deg(\mathbf{d})$ or at times $|\mathbf{d}|$, we mean $d_1 + \ldots + d_n$.

We set

$$\mathbb{Z}^n_{\deg 0} = \{\mathbf{d} \in \mathbb{Z}^n \mid \deg(\mathbf{d}) = 0\},$$

and for $a \in \mathbb{Z}$ we similarly set

$$\mathbb{Z}^n_{\deg a} = \{\mathbf{d} \in \mathbb{Z}^n \mid \deg(\mathbf{d}) = a\}, \quad \mathbb{Z}^n_{\deg \le a} = \{\mathbf{d} \in \mathbb{Z}^n \mid \deg(\mathbf{d}) \le a\}.$$

We use $\mathbf{e}_i \in \mathbb{Z}^n$ (with n understood) to denote the i-th standard basis vector (i.e., whose j-th component is 1 if $j = i$ and 0 otherwise), and for $I \subset [n]$ (with n understood) we set

$$\mathbf{e}_I = \sum_{i \in I} \mathbf{e}_i; \tag{11}$$

hence, in case $I = \emptyset$ is the empty set, then $\mathbf{e}_\emptyset = \mathbf{0} = (0, \ldots, 0)$, and similarly $e_{[n]} = \mathbf{1} = (1, \ldots, 1)$.

For $n \in \mathbb{N}$, we endow $\mathbb{Z}^n$ with the usual partial order, that is

$$\mathbf{d}' \le \mathbf{d} \quad \text{iff} \quad d'_i \le d_i \ \forall i \in [n],$$

where $[n] = \{1, 2, \ldots, n\}$.

2.2 Riemann Functions, Generalized Riemann-Roch Formulas, and Self-Duality

In this section, we define *Riemann functions* and *generalized Riemann-Roch formulas* and give some examples.

Definition 2.1. *We say that a function $f : \mathbb{Z}^n \to \mathbb{Z}$ is a Riemann function if for some $C, a, b \in \mathbb{Z}$ we have*

1. *$f(\mathbf{d}) = 0$ if $\deg(\mathbf{d}) \le a$; and*
2. *$f(\mathbf{d}) = \deg(\mathbf{d}) + C$ if $\deg(\mathbf{d}) \ge b$;*

we refer to C as the offset *of f.*

In our study of Riemann functions, it will be useful to introduce the following terminology.

Definition 2.2. *If f, g are functions $\mathbb{Z}^n \to \mathbb{Z}$, we say that f* equals g initially *(respectively,* eventually*) if $f(\mathbf{d}) = g(\mathbf{d})$ for $\deg(\mathbf{d})$ sufficiently small (respectively, sufficiently large); similarly, we say that that f is* initially zero *(respectively* eventually zero*) if $f(\mathbf{d}) = 0$ for $\deg(\mathbf{d})$ sufficiently small (respectively, sufficiently large).*

Therefore $f : \mathbb{Z}^n \to \mathbb{Z}$ is a Riemann function if and only if it is initially zero and it eventually equals the function $\deg(\mathbf{d}) + C$ for a constant $C \in \mathbb{Z}$ that we call the *offset* of f.

In particular, Riemann's theorem, which is a precursor to the classical Riemann-Roch theorem, gives examples of Riemann functions. In the next subsection, we will give a number of examples of Riemann functions, including those associated with the Baker-Norine rank function of a graph [5] and related functions. Before doing so, we give some of the basic properties of Riemann functions.

Definition 2.3. *Let $f : \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function with offset C, and $\mathbf{K} \in \mathbb{Z}^n$. The* $\mathbf{K}$-dual of f, *denoted $f^\wedge_{\mathbf{K}}$, refers to the function $\mathbb{Z}^n \to \mathbb{Z}$ given by*

$$f^\wedge_{\mathbf{K}}(\mathbf{d}) = f(\mathbf{K} - \mathbf{d}) - \deg(\mathbf{K} - \mathbf{d}) - C. \tag{12}$$

Replacing $\mathbf{d}$ with $\mathbf{K} - \mathbf{d}$ we equivalently write

$$f(\mathbf{d}) - f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C \tag{13}$$

and refer to this equation as a generalized Riemann-Roch formula. *We say that f* is self-dual at $\mathbf{K}$ *if $f^\wedge_{\mathbf{K}} = f$.*

If f is a Riemann functions that is self-dual at $\mathbf{K}$, then (13) reads

$$f(\mathbf{d}) - f(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C,$$

which resembles the classical Riemann-Roch formula and the Baker-Norine analog [5] and related formulas. We remark that in (13), $f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d})$ equals $f(\mathbf{d}) - \deg(\mathbf{d}) - C$, which is independent of $\mathbf{K}$.

Proposition 2.1. *Let $f : \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function with offset C, and $\mathbf{K} \in \mathbb{Z}^n$. Then:*

1. *$f^\wedge_{\mathbf{K}}$ is a Riemann function with offset $-\deg(\mathbf{K}) - C$;*
2. *$(f^\wedge_{\mathbf{K}})^\wedge_{\mathbf{K}} = f$;*
3. *for any other Riemann function, $g : \mathbb{Z}^n \to \mathbb{Z}$, $f = g$ if and only if for some (equivalently any) $\mathbf{K} \in \mathbb{Z}^n$, $f^\wedge_{\mathbf{K}} = g^\wedge_{\mathbf{K}}$.*

Proof. This proof is a straightforward calculation. For $\mathbf{d}$ sufficiently small we have

$$f(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{K} - \mathbf{d}) + C,$$

which by (12) implies that $f^\wedge_{\mathbf{K}}$ is initially zero. For $\mathbf{d}$ with $\deg(\mathbf{d})$ sufficiently large we have $f(\mathbf{K} - \mathbf{d}) = 0$, and hence (12) implies that for $\deg(\mathbf{d})$ sufficiently large

$$f^\wedge_{\mathbf{K}}(\mathbf{d}) = \deg(\mathbf{K} - \mathbf{d}) - C = \deg(\mathbf{d}) - \deg(\mathbf{K}) - C.$$

Hence, $f^\wedge_{\mathbf{K}}$ is a Riemann function with offset $-\deg(\mathbf{K}) - C$.

To see claim (2), since $f^\wedge_{\mathbf{K}}$ is a Riemann function with offset $-\deg(\mathbf{K}) - C$, by (12) we have

$$(f^\wedge_{\mathbf{K}})^\wedge_{\mathbf{K}}(\mathbf{d}) = f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) - \deg(\mathbf{K} - \mathbf{d}) - \big(\deg(\mathbf{K}) - C\big) = f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) + \deg(\mathbf{d}) + C,$$

which (12) implies

$$= f(\mathbf{d}) - \deg(\mathbf{d}) - C + \deg(\mathbf{d}) + C = f(\mathbf{d}).$$

To prove claim (3), if $f = g$ then we may apply ${}^\wedge_{\mathbf{K}}$ to both to conclude that $f^\wedge_{\mathbf{K}} = g^\wedge_{\mathbf{K}}$. Conversely, if $f^\wedge_{\mathbf{K}} = g^\wedge_{\mathbf{K}}$, then applying ${}^\wedge_{\mathbf{K}}$ and using claim (2) we get $f = g$. $\square$

2.3 Examples of Riemann functions

We briefly give some examples of Riemann functions. This section is not essential to the rest of this paper, although these examples are helpful for intuition; we will refer to some of these examples to illustrate some of our results.

Example 2.1. *Let $G = (V, E)$ be a connected graph with $V = \{v_1, \ldots, v_n\}$. Let $L = \mathrm{Image}(\Delta_G)$ be the image of the Laplacian of G, Δ_G. Say that $\mathbf{d} \in \mathbb{Z}^n$ is* effective *if there is a $\mathbf{d}' \geq \mathbf{0}$ (i.e., $d'_i \geq 0$ for all $i \in [n]$) such that $\mathbf{d} - \mathbf{d}' \in L$, and otherwise say that $\mathbf{d}$ is* not effective*. Let $\mathcal{N} \subset \mathbb{Z}^n$ be the subset of elements that are not effective. Let*

$$f(\mathbf{d}) = \mathrm{Distance}_{L^1}(\mathbf{d}, \mathcal{N}) = \min_{\mathbf{d}' \in \mathcal{N}} \|\mathbf{d} - \mathbf{d}'\|_{L^1}, \tag{14}$$

where L^1 is the usual L^1-norm,

$$\|\mathbf{x}\|_{L^1} = |x_1| + \cdots + |x_n|.$$

Then $r_{\mathrm{BN},G} = -1 + f$ is the usual Baker-Norine rank *[5] of G, and the Baker-Norine Graph Riemann-Roch formula [5] asserts that*

$$f(\mathbf{d}) - f(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + 1 - g$$

where $g = 1 + |E| - |V|$ (which is non-negative since G is connected), and where

$$\mathbf{K} = \big(\deg_G(v_1) - 2, \ldots, \deg_G(v_n) - 2\big).$$

Since $\mathcal{N}$ contains all elements of $\mathbb{Z}^n$ of negative degree, it follows that f is initially zero; the Baker-Norine formula implies that f is a Riemann function with offset $1 - g$ that is self-dual at K.

Example 2.2. *Let L be, more generally, any lattice of rank $n - 1$ in $\mathbb{Z}^n_0$. Then the same definitions work—effective, not effective, $\mathcal{N}$, and furthermore Amini and Manjurath [2] show that f as in (14) is a Riemann function with offset $1 - g_{\max}$ defined on page 5 there. They give conditions—which hold sometimes, but not always—for f to be self-dual at some $\mathbf{K} \in \mathbb{Z}^n$.*

One can slightly generalize this construction of Riemann functions, f, in (14) by allowing $\mathcal{N} \subset \mathbb{Z}^n$ to satisfy some weaker conditions; see [11].

Example 2.3. *Let $P_1, \ldots, P_n$ be n points of an algebraic curve over an algebraically closed field, k, and let K denote the function field of the curve. Let*

$$f(\mathbf{d}) = \dim_k\{g \in K \mid (g) \geq -d_1 P_1 - \cdots - d_n P_n\} \tag{15}$$

where (g) is the (Weil) divisor associated with g (and we view (0) as larger than any divisor). Then the classical Riemann theorem *states that f is a Riemann function, and that its offset equals $1 - g$, where g is the genus of the curve.*

The above example was our motivation for the name *Riemann function*.

2.4 Restrictions of Riemann functions and Alternating Sums

In this subsection, we give examples of obtaining Riemann functions and constructing new Riemann functions. Both ideas are fundamental to the way we construct the models in this article.

Example 2.4. *Let $f \colon \mathbb{Z}^n \to \mathbb{Z}$ be any Riemann function with $f(\mathbf{d}) = \deg(\mathbf{d}) + C$ for $\deg(\mathbf{d})$ sufficiently large. Then for any distinct $i, j \in [n]$ and $\mathbf{d} \in \mathbb{Z}^n$, the function $f_{i,j,\mathbf{d}} \colon \mathbb{Z}^2 \to \mathbb{Z}$ given as*

$$f_{i,j,\mathbf{d}}(a_i, a_j) = f\big(\mathbf{d} + a_i \mathbf{e}_i + a_j \mathbf{e}_j\big) \tag{16}$$

is a Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$, and for $a_i + a_j$ large we have

$$f_{i,j,\mathbf{d}}(a_i, a_j) = a_i + a_j + C', \quad \text{where} \quad C' = \deg(\mathbf{d}) + C.$$

We call $f_{i,j,\mathbf{d}}$ a two-variable restriction *of f; we may similarly restrict f to one variable or three or more variables; clearly, any restriction of a Riemann is again a Riemann function. (We write a_i, a_j as the arguments for $f_{i,j,\mathbf{d}}$ instead of, say, a_1, a_2, to stress that a_i corresponds to adding $a_i \mathbf{e}_i$ in (16), and similarly for a_j).*

In the above example, it will be crucial to us that C' depends only on $\mathbf{d}$ and not on i, j.

Example 2.5. *If for some $s, n \in \mathbb{N}$, $f_1, \ldots, f_s$ and $\tilde{f}_1, \ldots, \tilde{f}_{s-1}$ are Riemann functions $\mathbb{Z}^n \to \mathbb{Z}$, then so is*

$$f = f_1 + \cdots + f_s - (\tilde{f}_1 + \cdots + \tilde{f}_{s-1}).$$

Moreover, the offset, C, of f is given as

$$C = (C_1 + \cdots + C_s) - (\tilde{C}_1 + \cdots + \tilde{C}_{s-1}), \tag{17}$$

where C_i is the offset of f_i and $\tilde{C}_i$ is the offset of $\tilde{f}_i$.

2.5 The Weight of a Riemann function

Our main technique of modeling Riemann functions involves their weights. In this article, we are concerned with weights of Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$, but the foundations of weights apply to Riemann functions in any number of variables; see [11].

If $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is initially zero, then there is a unique initially zero $W\colon \mathbb{Z}^n \to \mathbb{Z}$ for which

$$\forall \mathbf{d} \in \mathbb{Z}^n, \quad f(\mathbf{d}) = \sum_{\mathbf{d}' \le \mathbf{d}} W(\mathbf{d}'), \tag{18}$$

since we can determine $W(\mathbf{d})$ inductively on $\deg(\mathbf{d})$, setting W initially zero (in degrees where f is initially zero), and using the equation

$$W(\mathbf{d}) = f(\mathbf{d}) - \sum_{\mathbf{d}' \le \mathbf{d},\ \mathbf{d}' \ne \mathbf{d}} W(\mathbf{d}').$$

Definition 2.4. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be initially zero. By the* weight of f *we mean the unique initially zero function $W\colon \mathbb{Z}^n \to \mathbb{Z}$ satisfying* (18).

Recall from (11) the notation $\mathbf{e}_I$ for $I \subset [n]$.

Proposition 2.2. *Consider the operator $\mathfrak{m}$ on functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$ defined via*

$$(\mathfrak{m}f)(\mathbf{d}) = \sum_{I \subset [n]} (-1)^{|I|} f(\mathbf{d} - \mathbf{e}_I),$$

and the operator on functions $W\colon \mathbb{Z}^n \to \mathbb{Z}$ that are initially zero given by

$$(\mathfrak{s}W)(\mathbf{d}) = \sum_{\mathbf{d}' \le \mathbf{d}} W(\mathbf{d}').$$

If f is any initially zero function, and W is the weight of f, then we have $f = \mathfrak{s}W$ and $W = \mathfrak{m}f$.

The proof is an easy computation; see [11] for details. One may also write

$$(\mathfrak{m}f) = (1 - \mathfrak{t}_1) \ldots (1 - \mathfrak{t}_n) f,$$

where $\mathfrak{t}_i$ is the operator taking f to the function

$$(\mathfrak{t}_i f)(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_i);$$

it easily follows that for $n \ge 2$, if $W\colon \mathbb{Z}^n \to \mathbb{Z}$ is the weight of any Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$, then W is eventually zero.

2.6 Weights and the Riemann-Roch formulas

Definition 2.5. *If $W\colon \mathbb{Z}^n \to \mathbb{Z}$ is any function and $\mathbf{L} \in \mathbb{Z}^n$, the* $\mathbf{L}$-dual weight of W, *denoted $W^*_{\mathbf{L}}$ refers to the function given by*

$$W^*_{\mathbf{L}}(\mathbf{d}) = W(\mathbf{L} - \mathbf{d}).$$

It is immediate that $(W^*_{\mathbf{L}})^*_{\mathbf{L}} = W$.

Theorem 2.1. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function, and $W = \mathfrak{m}f$. Let $\mathbf{K} \in \mathbb{Z}^n$ and let $\mathbf{L} = \mathbf{K} + \mathbf{1}$.*

1. *We have*
$$\mathfrak{m}\big(f^\wedge_{\mathbf{K}}\big) = (-1)^n W^*_{\mathbf{L}} = (-1)^n (\mathfrak{m}f)^*_{\mathbf{L}}.$$
2. *$f^\wedge_{\mathbf{K}} = f$ if and only if $W^*_{\mathbf{L}} = (-1)^n W$.*

The proof is a straightforward computation; see [11] for details.

We remark that it is immediate that the map $W \mapsto W^*_{\mathbf{L}}$ is an involution, i.e., applying it twice gives the same function; furthermore, the map $W \mapsto W^*_{\mathbf{L}}$ is defined on all functions $\mathbb{Z}^n \to \mathbb{Z}$. By contrast, the fact that $^\wedge_{\mathbf{K}}$ is an involution requires a bit more computation (in the proof of Proposition 2.1), and it is only defined on Riemann functions (at least as we have defined it) since the definition of $f^\wedge_{\mathbf{K}}$ in (12) requires us know the offset, C, of f. [In [11], $f^\wedge_{\mathbf{K}}$ is defined on a more general class of functions, f, namely f that are initially zero and whose weight, W, is eventually zero.] This gives two indications that working with the weight of a Riemann function has advantages over working with the Riemann function itself.

2.7 Translation Invariance and Periodicity

This subsection has two goals: first, in case $f = f^{\wedge}_{\mathbf{K}}$, then we study the uniqueness of such a $\mathbf{K}$. Second, we will introduce the related notation of *periodicity* which will be useful in future work to show that certain generalized Riemann-Roch formulas have a second type of Serre duality beyond what we cover in this article; we will briefly explain this in Section 10.

Definition 2.6. *Let $f : \mathbb{Z}^n \to \mathbb{Z}$ be any function. We say that f is* invariant by translation under $\mathbf{t} \in \mathbb{Z}^n$ *if*

$$\forall \mathbf{d} \in \mathbb{Z}^n, \quad f(\mathbf{d} + \mathbf{t}) = f(\mathbf{d}).$$

We define the set of invariant translations of f *to be the set of all such* $\mathbf{t}$.

In the above definition, we easily see that the set, T, of invariant translations of a function, f, is a lattice, i.e., T is closed under addition, and if $\mathbf{t} \in T$ then also $-\mathbf{t} \in T$. Furthermore, if f is translation invariant by $\mathbf{t}$, then for any $\mathbf{d} \in \mathbb{Z}^n$ we have $f(\mathbf{d}) = f(\mathbf{d} + m\mathbf{t})$ for any $m \in \mathbb{Z}$; it follows that if f is non-zero but initially zero, then any such $\mathbf{t}$ must lie in $\mathbb{Z}^n_{\deg 0}$.

Proposition 2.3. *Let $f : \mathbb{Z}^n \to \mathbb{Z}$ be any Riemann function, and $W = \mathfrak{m}f$ its weight. Let T be the set of invariant translations of f. Then*

1. *T equals the set of invariant translations of W;*
2. *for any $\mathbf{L}_1, \mathbf{L}_2 \in \mathbb{Z}^n$, $W^*_{\mathbf{L}_1} = W^*_{\mathbf{L}_2}$ if and only if $\mathbf{L}_1 - \mathbf{L}_2 \in T$;*
3. *for any $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{Z}^n$, $f^{\wedge}_{\mathbf{K}_1} = f^{\wedge}_{\mathbf{K}_2}$ if and only if $\mathbf{K}_1 - \mathbf{K}_2 \in T$;*
4. *if for some $\mathbf{K} \in \mathbb{Z}^n$ we have $f = f^{\wedge}_{\mathbf{K}}$, then for any $\mathbf{K}' \in \mathbb{Z}^n$, $f = f^{\wedge}_{\mathbf{K}'}$ if and only if $\mathbf{K}' - \mathbf{K} \in T$.*
5. *if for some $\mathbf{L} \in \mathbb{Z}^n$ we have $W = W^*_{\mathbf{L}}$, then for any $\mathbf{L}' \in \mathbb{Z}^n$, $W = W^*_{\mathbf{L}'}$ if and only if $\mathbf{L}' - \mathbf{L} \in T$.*

Proof. Claim (1) follows by observing that $\mathfrak{m}$ and $\mathfrak{s}$ commute with translation by $\mathbf{t}$, (i.e., the operator taking f to $\tilde{f}$ given by $\tilde{f}(\mathbf{d}) = f(\mathbf{d} + \mathbf{t})$).

To prove claim (2), we see that $W^*_{\mathbf{L}_1} = W^*_{\mathbf{L}_2}$ iff

$$(W^*_{\mathbf{L}_1})^*_{\mathbf{L}_2} = (W^*_{\mathbf{L}_2})^*_{\mathbf{L}_2} = W$$

iff, for all $\mathbf{d} \in \mathbb{Z}^n$ we have

$$W(\mathbf{d}) = (W^*_{\mathbf{L}_1})^*_{\mathbf{L}_2}(\mathbf{d}) = (W^*_{\mathbf{L}_1})(\mathbf{L}_2 - \mathbf{d}) = W\big(\mathbf{L}_1 - (\mathbf{L}_2 - \mathbf{d})\big) = W(\mathbf{d} + \mathbf{L}_1 - \mathbf{L}_2),$$

i.e., $\mathbf{L}_1 - \mathbf{L}_2 \in T$.

Claim (3) follows from claims (1) and (2). Claim (4) follows from claim (3), and claim (5) from claim (2). ☐

Definition 2.7. *We say that a function $f : \mathbb{Z}^n \to \mathbb{Z}$ is* r-periodic *for an $r \in \mathbb{Z}$ if for all $i, j \in [n]$ we have that f is invariant under translation by $r(\mathbf{e}_i - \mathbf{e}_j)$.*

Example 2.6. *In Examples 2.1 and 2.2, f as in (14) is translation invariant by $L \subset \mathbb{Z}^n_{\deg 0}$ where $\mathbb{Z}^n_{\deg 0}/L$ is finite (i.e., L is of rank $n-1$). If $p = |\mathbb{Z}^n_{\deg 0}/L|$, then any element of $\mathbb{Z}^n_{\deg 0}/L$ is of order divisible by p, and hence f is p-periodic.*

Example 2.7. *If in Example 2.3 we take an elliptic curve, and P_1 is any point, then $P_2 - P_1$ has finite order for only countably many P_2; hence f is r-periodic for some $r \geq 1$ for only countably many P_2.*

2.8 Weights of Riemann Functions $\mathbb{Z}^2 \to \mathbb{Z}$

We will be especially interested in Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ and their weights $W = \mathfrak{m}f$. It is useful to notice that for such functions we have that for any fixed d_1 and d_2' sufficiently large,

$$f(d_1, d_2') - f(d_1 - 1, d_2') = 1,$$

and that for any $d_1, d_2' \in \mathbb{Z}$ we have

$$f(d_1, d_2') - f(d_1 - 1, d_2') = \sum_{d_2 = -\infty}^{d_2'} W(d_1, d_2).$$

Hence, for fixed d_1,

$$\sum_{d_2=-\infty}^{\infty} W(d_1,d_2) = 1, \tag{19}$$

and similarly, for fixed d_2 we have

$$\sum_{d_1=-\infty}^{\infty} W(d_1,d_2) = 1. \tag{20}$$

We easily check the converse, i.e., if $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is initially and eventually zero and satisfies (19) and (20), then for $d_1'+d_2'$ fixed and sufficiently large we have

$$f(d_1',d_2')+1 = f(d_1'+1,d_2') = f(d_1',d_2'+1),$$

and we conclude that f is a Riemann function. Viewing W as a two-dimensional infinite array of numbers indexed in $\mathbb{Z}\times\mathbb{Z}$, one can therefore say that $W\colon \mathbb{Z}^2\to\mathbb{Z}$ is the weight of a Riemann function if and only if all its "row sums" (19) and all its "column sums" (20) equal 1.

2.9 Weights of Slowly Growing Riemann Functions $\mathbb{Z}^2 \to \mathbb{Z}$ and Perfect Matchings

In this subsection, we make some remarks on weights that we call "perfect matchings." In [11], these ideas were used to compute the weight of the Baker-Norine rank on graphs of two vertices (jointed by some number of edges). Here we will just state the definitions and an easy proposition.

Definition 2.8. *We say that a function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is* slowly growing *if for all $\mathbf{d}\in\mathbb{Z}^n$ and $i\in[n]$ we have*

$$f(\mathbf{d}) \le f(\mathbf{d}+\mathbf{e}_i) \le f(\mathbf{d})+1.$$

Definition 2.9. *Let W be a function $\mathbb{Z}^2\to\mathbb{Z}$ that is initially and eventually zero. We say that W is a* perfect matching *if there exists a permutation (i.e., a bijection) $\pi\colon\mathbb{Z}\to\mathbb{Z}$ such that*

$$W(i,j) = \begin{cases} 1 & \text{if } j=\pi(i), \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

It follows that for π as above, $\pi(i)+i$ is bounded above and below, since W is initially and eventually 0. Conversely, if $\pi\colon\mathbb{Z}\to\mathbb{Z}$ is a bijection with $\pi(i)+i$ bounded independently of i, then (21) is a perfect matching.

Proposition 2.4. *Let $f\colon\mathbb{Z}^2\to\mathbb{Z}$ be a slowly growing Riemann function. Let $W=\mathfrak{m}f$ be the weight of f. Then W takes only the values 0 and ±1. Furthermore, for any $\mathbf{d}\in\mathbb{Z}^2$, let $a=f(\mathbf{d})$; then*

$$W(\mathbf{d})=1 \iff f(\mathbf{d}-\mathbf{e}_1)=f(\mathbf{d}-\mathbf{e}_2)=f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)=a-1,$$

and

$$W(\mathbf{d})=-1 \iff f(\mathbf{d}-\mathbf{e}_1)=f(\mathbf{d}-\mathbf{e}_2)=a=f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)+1.$$

If W is everywhere non-negative, i.e., $W(\mathbf{d})\ge 0$ for all 0, then W is a perfect matching.

Proof. For the proof, see [11]; for ease for reading, we give the main idea: namely, since f is slowly growing, if $a=f(\mathbf{d})$, then

$$a-2\le f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)\le a.$$

In case $f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)=a-2$, then for $j=1,2$, $f(\mathbf{d}-\mathbf{e}_j)$ must equal $a-1$ (since f is slowly growing), in which case $W(\mathbf{d})=0$. Similarly if $f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)=a$ then $f(\mathbf{d}-\mathbf{e}_j)$ must equal a, and again $W(\mathbf{d})=0$. This leaves the case $f(\mathbf{d}-\mathbf{e}_1-\mathbf{e}_2)=a-1$, whereupon for $j=1,2$ each $f(\mathbf{d}-\mathbf{e}_j)$ is either a or $a-1$; this gives four cases to check, and after checking them, we see that $W(\mathbf{d})=1$ if and only if for $j=1,2$ both $f(\mathbf{d}-\mathbf{e}_j)$ are $a-1$, and $W(\mathbf{d})=-1$ if and only if for $j=1,2$ both $f(\mathbf{d}-\mathbf{e}_j)$ are a. ☐

Of course, if W is r-periodic, i.e., for all $\mathbf{d}\in\mathbb{Z}^2$, $W(\mathbf{d})=W(\mathbf{d}+(r,-r))$, then π is *skew-periodic* in the sense that $\pi(i+r)=\pi(i)-r$ for all $i\in\mathbb{Z}$.

2.10 Riemann Functions of Genus 1 and their Two-Variable Restrictions

There is a collection of Riemann functions that will be especially helpful to give concrete examples in Section 8. This section can be skipped until then; however, the reader may want to read this now to get some more concrete examples of Riemann functions.

Definition 2.10. *Let $b \in \mathbb{Z}$. We say that a Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is* of generalized genus 1 (of shift b) *if*

1. *for all $\mathbf{d} \in \mathbb{Z}^n$ with $\deg(\mathbf{d}) \neq b$,*
$$f(\mathbf{d}) = \max\bigl(0, \deg(\mathbf{d}) - b\bigr),$$
2. *for all $\mathbf{d} \in \mathbb{Z}^n$ with $\deg(\mathbf{d}) = b$, $f(\mathbf{d}) \in \{0,1\}$;*

if so, we say that f is of genus 1 *if $b = 0$.*

Figure 1 depicts a genus 1 Riemann function $\mathbb{Z}^n \to \mathbb{Z}$ with $n = 2$; Figure 2 depicts a generalized genus 1 Riemann function (of shift b) $\mathbb{Z}^n \to \mathbb{Z}$ with $n = 2$. The figures take $n = 2$ for ease of depiction.

0	1	2	3	4	5	6	
0	**1**	1	2	3	4	5	
0	0	**0**	1	2	3	4	
0	0	0	**0**	1	2	3	
0	0	0	0	**1**	1	2	
0	0	0	0	0	**1**	1	
0	0	0	0	0	0	**1**	$f(\mathbf{d}) = 0,1$ if $\deg(\mathbf{d}) = 0$

Figure 1: Riemann Functions of Genus 1: these functions, f, satisfy: (1) $f(\mathbf{d}) = 0$ if $\deg(\mathbf{d}) < 0$, (2) $f(\mathbf{d}) = \deg(\mathbf{d})$ if $\deg(\mathbf{d}) > 0$, and (3) are slowly growing, or, equivalently, satisfy $f(\mathbf{d}) = 0,1$ if $\deg(\mathbf{d}) = 0$. Here we give an example of such a function $\mathbb{Z}^2 \to \mathbb{Z}$, (although the definition more generally allows for functions $\mathbb{Z}^n \to \mathbb{Z}$). Hence $f(\mathbf{d})$ is uniquely determined unless $\deg(\mathbf{d}) = d_1 + d_2 = 0$. For example: if P_1, P_2 are points on an elliptic curve, and $f(\mathbf{d}) = f(d_1, d_2)$ in an (15) is a Riemann function of genus one. Another example: let G be a connected graph of Euler characteristic 0, let v_1, v_2 be distinct vertices of G, and let $f(\mathbf{d}) = f(d_1, d_2)$ be 1 plus the Baker-Norine rank of G at $d_1 v_1 + d_2 v_2$. Then f is Riemann function of genus 1.

0	1	2	3	4	5	6	
0	**1**	1	2	3	4	5	
0	0	**0**	1	2	3	4	
0	0	0	**0**	1	2	3	
0	0	0	0	**1**	1	2	
0	0	0	0	0	**1**	1	
0	0	0	0	0	0	**1**	$f(\mathbf{d}) = 0,1$ if $\deg(\mathbf{d}) = b$

Figure 2: Riemann Functions of Generalized Genus 1: This is equivalent to a Riemann function of genus 1 that has been "translated by b to the right" for some b: hence f is uniquely determined except when $\deg(\mathbf{d}) = b$, and in this case $f(\mathbf{d}) = 0,1$. The picture is almost identical, except for the shift in degree.

Example 2.8. *If in Example 2.3, the curve is of genus 1, and $P_1, \ldots, P_n$ are any points on the curve, then f in (15) is a Riemann function of genus 1. If $\mathbf{d} \in \mathbb{Z}$ with $\deg(\mathbf{d}) = 0$, then*

$$f(\mathbf{d}) = 1 \iff d_1 P_1 + \cdots + d_n P_n = 0$$

where $+$ is with respect to the group law in the curve with respect to some point (i.e., one chooses a point P_∞ and one defines $P_3 = P_1 + P_2$ if P_3 is linearly equivalent to $P_1 + P_2 - P_\infty$), and 0 is the identity element in the group law (i.e., the point P_∞).

Example 2.9. *Let G be a cycle of length $n \geq 2$, whose vertices in are, in cyclic order (of which there are $2n$ choices for $n \geq 3$) $v_1, \ldots, v_n$. If $r_{\rm BN}$ is the Baker-Norine rank, then $f(\mathbf{d}) = 1 + r_{\rm BN}(\mathbf{d})$ is a Riemann function of genus g. If $\deg(\mathbf{d}) = 0$, then we easily see that*

$$f(\mathbf{d}) = 0 \iff d_1 + 2d_2 + \cdots + (n-1)d_{n-1} \text{ is divisible by } n$$

(since $d_1 + \cdots + d_n = 0$, the above divisibility by n is independent of which cyclic order we choose for $v_1, \ldots, v_n$). Hence f is the same f as in Example 2.8 where P_1 is any point of order n on the curve, and $P_i = iP_1$ in the group law on the curve (hence we require the elliptic curve to have a point of order n, which is always the case in the classical case over the complex numbers).

Example 2.10. *The two-variable restriction of any function $\mathbb{Z}^n \to \mathbb{Z}$ that is of generalized genus 1 is again a function $\mathbb{Z}^2 \to \mathbb{Z}$ that is of generalized genus 1. (Similarly for a restriction to any number of variables.)*

Proposition 2.5. *Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a Riemann function of genus 1. Then f is slowly growing. Moreover, the weight, W, of f attains a negative value somewhere if and only if for some $d_1' \in \mathbb{Z}$, we have*

$$f(d_1', -d_1') = f(d_1' - 1, -d_1' + 1) = 1. \tag{22}$$

Hence W is everywhere non-negative if and only if

$$\forall \mathbf{d}, \quad \deg(\mathbf{d}) = 0 \text{ and } f(\mathbf{d}) = 1 \Rightarrow f(\mathbf{d} - \mathbf{e}_1 + \mathbf{e}_2) = 0.$$

Proof. We easily see that f is slowly growing.

Next, assume that $W(\mathbf{d}) < 0$ for some $\mathbf{d} \in \mathbb{Z}^2$; by Proposition (2.4), this is equivalent to

$$f(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_1) = f(\mathbf{d} - \mathbf{e}_2) = a = f(\mathbf{d} - \mathbf{e}_1 - \mathbf{e}_2) + 1. \tag{23}$$

holding for some $a \in \mathbb{Z}$. Note that $a \leq 0$ is impossible, for then

$$f(\mathbf{d} - \mathbf{e}_1 - \mathbf{e}_2) = a - 1 \leq -1,$$

which is impossible; hence $a \geq 1$. Also $a \geq 2$ is impossible, because otherwise $f(\mathbf{d} - \mathbf{e}_1) \geq 2$, and then the definition of a genus 1 function implies that the degree of $\mathbf{d} - \mathbf{e}_1$ is at least 2, and hence $f(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_1) + 1 \neq f(\mathbf{d} - \mathbf{e}_1)$, contradicting (23); hence $a \leq 1$.

Hence $1 \leq a \leq 1$, and so we necessarily have $a = 1$; hence

$$f(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_1) = f(\mathbf{d} - \mathbf{e}_2) = a = 1 \quad \text{and} \quad f(\mathbf{d} - \mathbf{e}_1 - \mathbf{e}_2) = 0.$$

Visualizing this as a grid, we have

$$\begin{matrix} f(d_1 - 1, d_2) = 1 & f(d_1, d_2) = 1 \\ f(d_1 - 1, d_2 - 1) = 0 & f(d_1, d_2 - 1) = 1 \end{matrix} \quad \text{and hence} \quad W(\mathbf{d}) = -1.$$

Moreover, implies that $d_1 + d_2 = 1$; hence, for $d_1' = d_1 - 1$ we have (22) holds. The converse is immediate, since (22) implies that

$$\begin{aligned} W(d_1' + 1, -d_1') &= f(d_1' + 1, -d_1') - f(d_1', -d_1') - f(d_1' + 1, -d_1' - 1) + f(d_1', -d_1' - 1) \\ &= 1 - 1 - 1 + 0 = -1. \end{aligned}$$

Finally, there exists a $d_1' \in \mathbb{Z}$ such that (22) if and only if

$$\exists \mathbf{d}, \quad \deg(\mathbf{d}) = 0 \text{ and } f(\mathbf{d}) = f(\mathbf{d} - \mathbf{e}_1 + \mathbf{e}_2) = 1.$$

This proves the last statement of the proposition. ☐

Corollary 2.1. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be of genus 1 such that for for some (necessarily distinct) $i, j \in [n]$ and all $\mathbf{d} \in \mathbb{Z}^n$ of degree 0 we have*

$$f(\mathbf{d}) = 1 \quad \Rightarrow \quad f(\mathbf{d} + \mathbf{e}_i - \mathbf{e}_j) = 0.$$

Then any restriction $f_{i,j,\mathbf{d}}$ has non-negative weight, i.e., its weight is a perfect matching.

Example 2.11. *Let $n = 4$ in Example 2.9 of a cycle of length 4, or the equivalent special case of Example 2.8, where P_1 is order 4, and $P_i = iP_1$ for $i = 2, 3, 4$. We easily see that f satisfies the hypothesis of Corollary 2.1 for all distinct $i, j \in [n]$. Then the two-variable restriction of f, $f_{1,2,\mathbf{0}}$, is a Riemann function, and we easily see that for $a_1 \in \mathbb{Z}$ we have*

$$f_{1,2,\mathbf{0}}(a_1, -a_1) = 1 \iff a_1 \bmod 4 = 0.$$

Similarly for $f_{1,3,\mathbf{0}}$, except that

$$f_{1,3,\mathbf{0}}(a_1, -a_1) = 1 \iff a_1 \bmod 2 = 0.$$

We easily see that the weight, W, of $f_{1,2,\mathbf{0}}$ has period 4 and $W(0,0) = W(1,1) = W(-1,2) = W(-2,1) = 1$ (which determines W everywhere). By contrast, the weight, W' of $f_{1,3,\mathbf{0}}$ has period 2 and satisfies $W'(0,0) = W'(1,1) = 1$. Hence $W'(-1,3) = W'(-2,2) = 1$, and so $W' \neq W$.

3 Weight Decomposition Theorems

In this section, we prove two theorems about decomposing the weight of Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$ into an alternating sum of perfect matchings. These are fundamental steps in modeling an arbitrary Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$.

3.1 r-fold Matchings and Infinite Versions of Hall's Theorem

In this subsection, we define r-fold matchings, which feature prominently in our models; we also give two infinite versions of Hall's Theorem which helps to understand r-fold matchings but are not essential to the rest of this article.

Definition 3.1. *Let $W : \mathbb{Z}^2 \to \mathbb{Z}$ be initially and eventually zero. For $i \in \mathbb{Z}$, the i-th* row sum *of W (respectively,* column sum*) is $\sum_{j\in\mathbb{Z}} W(i, j)$ (respectively, $\sum_{j\in\mathbb{Z}} W(j, i)$). For any $r \in \mathbb{N}$, we say that W is an r-*fold matching *if all values of W are non-negative, and all the row sums and all the column sums of W equal r.*

Of course, a perfect matching (Definition 2.9) is a 1-fold matching, and the sum of r perfect matchings is an r-fold matching. The rest of this subsection is devoted to proving the converse, both for general r-fold matchings and for p-periodic matchings (as equaling a sum of r p-periodic perfect matchings).

Definition 3.2. *If W is a perfect matching, we refer to the π satisfying* (21) *as the* bijection associated to W*; if $\pi : \mathbb{Z} \to \mathbb{Z}$ is a bijection with $\pi(i) + i$ bounded independent of i, we say that W in* (21) *is the* perfect matching associated to π.

Of course, if W is a perfect matching with associated bijection π, then for any $p \geq 1$, W is p-periodic if and only if for all $i \in \mathbb{Z}$ we have $\pi(i + p) = \pi(i) - p$.

Theorem 3.1. *Let W be an r-fold matching that is p-periodic. Then there exist p-periodic perfect matchings $W_1, \ldots, W_r$ whose sum is W.*

Proof. We will first prove that there is a perfect matching W_1 such that $W_1(i, j) \leq W(i, j)$ for all i, j; if so, then $W - W_1$ is an $(r - 1)$-fold matching, and hence we prove the theorem by induction on r.

Let $G = (V, E)$ be the bipartite graph, where $V = V_L \amalg V_R$ and all edges run from left to right, where $V_L = V_R = \{0, 1, \ldots, p - 1\}$, and the number of edges running from $i \in V_L$ to $j \in V_R$ is just

$$e(i, j) = \sum_{m\equiv j \bmod p} W(i, m).$$

Then G is a finite bipartite graph that is r-regular on both sides, i.e., each vertex is incident upon exactly r edges. It then follows that if $V' \subset V_L$, and $\Gamma(V')$ denotes the set of neighbours of V_L, i.e., of vertices (in V_R) adjacent to some vertex of V', then $|\Gamma(V')| \geq |V'|$ (since V' is incident upon $r|V'|$ edges, whose right endpoints span at least $|V'|$ vertices). Similarly if $V' \subset V_R$, then also $|\Gamma(V')| \geq |V'|$. Then Hall's theorem implies that G has a perfect matching, i.e., a subgraph $G' = (V, E')$ where each vertex is adjacent to exactly one vertex. This gives us a bijection $\pi : V_L \to V_R$ such that $e(i, \pi(i)) \geq 1$. For each $i = 0, \ldots, p-1$ choose a $j = \pi_1(i)$ such that $j \equiv \pi(i) \bmod p$ and such that $W(i, j) \geq 1$. Now extend $\pi_1(i) : \{0, \ldots, p - 1\} \to \mathbb{Z}$ to a function $\mathbb{Z} \to \mathbb{Z}$ by setting for all $m \in \mathbb{Z}$ and $i \in \{0, \ldots, p - 1\}$

$$\pi_1(i + pm) = \pi(i) - pm.$$

It follows that π is a bijection, and that its associated weight, W_1, satisfies $W(i, j) \geq 1$ whenever $W_1(i, j) = 1$. Hence $W_1 \leq W$ and we have our desired W_1. ☐

The next case we prove is the same theorem without the assumption of periodicity.

Theorem 3.2. *Let W be an r-fold matching. Then there exist perfect matchings $W_1, \ldots, W_r$ whose sum is W.*

The proof is well-known, based on the a general principle that Philip Hall's "marriage theorem" holds on a bipartite graph with countably many vertices on each side, provided that each vertex has finitely many neighbours (namely, in our case, at most r neighbours); see Marshall Hall's textbook (e.g., Theorem 5.1.2 of [14]); we give a proof—in terms of our language—for ease of reading this article.

Proof. Again, it suffices to show that there is a perfect matching W_1 such that $W_1(i, j) \leq W(i, j)$ for all i, j, and then to prove the above theorem by induction on r.

Consider the bipartite graph, $G = (V, E)$, $V = V_L \amalg V_R$ with $V_L = V_R = \mathbb{Z}$, and where the number of edges from $i \in V_L$ to $j \in V_R$ is $W(i, j)$. Let V be enumerated as $v_1, v_2, \ldots$. Again, $|\Gamma(V')| \geq |V'|$ since G is r-regular on both sides, and hence the augmenting path technique to prove Hall's theorem shows that for any m there is a matching $G_m = (V_m, E_m) \subset G$, meaning that each vertex of V_m is adjacent via E_m to exactly one other vertex of V_m, with the property that $\{v_1, \ldots, v_m\} \subset V_m$.

Now we use $G_1, G_2, \ldots$ to build a perfect matching $G' \subset G$. Namely, since $v_1 \subset V_m$ for $m \geq 2$, and since v_1 is adjacent to at most r-vertices in G, there is an (infinite) subsequence of $G_1, G_2, \ldots$ in which v_1 is adjacent to some fixed vertex of V; next, we choose a further infinite subsequence in which v_2 is adjacent to some fixed vertex of V; we similarly apply this process to $v_3, v_4, \ldots$. This gives a fixed matching defined on $v_1, v_2, \ldots$, and therefore on all of V. Hence, we get a bijection $\pi_1 \colon \mathbb{Z} \to \mathbb{Z}$ such that for all $i \in \mathbb{Z}$, $W(i, \pi_1(i)) \geq 1$. Hence, we take W to be the weight associated with π_1. ☐

3.2 Main Lemmas about Weights $\mathbb{Z}^2 \to \mathbb{Z}$ as an Alternating Sum of Perfect Matchings

In this subsection, we prove that any slowly growing Riemann function has a weight that can be written as the alternating sum of perfect matchings. It will be convenient to prove a more general result, namely Lemma 3.1 below.

Definition 3.3. *Let $r \in \mathbb{Z}$. By an* r-regular weight *we mean a function $W \colon \mathbb{Z}^2 \to \mathbb{Z}$ such that:*

1. *W is initially and eventually zero;*
2. *each row sum and each column sum of W equals r;*
3. *for some $C \in \mathbb{N}$, for all $\mathbf{d} \in \mathbb{Z}^2$ we have $|W(\mathbf{d})| \leq C$.*

The rest of this subsection is devoted to proving the following lemma.

Lemma 3.1. *Let $r \in \mathbb{Z}$, and let W be a r-regular weight. Then for some $\ell \in \mathbb{N}$ we may write*

$$W = (W_1 + W_2 + \cdots + W_\ell) - (\tilde{W}_1 + \cdots + \tilde{W}_{\ell - r})$$

where $W_1, \ldots, W_\ell$ and $\tilde{W}_1, \ldots, \tilde{W}_{\ell-r}$ are perfect matchings. Moreover, if W is p-periodic, then we may take each W_i and $\tilde{W}_i$ to be p-periodic.

Lemma 3.1 is immediate if there exists an $a \in \mathbb{Z}$, such that $W(\mathbf{d}) = 0$ whenever $\deg(\mathbf{d}) \neq a$, for then we must have $W(\mathbf{d}) = r$ whenever $\deg(\mathbf{d}) = a$. It will be helpful to introduce some definitions and notation related to this simple observation.

Definition 3.4. *For $b \in \mathbb{Z}$, the* perfect matching in degree b, *denoted W_b, refers to the perfect matching given by*

$$W_b(\mathbf{d}) = \begin{cases} 1 & \text{if } \deg(\mathbf{d}) = b, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Hence, each W_b is 1-periodic (and any perfect matching that is 1-periodic is of this form).

Definition 3.5. *Let $W \colon \mathbb{Z}^2 \to \mathbb{Z}$ be any function. The* support of W *is the set of $\mathbf{d} \in \mathbb{Z}^2$ such that $W(\mathbf{d}) \neq 0$. If $a, b \in \mathbb{Z}$ with $a < b$, we say that W is* supported in degrees a through b *if the support of W is a subset of those $\mathbf{d} \in \mathbb{Z}^2$ with $a \leq \deg(\mathbf{d}) \leq b$.*

Lemma 3.2. *Let W be an r-regular weight for some $r \in \mathbb{Z}$, and say that there exists an $a \in \mathbb{Z}$ such that W is supported in degrees a and $a+1$. Then for some $c \in \mathbb{Z}$ we have*

$$W(\mathbf{d}) = \begin{cases} c & \text{if } \deg(\mathbf{d}) = a, \\ r - c & \text{if } \deg(\mathbf{d}) = a+1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, W can be written as a difference of a sum of perfect matchings, each of which equals either W_a or W_{a+1} with notation as in (24).

Proof. Let $W(0, a) = c$. Then $W(0, a+1) = W(1, a) = r - c$, given that the 0-th column sum and a-th row sum of W both equal r. Similarly, $W(1, a-1) = W(-1, a+1) = c$. It then follows by induction on $m = 2, 3, \ldots$ that $W(m, a-m) = W(-m, a+m) = c$ and $W(m, a-m+1) = W(-m, a+m+1) = r - c$.

For the second claim, we have

$$W = cW_a + (r - c)W_{a+1}; \tag{25}$$

if $r, c - r$ are both non-negative, then (25) expresses W as a sum of r perfect matchings; otherwise (25) expresses W as a difference of two sums of perfect matchings (since $r \geq 0$, and hence at least one of $r, c - r$ is non-negative). ☐

To prove Lemma 3.1 we will use induction on $b-a$ where W is supported on elements of degrees between a and b. The discussion above deals with the cases where $b=a$ or $b=a+1$. Let us explain the inductive step. For this, it will be helpful to introduce the following notation: first, let $U\colon \mathbb{Z}^2 \to \mathbb{Z}$ be the function

$$U(\mathbf{d}) = \begin{cases} 1 & \text{if } \mathbf{d} = (0,0), (1,1), \\ -1 & \text{if } \mathbf{d} = (1,0), (0,1), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

[For intuition, note that all row and columns sums of U equal 0.] For a doubly-infinite sequence $S = \{\dots, s_{-1}, s_0, s_1, \dots\}$ of integers, we use the notation U_S to denote the function $\mathbb{Z}^2 \to \mathbb{Z}$ given by

$$U_S(\mathbf{d}) = \sum_{i\in\mathbb{Z}} U\bigl(\mathbf{d} + (i,-i)\bigr) s_i \tag{26}$$

[for intuition, it may help to observe that U_S is the convolution of U with the function supported in degree 0 taking $(i,-i)$ to s_{-i}]. Hence we have U_S is supported on $\mathbf{d}$ of degrees $0,1,2$, and for all $a \in \mathbb{Z}$, $U_S(a,-a) = s_a$. Clearly, all row and column sums of U are zero, and hence the same holds of U_S.

Here is the essential ingredient in our inductive step.

Lemma 3.3. *Let $S = \{\dots, s_{-1}, s_0, s_1, \dots\}$ be any doubly-infinite sequence of elements of $\{0,1\}$. Then U_S (as in (26)) can be written as the difference of a sum of perfect matchings supported in degrees 0 through 2. Furthermore, if for some $p \in \mathbb{N}$ we have $s_{a+p} = s_a$ for all $a \in \mathbb{Z}$, then the perfect matchings in the sums can be taken to be p-periodic.*

Proof. First, consider the lemma in the case where S is not assumed to be periodic, in the special case where for all $a \in \mathbb{Z}$ with a odd we have $s_a = 0$; let us prove the lemma in this situation. Let W_1 be as (24) and let W be given as follows: for all $t \in \mathbb{Z}$

1. if $s_{2t} = 1$, $W(2t,-2t) = W(2t+1,-2t+1) = 1$;
2. if $s_{2t} = 0$, $W(2t+1,-2t) = W(2t,-2t+1) = 1$;
3. all other values of W not specified above are zero.

We easily check that:

1. W and W_1 are perfect matchings;
2. W, W_1 are supported in degrees 0 through 2;
3. we have $W - W_1 = U_S$.

Hence, this proves the lemma in this special case of S.

We may similarly show the case where for all $a \in \mathbb{Z}$ with a even, we have $s_a = 0$ (i.e., by translating the construction in the last paragraph by $(1,-1)$).

In general we can write $S = S_{\text{even}} + S_{\text{odd}}$ (where + means adding the sequences element-by-element), where

$$S_{\text{even}} = (\dots, s_{-2}, 0, s_0, 0, s_2, 0 \dots)$$
$$S_{\text{odd}} = (\dots, 0, s_{-1}, 0, s_1, 0, s_3, \dots)$$

As such, we have

$$U_S = U_{S_{\text{even}}} + U_{S_{\text{odd}}},$$

and now we can write $U_{S_{\text{even}}}$ and $U_{S_{\text{odd}}}$ each as the difference of two perfect matchings. This proves the lemma in the non-periodic case.

Next, consider the case when S is p-periodic.

If $p=1$, then s_a are all 1 or 0; if they are all 0 then U_S is identically zero, and otherwise $U_S = W_0 - 2W_1 + W_2$ with notation as in (24).

Hence, we may assume $p \geq 2$. If p is even, then we can write U_S as above, and notice that S_{even} and S_{odd} are p-periodic. Then it follows that when we write $U_{S_{\text{even}}} = W - W_1$ as above, both W and W_1 are p-periodic (W_1 is 1-periodic), and similarly for $U_{S_{\text{odd}}}$. This solves the lemma in this case.

The only case that remains is when S is p-periodic when $p \geq 3$ is odd (in which case $S_{\text{even}}, S_{\text{odd}}$ are not p-periodic). In this case, we take a similar approach, being careful to have the matchings all p-periodic as follows: first, consider the special case of S for which $s_a = 0$ whenever $a \in \mathbb{Z}$ is not divisible by p. For each $t \in \mathbb{Z}$ we let W be the following perfect matching:

1. if $s_{pt} = 1$, $W(pt,-pt) = W(pt+1,-pt+1) = 1$;

2. if $s_{pt} = 0$, $W(pt+1, -pt) = W(pt, -pt+1) = 1$;

3. for all a not equal to pt or $pt+1$ for some t, i.e., with $a \bmod p \neq 0, 1$, (where $a \bmod p$ is meant as taking a value between 0 and $p-1$), $W(a, 1-a) = 1$;

4. all other values of W not specified above are zero.

We then have that W is p-periodic, and $U_S = W - W_1$. This solves the lemma in this case.

Similarly, we solve the lemma in the case for some $i = 1, \ldots, p-1$ we have $s_a = 0$ for all $a \in \mathbb{Z}$ with $a \bmod p \neq i$.

For the general case of S p-periodic with $p \geq 3$ odd, we write

$$S = S_0 + \cdots + S_{p-1},$$

where for $i = 0, \ldots, p-1$, $(S_i)_a = 0$ if $a \bmod p \neq i$. Then we write each U_{S_i} as a difference of periodic matchings supported in degrees 0 through 2, and use

$$U_S = U_{S_0} + \cdots + U_{S_{p-1}}$$

to write U_S as the difference of sum of p perfect matchings, each p-periodic. □

Proof of Lemma 3.1. First, let us prove the lemma in the case where W attains only non-negative values.

Let us prove the lemma in this case by induction on $m = 0, 1, \ldots$ for all W supported in degrees a through $a+m$. The cases $m = 0, 1$ are given in Lemma 3.2. Now consider the inductive step, where the lemma holds for $m-1$ for some $m \geq 2$ and we wish to prove it for m. By translating W, we may assume that it is supported in degrees 0 through m. Let

$$C = \max_{t \in \mathbb{Z}} W(t, -t).$$

Now let us prove our desired inductive step, i.e., that the lemma holds for W supported in degrees 0 through m, by using induction on $C = 0, 1, \ldots$ For $C = 0$, it follows that W is supported in degrees 1 through m, and hence by translation we can reduce the theorem to the case $m-1$.

Now, say the claim holds for some value of m and $C \geq 0$, and say that $W(t, -t) \leq C+1$. Let $S = \{\ldots, s_{-1}, s_0, s_1, \ldots\}$ be given by

$$s_t = \min\bigl(1, W(t, -t)\bigr).$$

Then $s_t \in \{0, 1\}$ for all t, and if W is p-periodic then $s_{t+p} = s_t$ for all $t \in \mathbb{Z}$. According to Lemma 3.3 we can find a difference of sums of perfect matchings supported in degrees 0 through 2—all p-periodic if W is p-periodic—whose value at $(a, -a)$ equal s_a. Subtracting this difference of sums from W we get W' where $W'(t, -t) = W(t, -t) - s_t$, to which we can apply the inductive claim.

This proves the lemma assuming $W(\mathbf{d}) \geq 0$ for all $\mathbf{d}$. If W is supported in degrees a and b and is bounded in absolute value by $C \in \mathbb{N}$, then with notation in (24) we have

$$W' = W + C(W_a + \cdots + W_b)$$

attains only non-negative values for some C sufficiently large, and is an r'-fold matching for $r' = r + C(b-a+1)$. Hence, we apply the lemma to W', and then subtract $C(W_a + \cdots + W_b)$. □

4 Diagrams, Betti Numbers, and Models for Riemann Functions Whose Weight is a Perfect Matchings

In this section, we introduce our basic models and develop some of their properties. We will especially study those related to Riemann functions $f : \mathbb{Z}^2 \to \mathbb{Z}$ whose weight is a perfect matching; such functions have a number of especially remarkable properties.

4.1 Conventions Regarding Linear Algebra: Cohomology, Betti Numbers, Fredholm Maps, and Direct Sums

In this subsection, we recall some basic concepts in linear algebra that we will need to compute the sheaf invariants of interest to us. Our motivation is that the invariants of the sheaves that we use can be computed as the kernel and the cokernel[1] of an associated linear map.

[1]If $\tau : B \to A$ is a linear map of vector spaces, the *cokernel* of τ is $A/\mathrm{Image}(\tau)$.

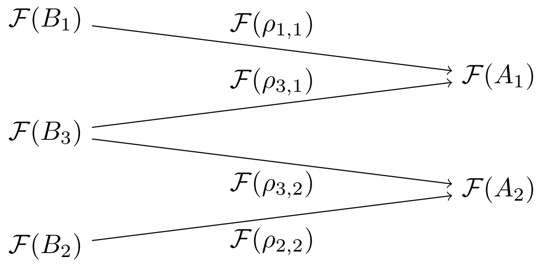


Figure 3: Our Diagrams

Let k be a field and $\tau\colon B \to A$ a linear map of k-vector spaces B, A. For $i = 0,1$ we define the *i-th cohomology group of τ* to be, respectively

$$H^0(\tau) \stackrel{\text{def}}{=} \ker(\tau), \quad H^1(\tau) \stackrel{\text{def}}{=} \operatorname{coker}(\tau),$$

and the *i-th Betti number* of τ to be $b^i(\tau) = \dim_k H^i(\tau)$; we say that τ is a *k-Fredholm map*, or simply *Fredholm*, if both Betti numbers are finite, and if so, we define the *Euler characteristic of τ* (also known is its *index*) to be

$$\chi(\tau) = b^0(\tau) - b^1(\tau);$$

if exactly one of $b^0(\tau)$ and $b^1(\tau)$ is infinite, we may also define $\chi(\tau)$ as $\pm\infty$ accordingly.

[Hence the i-th cohomology group and i-th Betti number of τ is the usual notion when we view τ as a chain

$$\cdots \to 0 \to B \to A \to 0 \to \cdots$$

with B positioned in degree 0.]

If $\{B_i\}_{i\in I}$ is a family of k-vector spaces indexed on a set I, we define its *direct sum*, denoted $\oplus_{i\in I} B_i$ as usual, i.e., the vector space of tuples $\{b_i\}_{i\in I}$ such that each $b_i \in B_i$ and all but finitely many of the b_i are zero. If $\tau_i\colon B_i \to A_i$ is a family of k-linear maps of vector spaces indexed on $i \in I$, we define the *direct sum* of $\{\tau_i\}_{i\in I}$ as usual, i.e., as the map

$$\bigoplus_{i\in I} \tau_i \colon \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} A_i;$$

we easily check that for $j = 0,1$ we have a simple isomororphism

$$H^j\left(\bigoplus_{i\in I} \tau_i\right) \simeq \bigoplus_{i\in I} H^j(\tau_i),$$

and hence Betti numbers

$$b^j\left(\bigoplus_{i\in I} \tau_i\right) = \sum_{i\in I} b^j(\tau_i),$$

so that if all the τ_i are Fredholm maps, where all but finitely many of the τ_i have both Betti numbers equal to zero, we get a finite and well-defined Euler characteristic

$$\chi\left(\bigoplus_{i\in I} \tau_i\right) = \sum_{i\in I} \chi(\tau_i).$$

4.2 k-Diagrams: Diagrams of k-Vector Spaces

Our models of Riemann functions will be k-linear maps $\tau\colon B \to A$, which are built from one fixed type of "diagram" of vector spaces, depicted in Figure 3 and which we now make precise.

Definition 4.1. *Let k be a field. By a* diagram of k-vector spaces*, or simply a* k-diagram *we mean a collection, $\mathcal{F}$, of data consisting of:*

1. *five k-vector spaces,*
$$\mathcal{F}(B_1), \mathcal{F}(B_2), \mathcal{F}(B_3), \mathcal{F}(A_1), \mathcal{F}(A_2)$$
called the values *of $\mathcal{F}$; and*

2. *k-linear maps $\mathcal{F}(\rho_{i,j})\colon \mathcal{F}(B_i) \to \mathcal{F}(A_j)$ for the pairs (i,j) where $(i,j) \in \{(1,1),(2,2),(3,1),(3,2)\}$ (i.e., $\mathcal{F}(\rho_{1,2})$ and $\mathcal{F}(\rho_{2,1})$ don't exist); we call the $\mathcal{F}(\rho_{ij})$ the* restriction maps *of $\mathcal{F}$.*

To this diagram, we associate the vector spaces

$$\mathcal{F}(B) = \mathcal{F}(B_1) \oplus \mathcal{F}(B_2) \oplus \mathcal{F}(B_3), \quad \mathcal{F}(A) = \mathcal{F}(A_1) \oplus \mathcal{F}(A_2),$$

and the linear transformation $\mathcal{F}(\partial)\colon \mathcal{F}(B) \to \mathcal{F}(A)$*, called the* differential of $\mathcal{F}$*, given by*

$$\mathcal{F}(\partial)(b_1, b_2, b_3) = \bigl(\mathcal{F}(\rho_{1,1})(b_1) - \mathcal{F}(\rho_{3,1})(b_3), \mathcal{F}(\rho_{2,2})(b_2) - \mathcal{F}(\rho_{3,2})(b_3)\bigr), \tag{27}$$

and define the zeroth and first cohomology groups of $\mathcal{F}$ *to be, respectively*

$$H^0(\mathcal{F}) \stackrel{def}{=} \ker\bigl(\mathcal{F}(\partial)\bigr), \quad H^1(\mathcal{F}) \stackrel{def}{=} \operatorname{coker}\bigl(\mathcal{F}(\partial)\bigr),$$

i.e., the kernel and cokernel of $\mathcal{F}(\partial)$*. If* $(b_1, b_2, b_3) \in H^0(\mathcal{F})$*, and* $a_j = \mathcal{F}(\rho_{jj})b_j$ *for* $j = 1, 2$*, then the tuple* $(b_1, b_2, b_3, a_1, a_2)$ *satisfies* $\mathcal{F}(\rho_{ij})b_i = a_j$ *whenever* $\mathcal{F}(\rho_{ij})$ *is defined, and we refer to* $(b_1, b_2, b_3, a_1, a_2)$ *as a* global section of $\mathcal{F}$*.*

Convention 4.1. *When we speak of a* k-vector space *or a* k-diagram *without prior reference to* k*, we understand* k *to be an arbitrary field.*

Note that if $(b_1, b_2, b_3, a_1, a_2)$ is a global section, then $(b_1, b_2, b_3) \in H^0(\mathcal{F})$; this therefore gives a bijection between global sections of $\mathcal{F}$ and $H^0(\mathcal{F})$, i.e., the kernel of the differential $\mathcal{F}(\partial)$ of $\mathcal{F}$. [Global sections tend to be conceptually more useful, but equivalent descriptions are useful in certain computations; we will later give another equivalent description of global sections as elements of $\operatorname{Hom}(\underline{k}, \mathcal{F})$.]

In Section 10, we will explain that our choice of $\mathcal{F}(\partial)$ and $H^1(\mathcal{F})$ are not canonical, but involve a choice of basis for each of two one-dimensional vector spaces; see the proof of Lemma 10.1 and the remark after its proof; however, this choice does not affect $H^0(\mathcal{F})$, i.e., the kernel of $\mathcal{F}(\partial)$.

4.3 Conventions on Sets, Multisets, Induced Vector Spaces, and $\mathcal{M}_{W,\mathbf{d}}$ for Non-Negative Weights

Definition 4.2. *Let* k *be a field. If* S *is a set, we use* $k^{\oplus S}$ *to denote the* k*-vector space that is the direct sum of one copy of* k *for each element of* S*, i.e., whose elements are collections* $\{v_s\}_{s\in S}$ *with* $v_s \neq 0$ *for at most finitely many values of* s*; for* $s \in S$*, we use* $\mathbf{e}_s$ *to denote the vector that is* 1 *in component* s *and* 0 *elsewhere. If* T *is another set and* $\alpha\colon S \to T$ *a map of sets, then* α *gives rise to a unique* k*-linear transformation, denoted* $k^{\oplus\alpha}$*, from* $k^{\oplus S} \to k^{\oplus T}$ *taking* $\mathbf{e}_s$ *to* $\mathbf{e}_{\alpha(s)}$*. If* $S \subset T$*, then the inclusion map* $\iota\colon S \to T$ *gives an injection* $k^{\oplus\iota}$ *which we call the* inclusion map (of $k^{\oplus S}$ to $k^{\oplus T}$).

In the above one, easily checks that if α is an injection, surjection, or bijection, then the same is true of $k^{\oplus\alpha}$. Next, we fix a convention for multisets (any reasonable convention would suffice).

Definition 4.3. *Let* S_1, S_2 *be sets, and* $W\colon S_1 \times S_2 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$*. The* multiset on $S_1 \times S_2$ with multiplicities W *refers to the set*

$$\operatorname{Multi}(W) = \{(s_1, s_2, i) \in S_1 \times S_2 \times \mathbb{N} \mid i \leq W(s_1, s_2)\}, \tag{28}$$

where if $W(s_1, s_2) = \infty$*, then we view all* i *as satisfying* $i \leq W(s_1, s_2)$*. We refer to the maps* $\operatorname{Multi}(W) \to S_1$ *and* $\operatorname{Multi}(W) \to S_2$ *taking* (s_1, s_2, i) *to, respectively,* s_1 *and* s_2*, as, respectively, the* first and second projections*. We use the notation* $k^{\oplus W}$ *to denote* $k^{\oplus \operatorname{Multi}(W)}$*, which comes with maps*

$$\operatorname{proj}_i\colon k^{\oplus W} \to k^{\oplus S_i} \tag{29}$$

induced by the first and second projections. The support of W *is the set of* $(s_1, s_2) \in S_1 \times S_2$ *such that* $W(s_1, s_2) \geq 1$*. When* W *takes on only the values* $\{0, 1\}$*, then with mild abuse of notation we may identify* $\operatorname{Multi}(W)$ *with its support, which is a subset of* $S_1 \times S_2$*, since in this case* W *is determined by its support.*

Example 4.1. *If* $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ *is a perfect matching, and* $\pi\colon \mathbb{Z} \to \mathbb{Z}$ *is its associated bijection, then* $k^{\oplus W}$ *has one copy of* k *for each pair* $(a_1, \pi(a_1)) \in \mathbb{Z}^2$ *varying over all* $a_1 \in \mathbb{Z}$*. In this case we may identify* $k^{\oplus W}$ *with* $k^{\oplus\mathbb{Z}}$*, where the first projection is the identity map on* $k^{\oplus\mathbb{Z}}$*, and the second projection is the map* $k^{\oplus\mathbb{Z}} \to k^{\oplus\mathbb{Z}}$*, takes* $\mathbf{e}_{a_1}$ *to* $\mathbf{e}_{\pi(a_1)}$*. Hence both maps* (29) *are isomorphisms.*

Definition 4.4. *Let* k *be a field,* $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$*, and* $\mathbf{d} \in \mathbb{Z}^2$*. We use* $\mathcal{M}_{W,\mathbf{d}}$ *to denote the following* k*-diagram (Definition 4.1):*

1. *for* $i = 1, 2$*,* $\mathcal{M}_{W,\mathbf{d}}(B_i) = k^{\oplus\mathbb{Z}_{\leq d_i}}$*,* $\mathcal{M}_{W,\mathbf{d}}(A_i) = k^{\oplus\mathbb{Z}}$*,* $\rho_{i,i}$ *is the inclusion,*
2. $B_3 = k^{\oplus W}$*, and for* $j = 1, 2$*,* $\rho_{3,j}$ *are the projection maps (as in* (29)*).*

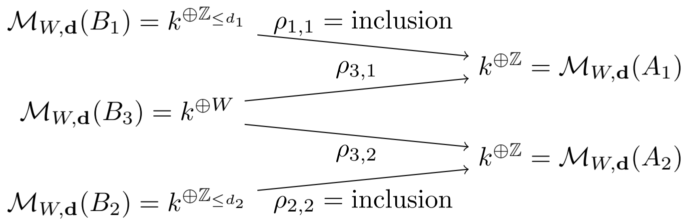


Figure 4: The k-Diagram $\mathcal{M}_{W,\mathbf{d}}$.

We depict these k-diagrams in Figure 4.

The cohomology groups of $\mathcal{M}_{W,\mathbf{d}}$ are therefore the kernel and cokernel of the maps

$$\tau_{W,\mathbf{d}} = \mathcal{M}_{W,\mathbf{d}}(\partial)\colon k^{\oplus \mathbb{Z}_{\le d_1}} \oplus k^{\oplus W} \oplus k^{\oplus \mathbb{Z}_{\le d_2}} \to k^{\oplus \mathbb{Z}} \oplus k^{\oplus \mathbb{Z}}$$

given as the map

$$(b_1, b_3, b_2) \mapsto \big(b_1 - k^{\mathrm{pr}_1}(b_3), b_2 - k^{\mathrm{pr}_2}(b_3)\big),$$

where pr_i denotes the i-th projection $k^{\oplus W} \to k^{\oplus \mathbb{Z}}$.

4.4 The Euler Characteristic of $\mathcal{M}_{W,\mathbf{d}}$ as a Function of d and Riemann Functions

Before computing the Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$ for specific W of interest, we wish to point out some general properties of their Betti numbers and Euler characteristics. In particular, we will prove that if for some $\mathbf{d}$ and W we have that $\chi(\mathcal{M}_{W,\mathbf{d}})$ is well-defined, i.e., at least one of the Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$ is finite, then

$$\chi(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_1}) = \chi(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_2}) = \chi(\mathcal{M}_{W,\mathbf{d}}) + 1. \tag{30}$$

This is an easy consequence of the following lemma.

Lemma 4.1. *Let $\tau\colon B \to A$ be a linear map of k-vector spaces, and let $B' \subset B$ be a subspace of codimension one, and let $\tau' = \tau|_{B'}$, i.e., the restriction of τ to B'. Then if either $\chi(\tau)$ or $\chi(\tau')$ is well defined, then so is the other, and*

$$\chi(\tau) = \chi(\tau') + 1. \tag{31}$$

In more detail, either

$$b^0(\tau) = b^0(\tau') + 1 \quad \text{and} \quad b^1(\tau) = b^1(\tau') \tag{32}$$

or

$$b^0(\tau) = b^0(\tau') \quad \text{and} \quad b^1(\tau) = b^1(\tau') - 1, \tag{33}$$

where we allow for these Betti numbers to equal ∞, in which case $\infty \pm 1$ is taken to ∞.

Proof. The proof of this lemma is straightforward: since B' has codimension 1 in B, $\ker(\tau') = \ker(\tau) \cap B'$ has either codimension 1 or 0 in $\ker(\tau)$; we easily verify that codimension 1 implies (32) and codimension 0 implies (33). Both cases (32) and (33) imply (31). □

Corollary 4.1. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$. Then for any $\mathbf{d} \in \mathbb{Z}^2$ and $i = 1, 2$, we have that the conclusions of Lemma 4.1 hold for $\tau = \mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}(\partial)$ and $\tau' = \mathcal{M}_{W,\mathbf{d}}(\partial)$. In particular, for any $\mathbf{d}$ and $i = 1, 2$ we have*

$$b^0(\mathcal{M}_{W,\mathbf{d}}) \le b^0(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}) \le b^0(\mathcal{M}_{W,\mathbf{d}}) + 1 \tag{34}$$

(which makes sense if these Betti numbers equal $+\infty$, in which case the above reads $+\infty \le +\infty \le +\infty$) and

$$b^1(\mathcal{M}_{W,\mathbf{d}}) + 1 \ge b^1(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}) \ge b^1(\mathcal{M}_{W,\mathbf{d}}).$$

In particular, if at least one of the Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$ is finite, or one of $\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_1}$ or $\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_2}$, then (30) *holds.*

Applying this corollary repeatedly we get the following result.

Theorem 4.1. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\ge 0}$ be a function such that $\chi(\mathcal{M}_{W,\mathbf{d}})$ is well defined for some $\mathbf{d} \in \mathbb{Z}^2$. Then for all $\mathbf{d}' \in \mathbb{Z}^2$, $\chi(\mathcal{M}_{W,\mathbf{d}'})$ is well defined, and*

$$\chi(\mathcal{M}_{W,\mathbf{d}+\mathbf{d}'}) = \chi(\mathcal{M}_{W,\mathbf{d}}) + \deg(\mathbf{d}'); \tag{35}$$

equivalently, for all $\mathbf{d}\in\mathbb{Z}^2$ *we have*

$$b^0(\mathcal{M}_{W,\mathbf{d}})-b^1(\mathcal{M}_{W,\mathbf{d}})=\ \chi(\mathcal{M}_{W,\mathbf{d}})=\deg(\mathbf{d})+C,\quad \textit{where}\quad C=\chi(\mathcal{M}_{W,\mathbf{0}}). \tag{36}$$

Furthermore, if for $\deg(\mathbf{d})$ *sufficiently small we have* $b^0(\mathcal{M}_{W,\mathbf{d}})=0$*, and for* $\deg(\mathbf{d})$ *sufficiently large we have* $b^1(\mathcal{M}_{W,\mathbf{d}})=0$*, then* $f(\mathbf{d})=b^0(\mathcal{M}_{W,\mathbf{d}})$ *is a slowly growing Riemann function, and for any* $\mathbf{K}$ *we have*

$$b^1(\mathcal{M}_{W,\mathbf{d}})=f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d}).$$

Proof. Applying Corollary 4.1 repeatedly we see that if $\chi(\mathcal{M}_{W,\mathbf{d}})$ is well defined, then (35) holds for all $\mathbf{d}'\ge\mathbf{d}$, and in particular for a_1,a_2 sufficiently large we have

$$\chi(\mathcal{M}_{W,a_1\mathbf{e}_1+a_2\mathbf{e}_2})=\chi(\mathcal{M}_{W,\mathbf{d}})+a_1+a_2-d_1-d_2. \tag{37}$$

Then applying Corollary 4.1 repeatedly to $\mathbf{a}=(a_1,a_2)$ we have that for all $\mathbf{d}'\le\mathbf{a}$

$$\chi(\mathcal{M}_{W,\mathbf{d}'})=\chi(\mathcal{M}_{W,a_1\mathbf{e}_1+a_2\mathbf{e}_2})+d_1'+d_2'-a_1-a_2,$$

which, in view of (37), equals the left-hand-side of (35). Hence (35) holds for all $\mathbf{d}'$. Applying this with $\mathbf{d}'$ replaced with an arbitrary $\mathbf{d}''\in\mathbb{Z}^2$ and subtracting yields

$$\chi(\mathcal{M}_{W,\mathbf{d}''})-\chi(\mathcal{M}_{W,\mathbf{d}'})=\deg(\mathbf{d}''-\mathbf{d}'),$$

for all $\mathbf{d}'',\mathbf{d}'$, and setting $\mathbf{d}'=\mathbf{0}$ and $\mathbf{d}''=\mathbf{d}$ yields (36).

Setting $f(\mathbf{d})=b^0(\mathcal{M}_{W,\mathbf{d}})$, we have that if $f(\mathbf{d})=0$ for $\deg(\mathbf{d})$ sufficiently small, then f is intially zero; according to Corollary 4.1, $f(\mathbf{d})$ is finite for all $\mathbf{d}$ and, by (34), it is slowly growing; if $b^1(\mathcal{M}_{W,\mathbf{d}})=0$ for $\deg(\mathbf{d})$ large, then for such $\mathbf{d}$ we have

$$f(\mathbf{d})=b^0(\mathcal{M}_{W,\mathbf{d}})=b^0(\mathcal{M}_{W,\mathbf{d}})-b^1(\mathcal{M}_{W,\mathbf{d}})=\deg(\mathbf{d})+C,$$

and hence $f(\mathbf{d})=b^0(\mathcal{M}_{W,\mathbf{d}})$ is a Riemann function. It is slowly growing by (34). In view of, (13), for any $\mathbf{K}\in\mathbb{Z}^2$ we have

$$f(\mathbf{d})-f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d})=\deg(\mathbf{d})+C$$

and so

$$f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d})=b^1(\mathcal{M}_{W,\mathbf{d}}).$$

□

[In terms of sheaf theory, the above lemmas and corollaries express the fact that the two k-diagrams $\mathcal{M}_{W,\mathbf{d}}$ and $\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}$ fit into a short exact sequence with a *skyscraper sheaf* supported at B_i whose value is k; see Subsection 10.9.]

4.5 Simple Examples of $\mathcal{M}_{W,\mathbf{d}}$ Betti Number Bounds

We remark that without assumptions on W, the cohomology groups and Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$ may not be finite (or particularly interesting).

Example 4.2. *Let* $W=0$*. Then the kernel of* $\tau_{W,\mathbf{d}}=\mathcal{M}_{W,\mathbf{d}}(\partial)$ *is zero, and its cokernel can be identified with*

$$k^{\oplus\mathbb{Z}_{\ge d_1+1}}\oplus k^{\oplus\mathbb{Z}_{\ge d_2+1}},$$

which is infinite-dimensional. Hence $b^0(\mathcal{M}_{W,\mathbf{d}})=0$*,* $b^1(\mathcal{M}_{W,\mathbf{d}})=+\infty$*.*

Example 4.3. *We easily see that if* $W(\mathbf{d})=2$ *for some* $\mathbf{d}\in\mathbb{Z}^2$*, then* Multi$(W)$ *contains some elements of the form* $(d_1,d_2,1),(d_1,d_2,2)$*, and if* $b_3=\mathbf{e}_{(d_1,d_2,2)}-\mathbf{e}_{(d_1,d_2,1)}$*, then* $(0,b_3,0)\in\ker(\tau_{W,\mathbf{d}})$*. Similarly if* $W(d_1,d_2)\ge m$ *for some* $m\ge3$*, with* $b_3=\mathbf{e}_{(d_1,d_2,m)}-\mathbf{e}_{(d_1,d_2,1)}$ *we see that*

$$b^0(\mathcal{M}_{W,\mathbf{d}})\ge\sum_{\mathbf{d}'\in\mathbb{Z}^2}\bigl(W(\mathbf{d}')-1\bigr). \tag{38}$$

Example 4.4. *We say that* $s_1\in\mathbb{Z}$ *is* isolated in the first component of W *if* $W(s_1,s_2)=0$ *for all* s_2*; we similarly define when an* $s_2\in\mathbb{Z}$ *is isolated from the second component of* W*. If* $s_1\ge d_1+1$*, then all elements of the image of* $\tau_{W,\mathbf{d}}$ *have a zero coefficient in the* $\mathbf{e}_{s_1}$ *component in the* $\mathcal{M}_{W,\mathbf{d}}(A_1)$ *summand of the codomain (or range) of* $\tau_{W,\mathbf{d}}$*; similarly if* $s_2\ge d_2+1$ *is missing form the second component of* W*. It follows that*

$$b^1(\mathcal{M}_{W,\mathbf{d}})\ge\bigl|\mathrm{Iso}_{1,\ge d_1+1}\bigr|+\bigl|\mathrm{Iso}_{2,\ge d_2+1}\bigr| \tag{39}$$

where $\mathrm{Iso}_{1,\ge d_1+1}$ *is the set of isolated* s_1 *in the first component of* W *with* $s_1\ge d_1+1$*, and similarly for* $\mathrm{Iso}_{2,\ge d_2+1}$*.*

Example 4.5. *Let* W *be given as* $W(d_1,d_2)=2$ *if* $d_1=0$*, and* $W(d_1,d_2)=0$ *if* $d_1\ne0$*. Then the bounds* (38) *and* (39) *show that* $b^0(\mathcal{M}_{W,\mathbf{d}})=b^1(\mathcal{M}_{W,\mathbf{d}})=+\infty$*.*

4.6 The Betti Numbers of $\mathcal{M}_{W,\mathbf{d}}$ for Perfect Matchings

In the case $W\colon \mathbb{Z}^2 \to \{0,1\}$ is a perfect matching, it is easy to determine its Betti numbers.

Theorem 4.2. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a perfect matching. Then*

1. *$b^0(\mathcal{M}_{W,\mathbf{d}})$ equals the number of $\mathbf{a} \in \mathbb{Z}^2$ such that $W(\mathbf{a}) = 1$ and $\mathbf{a} \le \mathbf{d}$, and hence*
$$f(\mathbf{d}) \stackrel{def}{=} b^0(\mathcal{M}_{W,\mathbf{d}}) = (\mathfrak{s}W)(\mathbf{d});$$
2. *more precisely, $H^0(\mathcal{M}_{W,\mathbf{d}})$ has a basis consisting of*
$$\bigl(\mathbf{e}_{a_1}, \mathbf{e}_{(a_1,a_2)}, \mathbf{e}_{a_2}\bigr) \in \ker\bigl(\mathcal{M}_{W,\mathbf{d}}(\partial)\bigr) \quad s.t. \quad \mathbf{a} \le \mathbf{d}; \tag{40}$$
3. *$b^1(\mathcal{M}_{W,\mathbf{d}})$ equals the number of $\mathbf{a} \in \mathbb{Z}^2$ such that $W(\mathbf{a}) = 1$ and $\mathbf{a} \ge \mathbf{d} + \mathbf{1}$; and*
4. *more precisely, if π is the bijection associated to W, then $H^1(\mathcal{M}_{W,\mathbf{d}})$ has a basis consisting of the images in $H^1(\mathcal{M}_{W,\mathbf{d}})$ of*
$$\mathbf{e}_{a_1} \in \mathcal{M}_{W,\mathbf{d}}(A_1) \quad s.t. \quad a_1 \ge d_1 + 1,\ \pi(a_1) \ge d_2 + 1. \tag{41}$$

In particular $b^i(\mathcal{M}_{W,\mathbf{d}})$ is finite for all $i = 0, 1$ and all $\mathbf{d} \in \mathbb{Z}^2$, and is zero when $i = 0$ and $\deg(\mathbf{d})$ is sufficiently small or when $i = 1$ and $\deg(\mathbf{d})$ is sufficiently large; furthermore for some $C \in \mathbb{Z}$ we have
$$\chi(\mathcal{M}_{W,\mathbf{d}}) = \deg(\mathbf{d}) + C,$$
and, moreover, $C = \chi(\mathcal{M}_{W,\mathbf{0}})$. Hence for any $\mathbf{K} \in \mathbb{Z}^2$ and $\mathbf{L} = \mathbf{K} + \mathbf{1}$ we have
$$b^1(\mathcal{M}_{W,\mathbf{d}}) = f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = (\mathfrak{s}W^*_{\mathbf{L}})(\mathbf{K} - \mathbf{d}).$$

Proof. Let us begin by proving claims (1)—(4) above. Note that (2) implies (1), and (4) implies (3), so it suffices to prove (2) and (4). The proofs of (2) and (4) straightforward; let us begin with (2).

To prove (2), we note that $\mathcal{M}_{W,\mathbf{d}}(B_3) = k^{\oplus W}$, so the vectors
$$\bigl(\mathbf{e}_{a_1}, \mathbf{e}_{(a_1,a_2)}, \mathbf{e}_{a_2}\bigr) \in \mathcal{M}_{W,\mathbf{d}}(B) = \mathcal{M}_{W,\mathbf{d}}(B_1) \oplus \mathcal{M}_{W,\mathbf{d}}(B_3) \oplus \mathcal{M}_{W,\mathbf{d}}(B_2)$$
ranging over all a_1, a_2 such that $W(a_1, a_2) = 1$ are linearly independent in $\mathcal{M}_{W,\mathbf{d}}(B)$ by considering merely their $\mathcal{M}_{W,\mathbf{d}}(B_3)$ component. Consider an element
$$(b_1, b_3, b_2) \in \ker\bigl(\mathcal{M}_{W,\mathbf{d}}(\partial)\bigr);$$
then
$$b_3 = \sum_{W(a_1,a_2)=1} \mathbf{e}_{(a_1,a_2)} c_{a_1,a_2}$$
for some $c_{a_1,a_2} \in k$; the condition that (b_1, b_3, b_2) lies in the kernel is equivalent to
$$b_1 = \sum_{W(a_1,a_2)=1} \mathbf{e}_{a_1} c_{a_1,a_2} \in \mathcal{M}_{W,\mathbf{d}}(B_1) = k^{\oplus \mathbb{Z}_{\le d_1}}, \quad b_2 = \sum_{W(a_1,a_2)=1} \mathbf{e}_{a_2} c_{a_1,a_2} \in \mathcal{M}_{W,\mathbf{d}}(B_2) = k^{\oplus \mathbb{Z}_{\le d_2}},$$
which holds if and only if $a_1 \le d_1$ and $a_2 \le d_2$ whenever $c_{a_1,a_2} \ne 0$. Hence each such triple (b_1, b_3, b_2) is a unique linear combination of the vectors in (40).

To prove (4), since $\mathcal{M}_{W,\mathbf{d}}(\rho_{31}), \mathcal{M}_{W,\mathbf{d}}(\rho_{32})$ are isomorphisms, it follows that
$$V = \bigl(\mathcal{M}_{W,\mathbf{d}}(A_1) \oplus \mathcal{M}_{W,\mathbf{d}}(A_2)\bigr)/\mathrm{Image}\bigl(\mathcal{M}_{W,\mathbf{d}}(B_3)\bigr)$$
has $(\mathbf{e}_{a_1}, 0)$ as a basis, where a_1 ranges over all of $\mathbb{Z}$. The image of $\mathcal{M}_{W,\mathbf{d}}(B_1)$ in V is precisely the span of all $(\mathbf{e}_{a_1}, 0)$ with $a_1 \le d_1$, and hence
$$V' = V/\mathrm{Image}\bigl(\mathcal{M}_{W,\mathbf{d}}(B_1)\bigr)$$
has a basis consisting of all the $(\mathbf{e}_{a_1}, 0)$ with $a_1 \ge d_1 + 1$; finally the image of $\mathcal{M}_{W,\mathbf{d}}(B_2)$ in V' is precisely the span of all $(0, \mathbf{e}_{a_2})$ with $a_2 \le d_2$, each of which equals $(-\mathbf{e}_{a_1}, 0)$ for the unique a_1 with $W(a_1, a_2) = 1$. Hence
$$H^1(\mathcal{F}) = V'/\mathrm{Image}\bigl(\mathcal{M}_{W,\mathbf{d}}(B_2)\bigr)$$
has a basis as claimed in (41).

This establishes (1)–(4) of the theorem. Next we prove the rest of the theorem. Since W is a perfect matching, by definition it is initially and eventually zero; hence the number of $\mathbf{a} \le \mathbf{d}$ with $W(\mathbf{a}) = 1$ is zero for $\deg(\mathbf{d})$ sufficiently small, and for such $f(\mathbf{d}) = b^0(\mathbf{d}) = 0$; similarly, for $\deg(\mathbf{d})$ sufficiently large the number of $\mathbf{a} \ge \mathbf{d} + \mathbf{1}$ with $W(\mathbf{a}) = 1$ is zero, and for such $\mathbf{d}$ we have $b^1(\mathbf{d}) = 0$. The remaining claims follow from Theorem 4.1 and the fact that $\mathfrak{m} f^{\wedge}_{\mathbf{K}} = (-1)^2 W^*_{\mathbf{L}} = W^*_{\mathbf{L}}$ (by Theorem 2.1). ☐

4.7 The Betti Numbers of $\mathcal{M}_{W,\mathbf{d}}$ for General W via an Associated Graph

One can give a description of the Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$ for any W in terms of a graph associated to W and $\mathbf{d}$. This formula is foundational and seems interesting, but it is independent of the rest of this article.

Given $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\mathbf{d} \in \mathbb{Z}^2$ we associate the following graph, which may have multiple edges and self-loops, $G = \mathrm{Graph}(W, \mathbf{d})$, which one can describe in two ways: first,

1. one forms the bipartite graph G' whose vertex set is $\mathbb{Z} \times \{1, 2\}$ and whose edge set has $W(s_1, s_2)$ edges joining $(s_1, 1)$ with $(s_2, 2)$;

2. one then takes G to be the graphs obtained from G' by collapsing the vertices $\mathbb{Z}_{\leq d_1} \times \{1\}$ and $\mathbb{Z}_{\leq d_2} \times \{2\}$ into a single vertex, v_0.

In particular, G is not generally bipartite, and has r self-loops at v_0 (and no self-loops about any other vertex), where

$$r = \sum_{(s_1,s_2)\leq \mathbf{d}} W(s_1, s_2);$$

each such self-loop adds 1 to the first Betti number of G.

The second way to describe G is more explicit: namely, G is the graph with vertex set $V_G = v_0 \amalg V_{\mathrm{first}} \amalg V_{\mathrm{second}}$[2] with $V_{\mathrm{first}} = \mathbb{Z}_{\geq d_1+1}$ and $V_{\mathrm{second}} = \mathbb{Z}_{\geq d_2+1}$, and whose edge set can be identified with $E_G = \mathrm{Multi}(W)$ (as in (28)), where each element $(s_1, s_2, i) \in \mathrm{Multi}(W)$ creates:

1. an edge joining $s_1 \in V_{\mathrm{first}}$ and $s_2 \in V_{\mathrm{second}}$ if $s_1 \geq d_1 + 1$ and $s_2 \geq d_2 + 1$;

2. a self-loop about v_0 if $s_1 \leq d_1$ and $s_2 \leq d_2$;

3. an edge joining $s_1 \in V_{\mathrm{first}}$ and v_0 if $s_1 \geq d_1 + 1$ and $s_2 \leq d_2$; and

4. an edge joining v_0 and $s_2 \in V_{\mathrm{second}}$ if $s_1 \leq d_1$ and $s_2 \geq d_2 + 1$.

It will be convenient to write G as the union of its connected components $G_i = (V_i, E_i)$, where i ranges over $\{0, \ldots, \ell\}$ if G has finitely many connected components, or i ranging over $\mathbb{Z}_{\geq 0}$ otherwise; we will also set G_0 to be the connected component of v_0.

We need to recall some convenient definitions of the Betti numbers of an infinite graph. If $G = (V, E)$ is a graph (with V, E not necessarily finite), then one can define its incidence matrix, ι_G, as usual, by orienting each edge arbitrarily, so that ι_G is a map $k^{\oplus E} \to k^{\oplus V}$, and then $b^0(G), b^1(G)$ are, respectively, the dimensions of the cokernel and kernel of ι_G. This will be useful to us. However, it will also be conventient to define the Betti numbers as follows: $b^0(G)$ is the number of connected components, i.e., equivalence classes of vertices, where two vertices are equivalent if they are connected by a walk of finite length. For each connected component, we choose a spanning tree (by fixing a vertex, r as the root of the tree, then adding one edge joining r to each of its neighbours, then one edge for each vertex of distance 2 to r, etc.). This gives a spanning forest of G. Then $b^1(G)$ is the cardinality of set of edges of G that don't lie in the spanning forest.

Theorem 4.3. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\mathbf{d} \in \mathbb{Z}^2$. Let $G = \mathrm{Graph}(W, \mathbf{d})$. Then*

$$b^1(\mathcal{M}_{W,\mathbf{d}}) = b^0(G) - 1$$
$$b^0(\mathcal{M}_{W,\mathbf{d}}) = b^1(G).$$

We remark that it is interesting how each Betti number of $\mathcal{M}_{W,\mathbf{d}}$ can be inferred from the opposite Betti number of G. We also remark that if W is a perfect matching, then Theorem 4.2 is a simpler description of the Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$. Hence, the seeming simplicity of the above theorem is not necessarily the most practical way to describe the Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$.

Proof. Let $\tau = \mathcal{M}_{W,\mathbf{d}}(\partial)$, which one can view as a map

$$\tau\colon k^{\oplus\left(\mathbb{Z}_{\leq d_1} \amalg \mathbb{Z}_{\leq d_2} \amalg \mathrm{Multi}(W)\right)} \to k^{\oplus(\mathbb{Z} \amalg \mathbb{Z})}.$$

In the notion introduced after the definition of G, we saw that G decomposes into its connected components $G_i = (V_i, E_i)$ where i ranges over I, where I equals $\{0, \ldots, \ell\}$ or $\mathbb{Z}_{\geq 0}$, and G_0 is the component containing v_0. Then $\mathrm{Multi}(W)$ is partitioned into multisets E_i with $i \in I$; setting

$$E_0' = \mathbb{Z}_{\leq d_1} \amalg \mathbb{Z}_{\leq d_2} \amalg E_0,$$

[2] We assume some reasonable convention for the meaning of the *disjoint union* $\amalg$, which is a limit and hence only defined up to unique isomorphism; e.g., for sets $A_1, \ldots, A_s$, the set $A_1 \amalg \ldots \amalg A_s$ refers to the union $\bigcup_i A_i \times \{i\}$.

we have that

$$\mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2} \amalg \text{Multi}(W) = E_0' \cup \bigcup_{i \in I\setminus\{0\}} E_i$$

where we identify $E_i \subset \text{Multi}(W)$ with the subset $\text{Multi}(W)$ as it lies in $\mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2} \amalg \text{Multi}(W)$. Similarly setting

$$V_0' = \big(\mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2}\big) \cup (V_0 \setminus \{v_0\}) \quad \text{(which is a subset of } \mathbb{Z} \amalg \mathbb{Z}),$$

we have

$$\mathbb{Z} \amalg \mathbb{Z} = V_0' \cup \bigcup_{i \in I\setminus\{0\}} V_i.$$

It follows that τ factors as a map

$$\tau = \bigoplus_{i \in I} \tau_i$$

where

$$\tau_0 \colon k^{\oplus E_0'} \to k^{\oplus V_0'}$$

and for $i \ne 0$,

$$\tau_i \colon k^{\oplus E_i} \to k^{\oplus V_i}.$$

Note that for $i \ne 0$, τ_i sends $(a_1, a_2, j) \in E_i \subset \text{Multi}(W)$ to $(\mathbf{e}_{a_1}, \mathbf{e}_{a_2})$; since G_i is bipartite, $b^0(\tau_i), b^1(\tau)$ are the same as b^0, b^1 of the map sending (a_1, a_2, j) to $(\mathbf{e}_{a_1}, -\mathbf{e}_{a_2})$, which is an incidence matrix of G_i. Hence for $j = 0, 1$,

$$b^j(\tau) = \sum_{i \in I} b^j(\tau_i) = b^j(\tau_0) + \sum_{i>0} b^{1-j}(G_i).$$

Since

$$b^{1-j}(G) = \sum_{i \in I} b^{1-j}(G_i),$$

it remains to show that $b^1(\tau_0) = b^0(G) - 1 = 0$ and $b^0(\tau_0) = b^1(G_0)$.

We claim that τ_0 is surjective; hence for $a \in V_0'$, we need to show that the standard basis vector $\mathbf{e}_a$ is in the image of τ; this is clear for $a \in \mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2}$. If the distance from a to v_0 is 1, and $a \in V_{\text{first}}$, then for some $a' \le d_2$ we have $(a, a') \in E_0$, and hence $\mathbf{e}_a + \mathbf{e}_{a'} \in \text{Image}(\tau)$, and hence $\mathbf{e}_a \in \text{Image}(\tau)$; similarly if $a \in V_{\text{second}}$. We then similarly get that $\mathbf{e}_a \in \text{Image}(\tau)$ if the distance from a to v_0 equals 2, since then $\mathbf{e}_a + \mathbf{e}_{a'} \in \text{Image}(\tau)$ for some a' of distance 1 to v_0. We then argue the general case by induction on its distance to v_0. Hence $b^1(\tau_0) = 0$.

Finally let us show that $b^0(\tau_0) = b^1(G_0)$. Let us give an isomorphism $\ker(\tau_0) \to \ker(\iota_{G_0})$ where ι_{G_0} is an incidence matrix of G_0. To describe such an incidence matrix, we orient each $w \in E_0$ arising from $(a_1, a_2, j) \in \text{Multi}(W)$ as running from a_1 to a_2. Next note that τ_0 is a map

$$\tau_0 \colon k^{\oplus(\mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2}) \cup E_0} \to k^{\oplus(\mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2}) \cup (V_0 \setminus\{v_0\})},$$

so each element of the domain of τ_0 can be viewed as a pair (α, β) which refers to the element

$$\sum_{u \in \mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2}} \alpha(u)\mathbf{e}_u \ + \sum_{w \in E_0} \beta(w)\mathbf{e}_w,$$

where α, β are functions that are zero for all but finitely many of their values. Since τ takes $(\alpha, 0)$ to

$$\sum_{u \in \mathbb{Z}_{\le d_1} \amalg \mathbb{Z}_{\le d_2}} \alpha(u)\mathbf{e}_u \in k^{\oplus(\mathbb{Z} \amalg \mathbb{Z})},$$

it follows that for each β there exists at most one α with $(\alpha, \beta) \in \ker(\tau_0)$, and such an α exists precisely when

$$\sum_{u=(a_1,a_2,\ell) \in E_0, a_1 \ge d_1+1} \beta(u)\mathbf{e}_{a_1} = 0 = \sum_{u=(a_1,a_2,\ell) \in E_0, a_2 \ge d_2+1} \beta(u)\mathbf{e}_{a_2}. \tag{42}$$

So we want to prove that the dimension of all β satisfying (42) equals $b^1(G_0)$.

First consider the case where $G_0 = (V_0, E_0)$ is a tree: we claim that (42) forces $\beta = 0$, since if $U \subset E_0$ is the support of β and $u' \in U$ is an edge of maximum distance to v_0, one vertex incident upon u' is not incident upon any other edge of U', which is a contradiction.

Next, consider the case that G_0 is a tree (V_0, T) plus an edge u_1: there exists a β as above with $\beta(u_1)$ nonzero, using the unique cycle created by u_1; since β can be increased by at most 1 with the addition of an edge, it follows that the dimension of β satisfying (42) is exactly 1. Similarly, if we add another edge, u_2, to

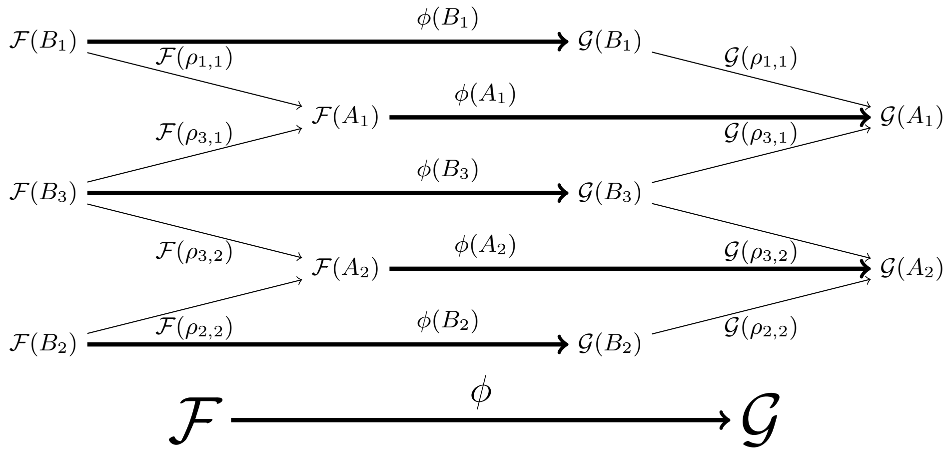


Figure 5: A morphism of diagrams $\phi\colon \mathcal{F} \to \mathcal{G}$, depicted in thick lines

G_0, the dimension of β satisfying (42) increases by 1, using any cycle created by u_2; since it can increase by at most 1, the dimension of such β is exactly 2. It similarly follows by induction on m, that if G_0 is a tree plus m edges, then the dimension of β satisfying (42) is exactly m. It follows by taking $m = b^1(G_0)$, or letting $m \to \infty$ if $b^1(G_0) = \infty$, that

$$b^0(\tau) = b^1(G_0).$$

□

5 Morphisms, Isomorphisms, and Direct Sum of k-Diagrams

For the rest of this article we will need to know when two k-diagrams are *isomorphic* and some other basic properties of (the category of) k-diagrams.

5.1 Morphisms of k-Diagrams

Definition 5.1. *Let $\mathcal{F}, \mathcal{G}$ be two k-diagrams. By a* morphism $\phi\colon \mathcal{F} \to \mathcal{G}$ *we mean the data, ϕ, of linear maps from each value of $\mathcal{F}$ to the corresponding value on $\mathcal{G}$ in a way that commutes with the restriction maps: i.e., ϕ consists of k-linear maps $\phi(B_i)\colon \mathcal{F}(B_i) \to \mathcal{G}(B_i)$ for $i = 1, 2, 3$ and $\phi(A_j)\colon \mathcal{F}(A_j) \to \mathcal{G}(A_j)$ for $j = 1, 2$ such that $\mathcal{G}(\rho_{ij})\phi(B_i) = \phi(A_j)\mathcal{F}(\rho_{ij})$ whenever $\mathcal{F}(\rho_{ij}), \mathcal{G}(\rho_{ij})$ exist (i.e., $i = j$ or $i = 3$ and any j). For k-diagrams $\mathcal{F}, \mathcal{G}$ we use $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ to denote the set of morphism $\mathcal{F} \to \mathcal{G}$; if $\phi, \phi' \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ and $\alpha, \alpha' \in \mathbb{F}$, then one can define $\alpha\phi + \alpha'\phi' \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ to be the map that is the value-by-value linear combination, i.e., for $P = A_j$ or $P = B_i$,*

$$\big(\alpha\phi + \alpha'\phi'\big)(P) = \alpha\phi(P) + \alpha'\phi'(P);$$

this gives $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ the structure of a k-vector space.

We illustrate a morphism of k-diagrams in Figure 5.

To turn the set of k-diagrams into a category, we need to define the *composition* of morphisms.

Definition 5.2. *Let $\phi_1\colon \mathcal{F} \to \mathcal{G}$ and $\phi_2\colon \mathcal{G} \to \mathcal{H}$ be two morphisms of k-diagrams. We define the* composition of ϕ_1 followed by ϕ_2, *denoted $\phi_2 \circ \phi_1$ or $\phi_2\phi_1$, to be the morphism $\mathcal{F} \to \mathcal{H}$ given by:*

$$\forall i = 1, 2, 3, \quad (\phi_2\phi_1)(B_i) = \phi_2\big(\phi_1(B_i)\big),$$
$$\forall j = 1, 2, \quad (\phi_2\phi_1)(A_j) = \phi_2\big(\phi_1(A_j)\big).$$

We easily check that in the above definition, $\mathcal{H}(\rho_{ij})(\phi_2\phi_1)(B_i) = (\phi_2\phi_1)(A_j)\mathcal{F}(\rho_{ij})$ whenever $\mathcal{F}(\rho_{ij}), \mathcal{G}(\rho_{ij})$ exist (i.e., $i = j$ or $i = 3$ and any j).

We remark that it is clear how to define the identity morphism, and hence Definition 5.2 endows the set of k-diagrams with the structure of a category.[3]

[3] This category is none other than the category $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ ([1], Exposé I, 1.1.1, just below Definition 1.1) where $\mathcal{C}$ is a category with 5 objects and $\mathcal{D}$ is the category of k-vector spaces. Since the category of k-vector spaces is an algebraic structure defined by finite projective limits, Definitions 4.1, 5.1, and 5.2 are really a consequence of [1], Exposé I, Corollaire 3.2.

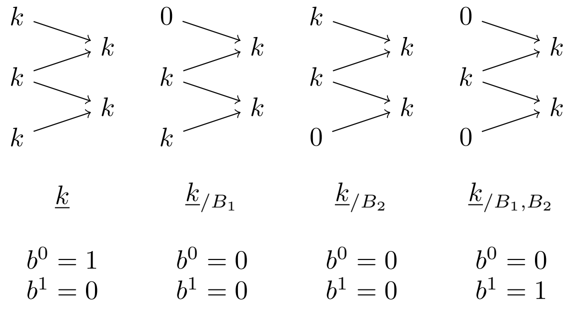


Figure 6: The Diagram $\underline{k}$ and Related Diagrams: these are the four basic diagrams into which our $\mathcal{M}_{W,\mathbf{d}}$ decompose as direct sums. Here we also include their Betti numbers (Subsection 4.1); the reason is that it is important to remember that b^0 and b^1, respectively, come only from copies of $\underline{k}$ and $\underline{k}_{/B_1,B_2}$, respectively. Hence any k-diagram that decomposes into our four basic diagrams, such as $\mathcal{M}_{W,\mathbf{d}}$, and has finite Betti numbers, must therefore contain only finitely many copies of each of $\underline{k}, \underline{k}_{/B_1,B_2}$.

5.2 Constant and the Four Basic k-Diagrams

Let us describe some simple k-diagrams that we will use.

Definition 5.3. *Let k be a field, and V a k-vector space. The* constant k-diagram V*, denoted $\underline{V}$, refers to the diagram whose values are all V, and whose restriction maps are the identity map on V.*

In particular, $\underline{k}$ is the constant diagram whose values are k, viewed as a k-vector space.

Let us describe a number of k-diagrams closely related to the diagram $\underline{k}$ that we will use; we will collectively refer to these k-diagrams as the *four basic k-diagrams*. Before giving the formal definition, we depict these diagrams in Figure 6 by their values, all of which are either 0 or k, and all maps $k \to k$ are the identity maps.

Definition 5.4. *Let k be a field. Consider the four possible diagrams, $\mathcal{F}$, such that:*

1. *all values of $\mathcal{F}$ equal k or 0;*
2. *$\mathcal{F}(B_3) = \mathcal{F}(A_1) = \mathcal{F}(A_2) = k$;*
3. *all restriction maps $k \to k$ are the identity map.*

We use the notation $\underline{k}$ to refer to $\mathcal{F}$ as above with $\mathcal{F}(B_1) = \mathcal{F}(B_2) = k$ and call it the constant diagram k*; for $i = 1, 2$ we use the notation $\underline{k}_{/B_i}$ to refer the same diagram except with $\mathcal{F}(B_i) = 0$; and we use the notation $\underline{k}_{/B_1,B_2}$ to refer to the remaining such diagram, i.e., with $\mathcal{F}(B_1) = \mathcal{F}(B_2) = 0$. We refer to these four diagrams collectively as the* four basic k-diagrams*.*

5.3 Simple Examples of Morphisms with the Four Basic Diagrams

The reader who has never worked with k-diagrams or the related notion of presheaves and sheaves are encouraged to consider morphisms between the four basic diagrams.

Example 5.1. *As k-vector spaces, we have*

$$\mathrm{Hom}(\underline{k}_{/B_1}, \underline{k}) \simeq k,$$

since for any $\alpha \in k$ there is a unique morphism $\phi\colon \underline{k}_{/B_1} \to \underline{k}$ such that for $P = A_1, A_2, B_2, B_3$, $\phi(P)\colon k \to k$ is multiplication by α (and for $P = B_1, B_2$, since $\underline{k}_{/B_1,B_2}(P) = 0$, $\phi(P)$ is the trivial k-linear transformation $\{0\} \to k$). By contrast

$$\mathrm{Hom}(\underline{k}, \underline{k}_{/B_1}) \simeq \{0\},$$

since if $\phi\colon \underline{k} \to \underline{k}_{/B_1}$, then

$$\phi(B_1)\colon \underline{k}(B_1) \to \underline{k}_{/B_1}(B_1) = 0$$

must be the zero map, but then (recall the meaning of $\rho_{i,j}$ from Definition 4.1)

$$\underline{k}(B_1) \xrightarrow{\underline{k}_{/B_1}(\rho_{1,1}) \circ \phi(B_1)} \phi(A_1)$$

must be the zero map, and since this must equal the map

$$\underline{k}(B_1) \xrightarrow{\phi(A_1) \circ \underline{k}(\rho_{1,1})} \phi(A_1)$$

this forces $\phi(A_1)$ to be multiplication by 0. Following Figure 5 around (with $\mathcal{F} = \underline{k}$ and $\mathcal{G} = \underline{k}_{/B_1}$) we see that ϕ must be everywhere zero.

[The experts will realize that the above example reflects the fact that there is a canonical inclusion $\mathcal{F}_U \to \mathcal{F}$ (and the fact there is typically no nonzero morphism $\mathcal{F} \to \mathcal{F}_U$) where $\mathcal{F}$ is a sheaf on a topological space, U an open subset, and $\mathcal{F}_U$ is the extension by zero of the restriction of $\mathcal{F}$ to U; we will explain this in more detail in Subsection 10.8.]

For future use, it will be helpful to note the following calculations.

Example 5.2. *Similar to Example 5.1, we see that*

$$\mathrm{Hom}(\underline{k}_{/B_1,B_2}, \underline{k}_{/B_1,B_2}) \simeq k,$$

and $\mathrm{Hom}(\mathcal{F}, \underline{k}_{/B_1,B_2}) = \{0\}$ *for* $\mathcal{F} = \underline{k}, \underline{k}_{/B_1}, \underline{k}_{/B_2}$.

Example 5.3. *More generally, there is a partial order of our four basic k-diagrams: denoting each of these diagrams by $\underline{k}_{/S}$ where S is some subset of $\{B_1, B_2\}$, where we understand $\underline{k}_{/\emptyset} = \underline{k}$. We have*

$$\mathrm{Hom}(\underline{k}_{/S}, \underline{k}_{/S'}) \simeq \begin{cases} k & \text{if } S \subset S', \text{ and} \\ \{0\} & \text{otherwise.} \end{cases}$$

5.4 Example: Global Sections

If $\phi \colon \mathcal{F} \to \mathcal{G}$, then we easily see that ϕ gives maps $\mathcal{F}(B) \to \mathcal{G}(B)$ and $\mathcal{F}(A) \to \mathcal{G}(A)$ that induce maps $H^i(\mathcal{F}) \to H^i(\mathcal{G})$ for $i = 0, 1$.

If $\mathcal{F}$ is any k-diagram, then if

$$\phi \in \mathrm{Hom}(\underline{k}, \mathcal{F}),$$

then $\phi(A_1)$ takes the element $1 \in k$ to an element $a_1 \in A_1$, and similarly for $\phi(A_2)$ and the $\phi(B_i)$; this gives a tuple $(b_1, b_2, b_3, a_1, a_2)$, and the fact that the restrictions of $\underline{k}$ are the identity maps implies that $(b_1, b_2, b_3, a_1, a_2)$ is a global section (Definition 4.1); conversely every global section $(b_1, b_2, b_3, a_1, a_2)$ determines a ϕ where $\phi(A_i)1 = a_i$ and $\phi(B_j)1 = b_j$ which we easily check is an element of $\mathrm{Hom}(\underline{k}, \mathcal{F})$. Hence, we see (as usual in sheaf theory)

$$H^0(\mathcal{F}) \simeq \mathrm{Hom}(\underline{k}, \mathcal{F}),$$

and we easily check that for any morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ the map $H^0(\mathcal{F}) \to H^0(\mathcal{G})$ is the same as the map

$$\mathrm{Hom}(\underline{k}, \mathcal{F}) \to \mathrm{Hom}(\underline{k}, \mathcal{G})$$

given by composition of the morphism $\underline{k} \to \mathcal{F}$ with ϕ; this is a standard fact about global sections of $\mathcal{F}$ in sheaf theory.

5.5 Isomorphisms and Direct Sums of k-Diagrams

We now give the notion of *isomorphisms* and *direct sums* for k-diagrams; later, in Subsection 10.2 we will explain that these notions really result once one specifies what is meant by a k-diagram and a morphism of k-diagrams.

Definition 5.5. *A morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ of k-diagrams is an* isomorphism *if all the $\phi(A_j)$ and $\phi(B_i)$ are isomorphisms.*

This is equivalent to saying that there exists an inverse morphism $\nu \colon \mathcal{G} \to \mathcal{F}$ such that $\phi\nu$ is the identity on $\mathcal{G}$ (i.e., all the $\phi\nu(A_j)$ and $\phi\nu(B_i)$ are the identity maps) and $\nu\phi$ is the identity morphism on $\mathcal{F}$.

Definition 5.6. *Let L be a set and for each $\ell \in L$, say that we are given a k-diagram, $\mathcal{F}_\ell$. The* direct sum *of $\{\mathcal{F}_\ell\}_{\ell \in L}$ refers to the k-diagram denoted*

$$\bigoplus_{\ell \in L} \mathcal{F}_\ell$$

whose values at the A_j (for $j = 1, 2$) are

$$\bigoplus_{\ell \in L} \mathcal{F}_\ell(A_j),$$

and similarly for the values at the B_i, and similarly the restriction maps are the direct sums of those of the $\mathcal{F}_\ell$. Similarly, if for each $\ell \in L$ we are given a morphism ϕ_ℓ of k-diagrams, the morphism $\oplus_{\ell \in L} \phi_\ell$ is the morphism of k-diagrams $\oplus_{\ell \in L} \mathcal{F}_\ell$ to $\oplus_{\ell \in L} \mathcal{G}_\ell$ given by direct sum of the ϕ_ℓ.

We easily check that all the constructions in Definition 4.1 commute with taking direct sums. In particular, for any direct sum $\{\mathcal{F}_\ell\}_{\ell\in L}$ we have

$$\left(\bigoplus_{\ell\in L}\mathcal{F}_\ell\right)(\partial)=\bigoplus_{\ell\in L}\mathcal{F}_\ell(\partial),$$

and for $j=0,1$ we have

$$H^j\left(\bigoplus_{\ell\in L}\mathcal{F}_\ell\right)=\bigoplus_{\ell\in L}H^j(\mathcal{F}_\ell)$$

and taking dimensions we have

$$b^j\left(\bigoplus_{\ell\in L}\mathcal{F}_\ell\right)=\sum_{\ell\in L}b^j(\mathcal{F}_\ell).$$

6 Sums of $\mathcal{M}_{W,\mathbf{d}}$ and Indicator k-Diagrams

The main point of this section is to prove the following theorem.

Theorem 6.1. *Let $W_1,\ldots,W_s$ and $\tilde{W}_1,\ldots,\tilde{W}_s$ be perfect matchings $\mathbb{Z}^2\to\mathbb{Z}$ such that*

$$W_1+\cdots+W_s=\tilde{W}_1+\cdots+\tilde{W}_s.$$

Then for any $\mathbf{d}$ we have

$$\mathcal{M}_{W_1,\mathbf{d}}\oplus\cdots\oplus\mathcal{M}_{W_s,\mathbf{d}}\simeq\mathcal{M}_{\tilde{W}_1,\mathbf{d}}\oplus\cdots\oplus\mathcal{M}_{\tilde{W}_s,\mathbf{d}}.$$

The proof will be given at the end of this section, i.e., after Proposition 6.3 and its proof.

Moreover, we will prove this theorem is true in a very strong sense: namely, if $W=W_1+\cdots+W_s$, then the direct sum

$$\mathcal{M}_{W_1,\mathbf{d}}\oplus\cdots\oplus\mathcal{M}_{W_s,\mathbf{d}}$$

is isomorphic to a sum, $\mathcal{I}^{\oplus W}_{\mathbf{d}}$ of what we call *indicator diagrams*, that can be inferred from W alone, without reference to the $W_1,\ldots,W_s$. This will provide additional intuition regarding Theorem 6.1.

6.1 Example: $\mathcal{M}_{W,\mathbf{d}}$ as a Direct Sum of Indicator Diagrams

The k-diagrams $\mathcal{M}_{W,\mathbf{d}}$ of the last section can be naturally viewed as a direct sum of our four basic diagrams. In fact, for W fixed, the family $\mathcal{M}_{W,\mathbf{d}}$ with $\mathbf{d}$ varying decomposes as a sum of a family of our four basic diagrams indexed on $\mathbf{d}$. This point of view will be useful to understand the virtual k-diagrams that we study later.

Definition 6.1. *Let k be a field and $\mathbf{a}\in\mathbb{Z}^2$. The $\geq\mathbf{a}$-*indicator family of k-diagrams *refers to the family of k-diagrams indexed on $\mathbf{d}\in\mathbb{Z}^2$, denoted $\{\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}\}_{\mathbf{d}\in\mathbb{Z}^2}$, where for each $\mathbf{d}\in\mathbb{Z}^2$ we set*

1. $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}=\underline{k}$ *if* $\mathbf{d}\geq\mathbf{a}$,
2. $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}=\underline{k}_{/B_2}$ *if* $d_1\geq a_1$ *and* $d_2<a_2$,
3. $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}=\underline{k}_{/B_1}$ *if* $d_2\geq a_2$ *and* $d_1<a_1$, *and*
4. $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}=\underline{k}_{/B_1,B_2}$ *if* $d_1<a_1$ *and* $d_2<a_2$.

Equivalently, for each $\mathbf{d}$, $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$ is equal to one of the four basic k-diagrams, where for each $j=1,2$, $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}(B_j)=k$ if and only if $a_j\leq d_j$.

We depict the indicator diagram in Figure 7.

Definition 6.2. *If $W\colon\mathbb{Z}^2\to\mathbb{Z}_{\geq 0}\cup\{\infty\}$, the W-*sum indicator k-diagram*, denoted $\mathcal{I}^{\oplus W}_{\mathbf{d}}$, refers to the direct sum*

$$\mathcal{I}^{\oplus W}_{\mathbf{d}}=\bigoplus_{\mathbf{a}\in\mathbb{Z}^2}\left(\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}\right)^{\oplus W(\mathbf{a})},$$

where if $W(\mathbf{a})=\infty$, then $\oplus W(\mathbf{a})$ refers to $\oplus\mathbb{N}$, i.e., the summand involved is the direct sum of a countably infinite number of copies of $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$.

The following proposition is immediate, but worth stating.

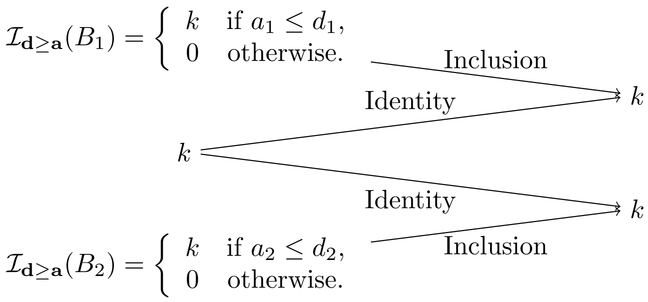


Figure 7: The Indicator k-Diagram $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$

Proposition 6.1. *Let $W\colon \mathbb{Z}^2 \to \{0,1\}$ be a perfect matching. Then there is an isomorphism*

$$\iota_{\mathbf{d}}\colon \mathcal{M}_{W,\mathbf{d}} \to \mathcal{I}_{\mathbf{d}}^{\oplus W}$$

given by the canonical isomorphisms for $i = 1, 2$

$$\mathcal{M}_{W,\mathbf{d}}(A_i) = k^{\oplus\mathbb{Z}} \to \mathcal{I}_{\mathbf{d}}^{\oplus W}(A_i), \quad \mathcal{M}_{W,\mathbf{d}}(B_i) = k^{\oplus\mathbb{Z}_{\leq d_i}} \to \mathcal{I}_{\mathbf{d}}^{\oplus W}(B_i), \tag{43}$$

and the isomorphisms

$$\mathcal{M}_{W,\mathbf{d}}(B_3) = k^{\oplus W} \to \mathcal{I}_{\mathbf{d}}^{\oplus W}(B_3); \tag{44}$$

moreover, and for $j = 1, 2$, the maps $\mathcal{M}_{W,\mathbf{d}}(\rho_{3j})$ are isomorphisms of $k^{\oplus W} \to k^{\mathbb{Z}}$.

Proof. The equalities in (43) and (44) are by definition (Definition 4.4). The fact that W is a perfect matching implies that for each a_1 there is a unique a_2 with $W(a_1, a_2) = 1$; this gives the isomorphism $k^{\oplus\mathbb{Z}} \to \mathcal{I}_{\mathbf{d}}^{\oplus W}(A_1)$ and $k^{\oplus\mathbb{Z}_{\leq d_1}} \to \mathcal{I}_{\mathbf{d}}^{\oplus W}(B_1)$; similarly for the subscript 1 replaced everywhere by 2. By Definition 6.1 each indicator diagram $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$ has $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}(\rho_{3j})$ being an isomorphism, and hence the same is true for $\mathcal{I}_{\mathbf{d}}^{\oplus W}$, and hence, by (43) and (44), it also holds for $\mathcal{M}_{W,\mathbf{d}}$. We easily check that the isomorphisms in (43) and (44) intertwine with the restriction maps, and hence gives the desired isomorphism $\iota_{\mathbf{d}}$. □

Proposition 6.2. *For any $W\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$,*

$$b^0(\mathcal{I}_{\mathbf{d}}^{\oplus W}) = \sum_{\mathbf{a}\leq\mathbf{d}} W(\mathbf{a}),$$
$$b^1(\mathcal{I}_{\mathbf{d}}^{\oplus W}) = \sum_{\mathbf{a}\geq\mathbf{d}+\mathbf{1}} W(\mathbf{a}),$$

and hence, if one of these two Betti numbers is finite, we have

$$\chi(\mathcal{I}_{\mathbf{d}}^{\oplus W}) = \left(\sum_{\mathbf{a}\leq\mathbf{d}} W(\mathbf{a})\right) - \left(\sum_{\mathbf{a}\geq\mathbf{d}+\mathbf{1}} W(\mathbf{a})\right).$$

Our main interest in Proposition 6.2 is for W that are s-fold perfect matchings in the following sense.

Definition 6.3. *We say that a function $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ is an s-fold matching if it can be written as the sum of s (bounded) perfect matchings.*

To build models for general Riemann functions $\mathbb{Z}^2 \to \mathbb{Z}$, we will require the following strengthening of Proposition 6.2.

Proposition 6.3. *Let $\{W_i\}_{i\in I}$ be a finite or countably infinite set of functions $W_i\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, and let $W = \sum_{i\in I} W_i$. Then*

$$\mathcal{I}_{\mathbf{d}}^{\oplus W} \simeq \bigoplus_{i\in I} \mathcal{I}_{\mathbf{d}}^{\oplus W_i}. \tag{45}$$

Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ be an s-fold matching, and $W = W_1 + \cdots + W_s$ be a decomposition of W into perfect matchings. Then for any $\mathbf{d} \in \mathbb{Z}^2$ there is an isomorphism

$$\mathcal{I}_{\mathbf{d}}^{\oplus W} \simeq \mathcal{M}_{W_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_s,\mathbf{d}}. \tag{46}$$

In particular, if $W = W_1' + \cdots + W_s'$ is another decomposition of W into perfect matchings, then we have

$$\mathcal{M}_{W_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_s,\mathbf{d}} \simeq \mathcal{M}_{W_1',\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_s',\mathbf{d}} \tag{47}$$

Proof. (45) follows from the easily verified fact that any direct sum of direct sums is the direct sum of all summands in the double summation (this is, more generally, valid in any additive category, since this is an inductive limit of inductive limits, see e.g., [1], Section I.2.5.0). (46) follows from (45) and Proposition 6.2. (47) follows from (46). □

Proof of Theorem 6.1. By Proposition 6.3, if

$$\begin{aligned} W &= W_1 + \cdots + W_s \\ W &= W_1' + \cdots + W_s' \end{aligned}$$

are two decompositions of W into perfect matchings, we have (47) holds; this is exactly what Theorem 6.1 asserts. □

7 Virtual Fredholm k-Diagrams and Riemann Functions for $n = 2$

If $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ is a general Riemann function, its weight, W, can attain negative values. In this case we don't know of a good way to model f as the zeroth Betti number of a family of k-diagrams; however, one can do so if we work with *virtual k-diagrams*. Our strategy is to use Lemma 3.1 to write

$$W = (W_1 + \cdots + W_s) - (\tilde{W}_1 + \cdots + \tilde{W}_{s-1}) \tag{48}$$

for some s, where the W_i and $\tilde{W}_i$ are perfect matchings. We then model f as b^0 of the *virtual k-diagram*

$$\left(\bigoplus_i \mathcal{M}_{W_i,\mathbf{d}}, \bigoplus_i \mathcal{M}_{\tilde{W}_i,\mathbf{d}} \right), \tag{49}$$

which is a "formal difference" of the first k-diagram "minus" the second.

Most of the work in this section is to iron out the notion of virtual vector spaces, virtual Fredholm maps, and virtual k-diagrams. [We borrow the term *virtual* from *virtual characters* in group theory.] We then show that the "virtual" or "formal difference of" k-diagrams (49) has the desired Betti numbers (this is immediate from our discussion of formal differences), and—more notably—is independent, up to equivalence, of the decomposition (48).

We caution the reader that by our conventions below, a *virtual k-vector space* will refer to a formal difference of finite-dimensional vector spaces (Convention 7.1) and a *virtual k-diagram* will refer to a formal difference of Fredholm k-diagrams, i.e., of k-diagrams whose Betti numbers are finite (Convention 7.2). These conventions are needed to get a well-defined notion of Betti numbers, as we explain below.

Our main modeling result is Proposition 7.1, which builds a virtual (Fredholm) k-digram that expresses any generalized Riemann-Roch formula (13) as an Euler characteristic formula. We formally take part of Proposition 7.1 and combine this with earlier material to get Theorem 7.1; one can view this theorem as the main result of this section. However, it is still important to understand all the results in this section, including the full statement of Proposition 7.1.

We remark that any function $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ can be canonically expressed as

$$W = W^+ - W^-, \quad \text{where} \quad W^+ = \max(W, 0),\ W^- = \max(-W, 0).$$

Although we can see (for example, Example 7.1 below) that $\mathcal{M}_{W^+,\mathbf{d}}, \mathcal{M}_{W^-,\mathbf{d}}$ are not generally Fredholm, even if $\mathfrak{s}W$ is a Riemann function, by contrast $\mathcal{I}_{\mathbf{d}}^{\oplus W^+}, \mathcal{I}_{\mathbf{d}}^{\oplus W^-}$ are always Fredholm, and this gives a sort of "canonical" or "minimal" way to express $\mathcal{M}_{W,\mathbf{d}}$ as a formal difference of (Fredholm) indicator k-diagrams (see Proposition 7.3).

7.1 Formal Definition of Virtual k-Diagrams (and Virtual Vector Spaces, Etc.)

We now introduce a group of formal differences—either of k-vector spaces, k-Fredholm maps, and k-diagrams—that one sees in, say, K-theory, or constructing the integers from the natural numbers. This general idea is often technically called the *Grothendieck completion* or the *Grothendieck group* (of a commutative monoid).

Remark 7.1. *Like virtual characters in group theory, our virtual k-vector spaces and virtual k-diagrams likely concern relatively coarse information (e.g., we don't care about categorizing anything), as compared with Grothendieck groups in other settings. See also Remark 9.1.*

$$
\begin{array}{ccc}
B & \xrightarrow{f} & A \\
{\scriptstyle \phi_B}\downarrow & & \downarrow{\scriptstyle \phi_A} \\
B' & \xrightarrow{f'} & A'
\end{array}
$$

Figure 8: A Morphism $f \to f'$.

Definition 7.1. *Let k be a field. By a* virtual k-diagram *(respectively,* virtual k-Fredholm map, virtual k-vector space, *etc.) we mean a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of k-diagrams (respectively, k-Fredholm map, k-vector space, etc.); we write $(\mathcal{F}_1, \mathcal{F}_2) \sim (\mathcal{F}_1', \mathcal{F}_2')$ — and say that $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}_1', \mathcal{F}_2')$ are* equivalent *— if there is an isomorphism ϕ with*

$$\phi \colon \mathcal{F}_1 \oplus \mathcal{F}_2' \oplus \mathcal{F}_0 \to \mathcal{F}_1' \oplus \mathcal{F}_2 \oplus \mathcal{F}_0$$

for some k-diagram $\mathcal{F}_0$ (respectively, k-Fredholm map, etc.). At times we use the notation $\mathcal{F}_1 \ominus \mathcal{F}_2$ to denote $(\mathcal{F}_1, \mathcal{F}_2)$. We also view a k-diagram (k-Fredholm map, etc.) $\mathcal{F}$ as the virtual diagram $\mathcal{F} \ominus \underline{0}$, where $\underline{0}$ is the zero k-diagram (and similarly for Fredholm k-map, etc.).

We easily see that $\sim$ is an equivalence relation, and that $\oplus$ and $\ominus$ can be extended to act on virtual k-diagrams (respectively, Fredholm k-diagram, etc.) taken to behave like $+$ and $-$ regarding parenthesis, e.g.,

$$(\mathcal{F}_1 \ominus \mathcal{F}_2) \ominus (\mathcal{F}_3 \ominus \mathcal{F}_4) \quad \text{refers to} \quad (\mathcal{F}_1 \oplus \mathcal{F}_4) \ominus (\mathcal{F}_2 \oplus \mathcal{F}_3).$$

7.2 Virtual Vector Spaces

It is important to understand the difference between virtual k-vector spaces and virtual k-vector spaces of finite dimension.

As virtual k-vector spaces, $k \sim 0$ since there is an isomorphism $k \oplus k^{\mathbb{N}} \to k^{\mathbb{N}}$; however, we will easily prove that for virtual finite-dimensional k-vector spaces, we have $k^b \ominus k^a \sim k^{b'} \ominus k^{a'}$ if and only if $b - a = b' - a'$. To prove this, one notices that:

1. if V, V' are isomorphic finite dimensional k-vector spaces, then (clearly) $\dim(V) = \dim(V')$; and
2. therefore if $V_1 \ominus V_2$ is equivalent to $V_3 \ominus V_4$ as finite-dimensional k-vector spaces, then for some finite dimensional k-vector space V_0 we have that

$$V_1 \oplus V_4 \oplus V_0 \simeq V_2 \oplus V_3 \oplus V_0,$$

and taking dimensions, we have an equality of finite integers

$$\dim(V_1) + \dim(V_4) + \dim(V_0) = \dim(V_2) + \dim(V_3) + \dim(V_0),$$

and hence

$$\dim(V_1) - \dim(V_2) = \dim(V_3) - \dim(V_4).$$

3. Hence, we can define

$$\dim(V_1 \ominus V_2) = \dim(V_1) - \dim(V_2) \in \mathbb{Z},$$

which is well defined in the equivalence class of $V_1 \ominus V_2$.

Convention 7.1. *By a* virtual k-vector space *we always mean a virtual k-vector space of finite dimension, unless we specify otherwise.*

7.3 Virtual k-Fredholm Maps

Let $f \colon B \to A$ and $f' \colon B' \to A'$ be morphisms (i.e., linear transformations) of k-vector spaces. By a *morphism from f to f'* we mean a pair $\phi = (\phi_B, \phi_A)$ of morphisms $\phi_B \colon B \to B'$ and $\phi_A \colon A \to A'$ that commute in the evident fashion, i.e., $\phi_A f = f' \phi_B$; see Figure 8. A morphism $\phi = (\phi_B, \phi_A) \colon f \to f'$ is an *isomorphism* if ϕ_B and ϕ_A are isomorphisms[4]. We easily verify the following facts.

1. If $\phi = (\phi_B, \phi_A) \colon f \to f'$ is an isomorphism, then ϕ_B restricts to an isomorphism $\ker(f) \to \ker(f')$, and similarly ϕ_A restricts to an isomorphism $\operatorname{coker}(f) \to \operatorname{coker}(f')$.

[4]We remark that elsewhere, one considers morphisms up to homotopy or localizes at quasi-isomorphisms; in this article we have no need for this.

2. It follows that if we define for a virtual Fredholm map $f = f_1 \ominus f_2$ its cohomology groups for $i = 0, 1$ to be the virtual (finite-dimensional) k-vector spaces

$$H^i\big(f_1 \ominus f_2\big) = H^i(f_1) \ominus H^i(f_2), \tag{50}$$

then if $f_1 \ominus f_2$ is equivalent to $f_3 \ominus f_4$ as virtual k-Fredholm maps, we have

$$H^i(f_1) \ominus H^i(f_2) \sim H^i(f_3) \ominus H^i(f_4)$$

as virtual finite-dimensional k-vector spaces.

3. Hence (50) for $i = 0, 1$ are well-defined virtual finite-dimensional k-vector spaces, and hence setting

$$b^i\big(f_1 \ominus f_2\big) = \dim H^i\big(f_1 \ominus f_2\big) = \dim H^i(f_1) - \dim H^i(f_2) \in \mathbb{Z}$$

gives well-defined Betti numbers of a virtual k-Fredholm map.

Note that if we work with virtual k-linear transformations, without insisting that they are Fredholm maps, then there seems to be no good way to define their Betti numbers (and hence cohomology groups), since dimensions are not well defined for virtual k-vector spaces when we allow the spaces to be of infinite dimension.

7.4 Virtual k-Diagrams with Finite Betti Numbers

If $\mathcal{F}$ is a k-diagram, then $\mathcal{F}(\partial)$ is a k-Fredholm map if and only if $b^i(\mathcal{F})$ is finite for both $i = 0, 1$. Hence, in the category of "k-diagrams with finite Betti numbers," the notion of a (virtual) cohomology group and (virtual) Betti numbers are well-defined, by associating with $\mathcal{F}_1 \ominus \mathcal{F}_2$ the virtual k-Fredholm map

$$(\mathcal{F}_1 \ominus \mathcal{F}_2)(\partial) \stackrel{\text{def}}{=} \mathcal{F}_1(\partial) \ominus \mathcal{F}_2(\partial).$$

Definition 7.2. *Let k be a field. By a* Fredholm k-diagram*, we mean a k-diagram with both Betti numbers finite.*

It follows that virtual Fredholm k-diagrams have well-defined virtual cohomology groups and virtual Betti numbers.

Convention 7.2. *By a* virtual k-diagram *we mean a virtual Fredholm k-diagram unless we specify otherwise, i.e., we are working with k-diagrams with both Betti numbers finite. Hence, a virtual k-diagram has well-defined cohomology groups (which are virtual k-vector spaces of finite dimension) and therefore well-defined Betti numbers.*

7.5 Riemann Functions as Virtual Direct Sums

Our convention is that a virtual k-diagram refers to k-diagrams that are Fredholm; this is necessary to get well-defined Betti numbers. Hence, we need the following easy lemma.

Lemma 7.1. *Let $W : \mathbb{Z}^2 \to \mathbb{Z}$ be any perfect matching. Then for any $\mathbf{d} \in \mathbb{Z}^2$, $\mathcal{M}_{W,\mathbf{d}}$ has finite Betti numbers. Similarly, for any $\mathbf{d} \in \mathbb{Z}^2$, the number of $\mathbf{a}$ with $W(\mathbf{a}) = 1$ and $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$ having at least one non-zero Betti number is finite.*

Proof. Let W be a perfect matching. Theorem 4.2 shows that $\mathcal{M}_{W,\mathbf{d}}$ has finite Betti numbers. The claim about $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$ follows since

$$\mathcal{M}_{W,\mathbf{d}} = \mathcal{I}_{\mathbf{d}}^{\oplus W} = \bigoplus_{W(\mathbf{a})=1} \mathcal{I}_{\mathbf{d}\geq\mathbf{a}},$$

and hence for $i = 0, 1$ we have

$$b^i(\mathcal{M}_{W,\mathbf{d}}) = \sum_{W(\mathbf{a})=1} b^i(\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}).$$

□

One can alternatively prove the claim about $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$ above by noting that the only non-zero Betti numbers of our four basic diagrams are b^0 of $\underline{k}$ and b^1 of $\underline{k}_{/B_1,B_2}$; furthermore we have $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}} = \underline{k}$ if and only if $\mathbf{d} \leq \mathbf{a}$, and $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}} = \underline{k}_{/B_1,B_2}$ if and only if $\mathbf{d} + \mathbf{1} \geq \mathbf{a}$, and each of these conditions on $\mathbf{a}$ occurs for only finitely many $\mathbf{a}$ for which $W(\mathbf{a}) = 1$, since W is supported in degree bounded from above and below.

Definition 7.3. *Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a Riemann function, and $W = \mathfrak{m}f$. For any way of writing*

$$W = (W_1 + \cdots + W_s) - (\tilde{W}_1 + \cdots + \tilde{W}_{s-1}) \tag{51}$$

as the difference of a sum of perfect matchings, we use $\mathcal{M}_{W,\mathbf{d}}$ (with (51) *understood) to denote the virtual (Fredholm) k-diagram*

$$\bigl(\mathcal{M}_{W_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_s,\mathbf{d}}, \mathcal{M}_{\tilde{W}_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{\tilde{W}_{s-1},\mathbf{d}}\bigr).$$

The following proposition immediately implies the main result of this section, Theorem 7.1.

Proposition 7.1. *Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a Riemann function, and W its weight. For any way of writing W as* (51)*, the equivalence class $[\mathcal{M}_{W,\mathbf{d}}]$, of the virtual k-diagram $\mathcal{M}_{W,\mathbf{d}}$ is independent of the way we write W in* (51)*. Furthermore, for any $\mathbf{d} \in \mathbb{Z}^2$,*

1. *$f(\mathbf{d}) = b^0([\mathcal{M}_{W,\mathbf{d}}])$,*
2. *for any $\mathbf{K} \in \mathbb{Z}^2$, $f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d}) = b^1([\mathcal{M}_{W,\mathbf{d}}])$, and*
3. *$\chi([\mathcal{M}_{W,\mathbf{d}}]) = \deg(\mathbf{d}) + C$ where C is the offset of f.*

Proof. Say that we write W as

$$W = W_1 + \cdots + W_s - \bigl(\tilde{W}_1 + \cdots + \tilde{W}_{s-1}\bigr)$$

with all W_i and $\tilde{W}_i$ as perfect matchings, and also as a difference of the sums of perfect matchings

$$W = W'_1 + \cdots + W'_{s'} - \bigl(\tilde{W}'_1 + \cdots + \tilde{W}'_{s'-1}\bigr).$$

Then we have

$$W_1 + \cdots + W_s + \tilde{W}'_1 + \cdots + \tilde{W}'_{s'-1} = \tilde{W}_1 + \cdots + \tilde{W}_{s-1} + W'_1 + \cdots + W'_{s'}.$$

It follows from Proposition 6.3, specifically (47), that for any $\mathbf{d} \in \mathbb{Z}^2$

$$\left(\bigoplus_{i=1}^{s} \mathcal{M}_{W_i,\mathbf{d}}\right) \oplus \left(\bigoplus_{i=1}^{s'-1} \mathcal{M}_{\tilde{W}'_i,\mathbf{d}}\right) \simeq \left(\bigoplus_{i=1}^{s-1} \mathcal{M}_{\tilde{W}_i,\mathbf{d}}\right) \oplus \left(\bigoplus_{i=1}^{s'} \mathcal{M}_{W'_i,\mathbf{d}}\right)$$

and hence

$$\left(\bigoplus_{i=1}^{s} \mathcal{M}_{W_i,\mathbf{d}}\right) \ominus \left(\bigoplus_{i=1}^{s-1} \mathcal{M}_{\tilde{W}_i,\mathbf{d}}\right) \simeq \left(\bigoplus_{i=1}^{s'} \mathcal{M}_{W'_i,\mathbf{d}}\right) \ominus \left(\bigoplus_{i=1}^{s'-1} \mathcal{M}_{\tilde{W}'_i,\mathbf{d}}\right)$$

as virtual k-diagrams. Hence the class $[\mathcal{M}_{W,\mathbf{d}}]$ is independent of how we write W as a difference of (finite) sums of perfect matchings.

For the second part of the proposition, we write W as in (51), and note that for any i,

$$b^0(\mathcal{M}_{W_i,\mathbf{d}}) = (\mathfrak{s}W_i)(\mathbf{d}),$$

and similarly for $\tilde{W}_i$, and hence

$$b^0(\mathcal{M}_{W,\mathbf{d}}) = \bigl(\mathfrak{s}(W_1 + \cdots + W_s)\bigr)(\mathbf{d}) - \bigl(\mathfrak{s}(\tilde{W}_1 + \cdots + \tilde{W}_{s-1})\bigr)(\mathbf{d}) = (\mathfrak{s}W)(\mathbf{d}),$$

where the last equality holds by applying $\mathfrak{s}$ to both sides of (51) and using the linearity of $\mathfrak{s}$. We reason similarly with b^0 replaced with χ, in view of (17); finally we use $b^1 = b^0 - \chi$ to reason about b^1. ☐

It is worth collecting together an easy consequence of Proposition 7.1 and material in previous sections; this can be viewed as the main result of this section.

Theorem 7.1. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function with offset C, and $\mathbf{K} \in \mathbb{Z}^n$. Then the generalized Riemann-Roch formula*

$$f(\mathbf{d}) - f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d}) = \deg(\mathbf{d}) + C \tag{52}$$

(see (13) *and Definition 2.3) can be modeled as expressing the Euler characteristic of a virtual k-diagram as follows:*

1. *Let W be the weight of f, i.e., $W = \mathfrak{m}f$, or, equivalently, $f = \mathfrak{s}W$ and W is initially zero (Proposition 2.2).*

2. *Write W as a difference of a sum of perfect matchings*

$$W = (W_1 + \cdots + W_s) - (\tilde{W}_1 + \cdots + \tilde{W}_{s-1}) \tag{53}$$

for some s (this is (48); see Lemma 3.1 for justification).

3. *Let $\mathcal{M}_{W,\mathbf{d}}$ be the virtual k-diagram:*

$$\mathcal{M}_{\mathbf{d}} = \left(\bigoplus_i \mathcal{M}_{W_i,\mathbf{d}}, \bigoplus_i \mathcal{M}_{\tilde{W}_i,\mathbf{d}} \right),$$

or, written equivalently,

$$\mathcal{M}_{\mathbf{d}} = \left(\bigoplus_i \mathcal{M}_{W_i,\mathbf{d}} \right) \ominus \left(\bigoplus_i \mathcal{M}_{\tilde{W}_i,\mathbf{d}} \right).$$

4. *Note that although $\mathcal{M}_{W,\mathbf{d}}$ depends on the choice of W_i's and $\tilde{W}_i$'s in (53), the isomorphism class, $[\mathcal{M}_{W,\mathbf{d}}]$, of $\mathcal{M}_{W,\mathbf{d}}$ is independent of this choice (Proposition 7.1).*

5. *Then (52) can be interpreted as follows:*

$$f(\mathbf{d}) = b^0([\mathcal{M}_{W,\mathbf{d}}]), \quad f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = b^1([\mathcal{M}_{W,\mathbf{d}}]),$$

and therefore

$$\chi([\mathcal{M}_{W,\mathbf{d}}]) = \deg(\mathbf{d}) + C.$$

In Section 10 we will see that $\mathcal{M}_{W,\mathbf{d}}$ inherits the usual properties of line bundles in the classical Riemann-Roch theorem: i.e., if $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors in $\mathbb{Z}^2$, then for each $i = 1, 2$ there is a short exact sequence of virtual k-diagrams

$$0 \to \mathcal{M}_{W,\mathbf{d}} \to \mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i} \to \mathcal{S}_i(k) \to 0$$

where $\mathcal{S}_i$ is a virtual k-diagram isomorphic to the usual skyscraper (sheaf or) k-diagram, $\mathrm{Sky}_{A_i}(k)$. [This is not entirely true, since we are working with virtual k-diagrams, and hence we have a "difference" of two short exact sequences, whose third elements are, respectively, s and $s-1$ copies of $\mathcal{S}_i(k)$.]

Example 7.1. *Let $W \colon \mathbb{Z}^2 \to \mathbb{Z}$ be 3-periodic and satisfy $W(1,0) = W(1,2) = W(0,1) = W(2,1) = 1$ and $W(1,1) = -1$. Therefore, for all $m \in \mathbb{Z}$,*

$$W(3m+1, -3m) = W(3m+1, -3m+2) = W(3m, -3m+1) = W(3m+2, -3m+1) = 1, \tag{54}$$

and

$$W(3m+1, -3m+1) = -1. \tag{55}$$

We easily see that the values of $\mathbf{a} = (a_1, a_2)$ at which $W(\mathbf{a}) \neq 0$ and $0 \le a_1 \le 2$ or $0 \le a_2 \le 2$ correspond to the values in (54) and (55) with $m = 0$. It follows that the $0,1,2$-th row sums and the $0,1,2$-th column sums equal 1, and hence all row sums and all column sums of W equal 1, and hence $\mathfrak{s}W$ is a Riemann function (one can also check that W is slowly growing, *as in Definition 2.8, by examining the values of W in rows $0,1,2$ and columns $0,1,2$)*[5]. *One can write $W = W_1 - W_2 + W_3$ in a number of ways, even where each W_i is 3-periodic, but there seems to be no canonical way of doing so. One can also "canonically" write $W = W^+ - W^-$ with $W^+ = \max(W,0)$ and $W^- = \max(-W,0)$, but then we easily see that $b^0(\mathcal{M}_{W^+,\mathbf{d}}) = +\infty$*[6] *for any $\mathbf{d}$ and $b^1(\mathcal{M}_{W^-,\mathbf{d}}) = +\infty$ for any $\mathbf{d}$*[7]. *Hence the virtual diagram $(\mathcal{M}_{W^+,\mathbf{d}}, \mathcal{M}_{W^-,\mathbf{d}})$ doesn't have a well-defined Euler characteristic (which should equal $\deg(\mathbf{d}) + C$ for some C).*

Example 7.2. *We remark that there are virtual k-diagrams $[\mathcal{M}_{W,\mathbf{d}}]$ that can be realized as a formal difference of Fredholm k-diagrams in uncountably many ways. Indeed, let $W(i,j) = 1$ if and only if for some $t \in \mathbb{Z}$ we have $i \in \{2t, 2t+1\}$ and $j \in \{-2t, -2t+1\}$. Hence W is a 2-fold matching. We claim there are uncountably many perfect matchings W_1 such that $W_2 = W - W_1$ is also a perfect matching: indeed, consider any perfect matching, W_1, such that for each $t \in \mathbb{Z}$ either $W_1(2t, -2t) = W_1(2t+1, -2t+1) = 1$ or $W_1(2t+1, -2t) = W(2t, -2t+1) = 1$; we easily see that $W_2 = W - W_1$ is also a perfect matching, and there are uncountably many such W_1 (and W_1, W_2 are supported in degrees 0,1,2, so they are, indeed, perfect matchings). By contrast, if we insist that W_1*

[5]For any $W \colon \mathbb{Z}^2 \to \mathbb{Z}$ we easily see that $\mathfrak{s}W$ is slowly growing if and only if the sign pattern in each row and each column is an alternating sequence of +'s and −'s, beginning and ending in +'s. For the W in this example, we easily check this to be the case.

[6]To see that $b^0(\mathcal{M}_{W^+,\mathbf{d}}) = +\infty$, note that for each m large, $G = \mathrm{Graph}(W^+, \mathbf{d})$ as in Theorem 4.3 has a cycle created by the W-values $W(3m+1, -3m) = W(3m+1, -3m+2) = 1$ that give a multiple edge from v_0 to the vertex $3m \in V_{\mathrm{first}}$.

[7]since for any $m \in \mathbb{Z}$ the vertices $3m, 3m+2 \in V_{\mathrm{first}}$ are isolated in the graph $\mathrm{Graph}(W^-, \mathbf{d})$ as in Theorem 4.3.

is r-periodic for some $r \geq 1$, then there are only finitely many possible W_1 as above. Hence, for fixed perfect matchings $\tilde{W}_1, \tilde{W}_2$ with $\tilde{W}_1 + \tilde{W}_2 = W$, we have

$$\mathcal{M}_{\tilde{W}_1,\mathbf{d}} \oplus \mathcal{M}_{\tilde{W}_2,\mathbf{d}} \simeq \mathcal{M}_{W_1,\mathbf{d}} \oplus \mathcal{M}_{W_2,\mathbf{d}}$$

whenever $W_1 + W_2 = W$.

Example 7.2 shows that unless we assume periodicity, the virtual k-diagram $[\mathcal{M}_{W,\mathbf{d}}]$ can be realized as a virtual k-diagram in uncountably many ways. It similarly follows (unless we make some periodicity assumptions) that whenever (48) holds, there are uncountably many ways of writing W as (48), if we replace s with $s+2$ there, and let $W_{s+1}, W_{s+2}, \tilde{W}_s, \tilde{W}_{s+1}$ be, respectively, $W_1, W_2, \tilde{W}_1, \tilde{W}_2$ in Example 7.2 above.

One can generalize the above proposition to indicator k-diagram sums.

Proposition 7.2. *Let $W_1, \ldots, W_4$ be functions $\mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ such that $W_1 - W_2 = W_3 - W_4$. Then for all $\mathbf{d}$ we have*

$$\mathcal{I}_{\mathbf{d}}^{W_1} \ominus \mathcal{I}_{\mathbf{d}}^{W_2} \sim \mathcal{I}_{\mathbf{d}}^{W_3} \ominus \mathcal{I}_{\mathbf{d}}^{W_4}.$$

Moreover, if $\mathcal{I}_{\mathbf{d}}^{W_i}$ is a Fredholm k-diagram for all i, then this equivalence holds as virtual Fredholm k-diagrams.

Proof. Indeed, if $W = W_1 + W_4 = W_2 + W_3$, then clearly

$$\mathcal{I}_{\mathbf{d}}^{W_1} \oplus \mathcal{I}_{\mathbf{d}}^{W_4} \simeq \mathcal{I}_{\mathbf{d}}^{W} \simeq \mathcal{I}_{\mathbf{d}}^{W_2} \oplus \mathcal{I}_{\mathbf{d}}^{W_3}.$$

□

7.6 A Canonical Virtual k-Diagram of a Riemann Function

In Example 7.1 we remarked that one can canonically write

$$W = W^+ - W^-, \quad \text{where} \quad W^+ = \max(W, 0),\ W^- = \max(-W, 0), \tag{56}$$

but this doesn't generally express W as a difference of perfect matchings; furthermore, as Example 7.1 shows, even if $\mathfrak{s}W$ is a Riemann function, $(\mathcal{M}_{W^+,\mathbf{d}}, \mathcal{M}_{W^-,\mathbf{d}})$ may not have a well defined Euler characteristic. In this subsection, we remark that the (56) does lead to a canonical way to write $[\mathcal{M}_{W,\mathbf{d}}]$ as a virtual k-diagram composed of indicator k-diagrams.

Proposition 7.3. *Let $W : \mathbb{Z}^2 \to \mathbb{Z}$ be initially and eventually zero. With notation as in (56), for each $\mathbf{d} \in \mathbb{Z}^2$, the formal difference*

$$\mathcal{I}_{\mathbf{d}}^{\oplus W} \stackrel{\text{def}}{=} \mathcal{I}_{\mathbf{d}}^{\oplus W^+} \ominus \mathcal{I}_{\mathbf{d}}^{\oplus W^-}$$

is a virtual k-diagram, i.e., both $\mathcal{I}_{\mathbf{d}}^{\oplus W^+}$ and $\mathcal{I}_{\mathbf{d}}^{\oplus W^-}$ are Fredholm k-diagrams. Furthermore, if $\mathfrak{s}W$ is a Riemann function, then as virtual (Fredholm) k-diagrams

$$[\mathcal{I}_{\mathbf{d}}^{\oplus W}] = [\mathcal{M}_{W,\mathbf{d}}]. \tag{57}$$

Proof. Since W is initially and eventually zero, for any $\mathbf{d}$, the number of $\mathbf{a}$ with $\mathbf{d} \leq \mathbf{a}$ and $W(\mathbf{a}) \neq 0$ is finite, and similarly for the number with $\mathbf{d} + \mathbf{1} \geq \mathbf{a}$. Hence both $\mathcal{I}_{\mathbf{d}}^{\oplus W^+}$ and $\mathcal{I}_{\mathbf{d}}^{\oplus W^-}$ are Fredholm k-diagrams. To see (57), we see that for any equality (48) we have

$$(W_1 + \cdots + W_s) - (\tilde{W}_1 + \cdots + \tilde{W}_{s-1}) = W = W^+ - W^-$$

and hence

$$W^- + (W_1 + \cdots + W_s) = W^+ + (\tilde{W}_1 + \cdots + \tilde{W}_{s-1})$$

and hence for all $\mathbf{d} \in \mathbb{Z}$ we have

$$\mathcal{I}_{\mathbf{d}}^{\oplus W^-} \oplus \mathcal{I}_{\mathbf{d}}^{\oplus W_1} \oplus \cdots \oplus \mathcal{I}_{\mathbf{d}}^{\oplus W_s} \simeq \mathcal{I}_{\mathbf{d}}^{\oplus W^+} \oplus \mathcal{I}_{\mathbf{d}}^{\oplus \tilde{W}_1} \oplus \cdots \oplus \mathcal{I}_{\mathbf{d}}^{\oplus \tilde{W}_{s-1}},$$

and hence

$$\mathcal{I}_{\mathbf{d}}^{\oplus W^-} \oplus \mathcal{M}_{W_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_s,\mathbf{d}} \simeq \mathcal{I}_{\mathbf{d}}^{\oplus W^+} \oplus \mathcal{M}_{\tilde{W}_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{\tilde{W}_{s-1},\mathbf{d}},$$

and since all k-diagrams are Fredholm, this implies (57) as virtual (Fredholm) k-diagrams. □

8 Modeling General Riemann Functions

In this section, we model any Riemann function $f : \mathbb{Z}^n \to \mathbb{Z}$ by gluing together the models we have developed for $n = 2$. We begin by stating the main results, leaving the proofs of the more difficult theorems for later subsections.

8.1 Main Modeling Results

Definition 8.1. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function. For any $i,j \in [n]$ with $i \neq j$ and any $\mathbf{d} \in \mathbb{Z}^n$, let $f_{i,j,\mathbf{d}} = f_{i,j,\mathbf{d}}(a_i,a_j)\colon \mathbb{Z}^2 \to \mathbb{Z}$ be the two-variable restriction (16). (We write a_i,a_j as the arguments for $f_{i,j,\mathbf{d}}$ instead of, say, a_1,a_2, to stress that a_i corresponds to adding $a_i\mathbf{e}_i$ in (16), and similarly for a_j). We set $W = W_{f;i,j,\mathbf{d}}$ to be the weight of $f_{i,j,\mathbf{d}}$, and define the* virtual k-diagram associated to f and $\mathbf{d}$ at coordinates *i,j to be the class*

$$[\mathcal{M}_{f;i,j,\mathbf{d}}] \stackrel{def}{=} [\mathcal{M}_{W,\mathbf{0}}] = [\mathcal{M}_{W_{f;i,j,\mathbf{d}},\mathbf{0}}]$$

(which we know is a single equivalence class of virtual (Fredholm) k-diagrams).

The merit of the above definition is described in the following theorem, that is really a straightforward consequence of Proposition 7.1.

Theorem 8.1. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function. Then for any distinct $i,j \in [n]$ and $\mathbf{d} \in \mathbb{Z}^n$ we have*

$$b^0([\mathcal{M}_{f;i,j,\mathbf{d}}]) = f_{i,j,\mathbf{d}}(\mathbf{0}) = f(\mathbf{d}), \tag{58}$$

$$\chi([\mathcal{M}_{f;i,j,\mathbf{d}}]) = \chi(\mathcal{M}_{W,\mathbf{0}}) = \deg(\mathbf{d}) + C \tag{59}$$

where C is the offset of f, and for every $\mathbf{K} \in \mathbb{Z}^n$ we have

$$b^1([\mathcal{M}_{f;i,j,\mathbf{d}}]) = f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d}). \tag{60}$$

In particular, it follows that for any distinct $i',j' \in [n]$ we have

$$b^1([\mathcal{M}_{f;i,j,\mathbf{d}}]) = f^{\wedge}_{\mathbf{K}}(\mathbf{K}-\mathbf{d}) = b^0([\mathcal{M}_{f^{\wedge}_{\mathbf{K}};i',j',\mathbf{K}-\mathbf{d}}]). \tag{61}$$

Proof. By definition, $\mathcal{M}_{f;i,j,\mathbf{d}} = \mathcal{M}_{W,\mathbf{0}}$ with $W = W_{f;i,j,\mathbf{d}}$ equal to the weight of $f_{i,j,\mathbf{d}}$. Hence Proposition 7.1 implies that

$$b^0[\mathcal{M}_{W,\mathbf{0}}] = (\mathfrak{s}W)(\mathbf{0}) = f_{i,j,\mathbf{d}}(\mathbf{0}) = f(\mathbf{d})$$

and hence (58) holds. If C is the offset of f, then for sufficiently large $a_i + a_j$ we have

$$f_{i,j,\mathbf{d}}(a_i,a_j) = f(\mathbf{d} + a_i\mathbf{e}_i + a_j\mathbf{e}_j) = \deg(\mathbf{d} + a_i\mathbf{e}_i + a_j\mathbf{e}_j) + C = a_i + a_j + \deg(\mathbf{d}) + C$$

and it follows that the offset of $f_{i,j,\mathbf{d}}$ is $C' = \deg(\mathbf{d}) + C$. So Proposition 7.1 implies that

$$\chi([\mathcal{M}_{W,\mathbf{0}}]) = \deg(\mathbf{0}) + C' = C' = \deg(\mathbf{d}) + C$$

and (59) follows. It follows that

$$b^1[\mathcal{M}_{W,\mathbf{0}}] = b^0[\mathcal{M}_{W,\mathbf{0}}] - \chi[\mathcal{M}_{W,\mathbf{0}}] = f(\mathbf{d}) - \deg(\mathbf{d}) - C,$$

and Proposition 7.1 and (13) implies (60).

Finally, (61) follows from (60) and from (58) with f replaced with $f^{\wedge}_{\mathbf{K}}$ and with $\mathbf{d}$ replaced with $\mathbf{K}-\mathbf{d}$. □

The main goal of this section is to prove the following two theorems that state that the equivalence class of the virtual Fredholm k-diagram $[\mathcal{M}_{f;i,j,\mathbf{d}}]$ is independent of the choice of i and j.

Theorem 8.2. *Let $n \in \mathbb{N}$ with $n \geq 3$, $\mathbf{d} \in \mathbb{Z}^n$, $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function, and $i,j,j' \in [n]$ be three distinct integers. Then $[\mathcal{M}_{f;i,j,\mathbf{d}}] = [\mathcal{M}_{f;i,j',\mathbf{d}}]$.*

This theorem is more technical, and will be proven in Subsection 8.5. The above theorem easily yields one of the main results in the section.

Corollary 8.1. *Let $n \in \mathbb{N}$ with $n \geq 2$, $\mathbf{d} \in \mathbb{Z}^n$, and $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function. Then the equivalence class $[\mathcal{M}_{f;i,j,\mathbf{d}}]$ of virtual (Fredholm) k-diagrams is independent of the choice of distinct $i,j \in [n]$.*

Proof. For distinct $i,j \in [n]$ we have $\mathcal{M}_{f;i,j,\mathbf{d}} \simeq \mathcal{M}_{f;j,i,\mathbf{d}}$ by the evident morphism that exchanges B_1,A_1 values respectively with B_2,A_2 values. Hence for any $i \neq j$ we have

$$[\mathcal{M}_{f;i,j,\mathbf{d}}] = [\mathcal{M}_{f;j,i,\mathbf{d}}]. \tag{62}$$

If $n = 2$, then the only choices of distinct $i,j \in [2]$ are (i,j) equal to either $(1,2)$ or $(2,1)$. Since (62) shows that

$$[\mathcal{M}_{f;1,2,\mathbf{d}}] = [\mathcal{M}_{f;2,1,\mathbf{d}}],$$

this proves the corollary in the case $n = 2$.

Hence, it suffices to prove the corollary when $n \geq 3$.

According to Theorem 8.2 we have $[\mathcal{M}_{f;1,2,\mathbf{d}}] = [\mathcal{M}_{f;1,j,\mathbf{d}}]$ for any $j \geq 3$, and similarly for any $i \neq j$ we have $[\mathcal{M}_{f;j,1,\mathbf{d}}] = [\mathcal{M}_{f;j,i,\mathbf{d}}]$. Combining these two equalities with (62) we have

$$[\mathcal{M}_{f;1,2,\mathbf{d}}] = [\mathcal{M}_{f;1,j,\mathbf{d}}] = [\mathcal{M}_{f;j,1,\mathbf{d}}] = [\mathcal{M}_{f;j,i,\mathbf{d}}] = [\mathcal{M}_{f;i,j,\mathbf{d}}].$$

□

The above corollary makes the following definition well-defined.

Definition 8.2. *For any Riemann function* $f\colon \mathbb{Z}^n \to \mathbb{Z}$*, and any* $\mathbf{d} \in \mathbb{Z}^n$*, we define the* virtual k-diagram of f at $\mathbf{d}$*, denoted* $[\mathcal{M}_{f \text{ at } \mathbf{d}}]$ *to be the class of virtual k-diagram* $[\mathcal{M}_{f;i,j,\mathbf{d}}]$ *for any distinct* $i, j \in [n]$ *(which is a single equivalence class of virtual k-diagrams in view of Corollary 8.1).*

Stating Theorem 8.1 in terms of Definition 8.2 immediately implies the following theorem.

Theorem 8.3. *Let* $f\colon \mathbb{Z}^n \to \mathbb{Z}$ *be any Riemann function with offset* C*, and let* $\mathbf{d}, \mathbf{K} \in \mathbb{Z}$*. Then we have*

$$b^0\big([\mathcal{M}_{f \text{ at } \mathbf{d}}]\big) = f(\mathbf{d}),$$

$$\chi\big([\mathcal{M}_{f \text{ at } \mathbf{d}}]\big) = \deg(\mathbf{d}) + C,$$

and

$$b^1\big([\mathcal{M}_{f \text{ at } \mathbf{d}}]\big) = f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = b^0\big([\mathcal{M}_{f^{\wedge}_{\mathbf{K}} \text{ at } \mathbf{K}-\mathbf{d}}]\big).$$

In the following special case of Theorem 8.2 one can prove a much stronger result.

Theorem 8.4. *Let* $n \in \mathbb{N}$ *with* $n \geq 3$*,* $\mathbf{d} \in \mathbb{Z}^n$*,* $f\colon \mathbb{Z}^n \to \mathbb{Z}$ *be a Riemann function, and* $i, j, j' \in [n]$ *be three distinct integers. Say that the weights of* $f_{i,j,\mathbf{d}}$ *and* $f_{i,j',\mathbf{d}}$*, respectively* $W_{f;i,j,\mathbf{d}}$ *and* $W_{f;i,j',\mathbf{d}}$*, are non-negative, and hence both perfect matchings. Then we have*

$$\mathcal{M}_{f;i,j,\mathbf{d}} \simeq \mathcal{M}_{f;i,j',\mathbf{d}}.$$

The proof of this theorem will be given in Subsection 8.3.

Corollary 8.2. *Let* $n \in \mathbb{N}$ *with* $n \geq 3$*,* $\mathbf{d} \in \mathbb{Z}^n$*, and* $f\colon \mathbb{Z}^n \to \mathbb{Z}$ *be a Riemann function. Say that for some* $I \subset [n]$ *we have that for all* $i, j \in I$*,* $W_{f;i,j,\mathbf{d}} = \mathfrak{m}f_{i,j,\mathbf{d}}$ *is everywhere non-negative, and is therefore a perfect matching. Then all the k-diagrams* $\mathcal{M}_{f;i,j,\mathbf{d}}$ *varying over distinct* $i, j \in I$ *are isomorphic (as k-diagrams).*

The proof is the same as that of Corollary 8.1.

In particular, if $I = [n]$ in the above corollary, then for fixed f and all $\mathbf{d} \in \mathbb{Z}^n$, the k-diagrams $\mathcal{M}_{f;i,j,\mathbf{d}}$ are all isomorphic, and one can define $[\mathcal{M}_{f \text{ at } \mathbf{d}}]$ as an equivalence class of k-diagrams. Then Corollary 8.2 yields the following result, which gives a stronger conclusion than Theorem 8.3 in a special case thereof.

Theorem 8.5. *Let* $f\colon \mathbb{Z}^n \to \mathbb{Z}$ *be any Riemann function with offset* C*, and let* $\mathbf{K} \in \mathbb{Z}$*. Assume that for all distinct* $i, j \in [n]$ *and all* $\mathbf{d} \in \mathbb{Z}^n$*,* $\mathfrak{m}f_{i,j,\mathbf{d}}$ *and* $\mathfrak{m}(f^{\wedge}_{\mathbf{K}})_{i,j,\mathbf{d}}$ *are perfect matchings. Then the conclusions of Theorem 8.3 hold where we understand that* $[\mathcal{M}_{f \text{ at } \mathbf{d}}]$ *and* $[\mathcal{M}_{f^{\wedge}_{\mathbf{K}} \text{ at } \mathbf{K}-\mathbf{d}}]$ *refer to equivalence classes of k-diagrams.*

We remark that (89) of Section 9 shows that if $\mathfrak{m}f_{i,j,\mathbf{d}}$ are non-negative for all $i, j, \mathbf{d}$, then so all the $\mathfrak{m}(f^{\wedge}_{\mathbf{K}})_{i,j,\mathbf{d}}$, and conversely.

It is helpful to first prove Theorem 8.4 first, as it is simpler to prove, but illustrates the main idea in the proof of Theorem 8.2

The rest of this section is dedicated to proving these two theorems.

One crucial ingredient of the proofs of both theorems is the equality

$$\forall a \in \mathbb{Z}, \quad f_{i,j,\mathbf{d}}(a, 0) = f(\mathbf{d} + a\mathbf{e}_i) = f_{i,j',\mathbf{d}}(a, 0). \tag{63}$$

The other idea in both proofs is to look for an isomorphism that is very simple along the A_1 values of $\mathcal{M}_{W,\mathbf{d}}$, and to see what conditions this requires elsewhere; it turns out that it is only along the B_2 value that one needs some conditions, and those conditions turn out to be exactly (63). Let us give the details.

8.2 Isomorphisms That Are Simple Along the A_1 Values

To prove Theorem 8.4, we will use the following lemma.

Lemma 8.1. *Let* W, W' *be perfect matchings, and* π, π' *their associated bijections. Then for any* $\mathbf{d} \in \mathbb{Z}^2$ *the following are equivalent:*

1. *there exists an isomorphism of k-diagrams* $\phi\colon \mathcal{M}_{W,\mathbf{d}} \to \mathcal{M}_{W',\mathbf{d}}$ *such that* $\phi(A_1)$ *is the identity; and*
2. $$\forall a \in \mathbb{Z}_{\leq d_1}, \quad \pi(a) \leq d_2 \iff \pi'(a) \leq d_2. \tag{64}$$

We remark that to prove Theorem 8.2, we need know only that (2) $\Rightarrow$ (1).

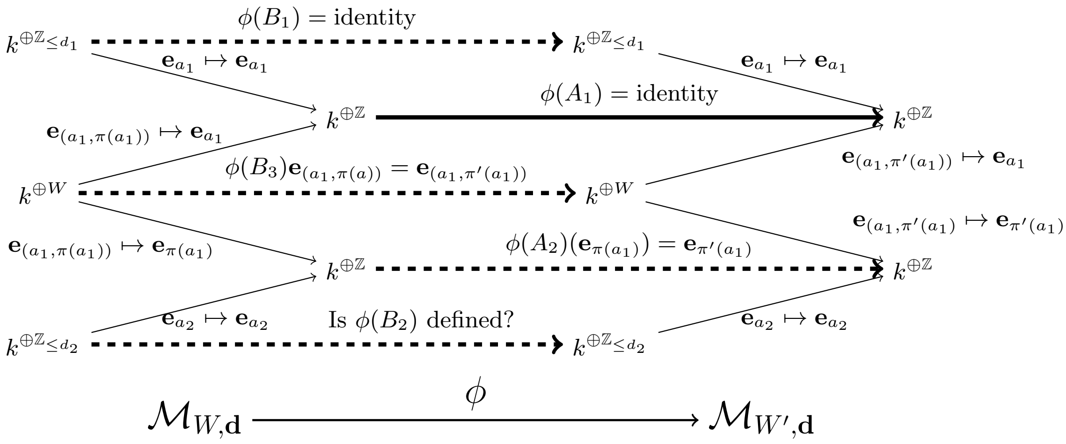


Figure 9: Implications of $\phi(A_1) = \text{identity}$, for a $\phi\colon \mathcal{M}_{W,\mathbf{d}} \to \mathcal{M}_{W',\mathbf{d}}$ (where we omit the quantifiers $\forall a_1 \in \mathbb{Z}$ and $\forall a_2 \in \mathbb{Z}$)

Proof. Consider a morphism ϕ with $\phi(A_1)$ the identity. In Figure 9, $\phi(A_1)$, i.e., identity map, is depicted in a thick line, and the inferences about the other values of ϕ are depicted in dashed lines. We easily see that (considering Figure 9):

1. $\phi(B_1)$ is forced to be the identity,
2. $\phi(B_3)$ must take $\mathbf{e}_{(a_1,\pi(a_1))}$ to $\mathbf{e}_{(a_1,\pi'(a_1))}$ for all $a_1 \in \mathbb{Z}$,
3. $\phi(A_2)$ is forced to take $\mathbf{e}_{\pi(a_1)}$ to $\mathbf{e}_{\pi'(a_1)}$ for all $a_1 \in \mathbb{Z}$, and
4. $\phi(B_2)$ is uniquely determined if it exists, and it exists if and only if for all $a_1 \in \mathbb{Z}$ with $\pi(a_1) \le d_2$, the vector the $\mathbf{e}_{\pi'(a_1)}$ lies in $k^{\oplus\mathbb{Z}_{\le d_2}}$.

Hence $\phi(B_2)$ exists if and only if

$$\forall a_1 \in \mathbb{Z}_{\le d_1}, \quad \pi(a_1) \le d_2 \Rightarrow \pi'(a_1) \le d_2;$$

and hence $\phi(B_2)$ exists and is an isomorphism if and only if (64) holds. □

We similarly prove the following generalization that we will use to prove Theorem 8.2.

Lemma 8.2. *Let $W_1, \cdots, W_s$ and $W'_1, \cdots, W'_s$ be two sequences of perfect matchings $\mathbb{Z}^2 \to \mathbb{Z}$. Let $\mathbf{d} \in \mathbb{Z}^2$, and for each $a_1 \in \mathbb{Z}$ and $j \in [s]$, let $\mathbf{e}_{a_i,j}$ and $\mathbf{e}'_{a_i,j}$ denote, respectively, the standard basis vector $\mathbf{e}_{a_1} \in k^{\mathbb{Z}}$ in, respectively $\mathcal{M}_{W_j,\mathbf{d}}(A_1)$ and $\mathcal{M}_{W'_j,\mathbf{d}}(A_1)$. Let*

$$W = W_1 + \cdots + W_s, \quad W' = W'_1 + \cdots + W'_s,$$

and

$$\mathcal{M}_{\mathbf{d}} = \mathcal{M}_{W_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_s,\mathbf{d}}, \quad \mathcal{M}'_{\mathbf{d}} = \mathcal{M}_{W'_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W'_s,\mathbf{d}}.$$

Then for any $\mathbf{d} \in \mathbb{Z}^2$, the following are equivalent:

1. *there exists an isomorphism $\phi\colon \mathcal{M} \to \mathcal{M}'$ that for each $a_1 \in \mathbb{Z}_{\le d_1}$, $\phi(A_1)$ restricted to $\mathbf{e}_{a_1,1}, \ldots, \mathbf{e}_{a_1,s}$ yields a bijection from this set to $\mathbf{e}'_{a_1,1}, \ldots, \mathbf{e}'_{a_1,s}$;*
2. *letting π_r, π'_r for $r \in [s]$ denote the bijections associated to W_r, W'_r, for all $a_1 \le d_1$,*

$$\bigl|\{r \mid \pi_r(a_1) \le d_2\}\bigr| = \bigl|\{r' \mid \pi'_{r'}(a_1) \le d_2\}\bigr|.$$

Proof. We have

$$\mathcal{M}_{\mathbf{d}}(A_1) = \bigoplus_{a_1 \in \mathbb{Z}} \operatorname{Span}(\mathbf{e}_{a_1,1}, \ldots, \mathbf{e}_{a_1,s})$$

and similarly with $\mathcal{M}'$ and the $\mathbf{e}'_{a_1,i}$. Condition (1) says that for each $a_1 \in \mathbb{Z}$ there is a permutation $\sigma = \sigma_{a_1}$ on $[s]$ such that

$$\phi(A_1)(\mathbf{e}_{a_1,i}) = \mathbf{e}'_{a_1,\sigma_{a_1}(i)}.$$

But the isomorphism $\mathcal{M}_{W_i,\mathbf{d}} \simeq \mathcal{I}_{\mathbf{d}}^{\oplus W_i}$ allows us to write

$$\mathcal{M}_{\mathbf{d}} \simeq \bigoplus_{a_1\in\mathbb{Z},\ i\in[s]} \mathcal{I}_{\mathbf{d}\geq(a_1,\pi_i(a_1))}$$

in a way that $\mathbf{e}_{a_1,i} \in \mathcal{M}_{\mathbf{d}}(A_1)$ corresponds to the standard basis vector in vector (with the same indices) $\mathbf{e}_{a_1,i}$. Since the same is true of $\mathcal{M}'_{\mathbf{d}}$ and σ_{a_1}, that gives that the desired isomorphism must have that for all $a_1 \in \mathbb{Z}$ and $i \in [s]$

$$\mathcal{I}_{\mathbf{d}\geq(a_1,i)} \simeq \mathcal{I}_{\mathbf{d}\geq(a_1,\sigma_{a_1}(i))};$$

similar to the argument in the proof of Lemma 8.1, this holds automatically for the B_1, B_3, A_2 values, and holds at the B_2 value if and only if

$$\pi_i(a_1) \leq d_2 \iff \pi'_{\sigma_{a_1}(i)}(a_1) \leq d_2. \tag{65}$$

Hence, for each $a_1 \in \mathbb{Z}$, such a σ_{a_1} exists if and only if (65) holds, and if so for each $a_1 \in \mathbb{Z}$ we can choose any permutation σ_{a_1} on $[s]$ that maps

$$\{r \mid \pi_r(a_1) \leq d_2\} \quad \text{to} \quad \{r' \mid \pi'_{r'}(a_1) \leq d_2\}$$

(and therefore σ_{a_1} also maps the same with $\leq d_2$ replaced everywhere with $> d_2$). ☐

8.3 Proof of Theorem 8.4 and Examples

In this section, we will prove Theorem 8.4, which follows almost immediately from the lemma below (which adds a third equivalent condition to Lemma 8.1).

Lemma 8.3. *Let W, W' be perfect matchings, and π, π' their associated bijections. Then for any $\mathbf{d} \in \mathbb{Z}^2$ the following are equivalent:*

1. *there exists an isomorphism of k-diagrams $\phi\colon \mathcal{M}_{W,\mathbf{d}} \to \mathcal{M}_{W',\mathbf{d}}$ such that $\phi(A_1)$ is the identity; and*
2. $$\forall a \in \mathbb{Z}_{\leq d_1}, \quad \pi(a) \leq d_2 \iff \pi'(a) \leq d_2;$$
   and
3. *setting $f = \mathfrak{s}W$ and $f' = \mathfrak{s}W'$ we have*
   $$\forall a \in \mathbb{Z}_{\leq d_1}, \quad f(a, d_2) = f'(a, d_2). \tag{66}$$

Proof. The equivalence of (1) and (2) is just Lemma 8.1.

(2) ⇒ (3): for any $a \in \mathbb{Z}$ we have

$$f(a,d_2) - f(a-1,d_2) = \sum_{a_2\leq d_2} W(a,a_2) = \begin{cases} 1 & \text{if } \pi(a) \leq d_2, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{67}$$

and similarly

$$f'(a,d_2) - f'(a-1,d_2) = \begin{cases} 1 & \text{if } \pi'(a) \leq d_2, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{68}$$

Now $f(a, d_2) = f'(a, d_2) = 0$ for a sufficiently small, and hence (66) holds for $a \leq a'$ for some a'. Assuming (2), (67) and (68) imply that for all $a \leq d_1$ we have

$$f(a,d_2) - f(a-1,d_2) = f'(a,d_2) - f'(a-1,d_2)$$

and therefore

$$f(a,d_2) - f'(a,d_2) = f(a-1,d_2) - f'(a-1,d_2).$$

Hence, we can use induction on a from $a'+1$ to d_1 to infer that (66) holds for all $a \leq d_1$.

(3) ⇒ (2): (66) implies that for any $a \leq d_1$ we have

$$f(a,d_2) - f(a-1,d_2) = f'(a,d_2) - f'(a-1,d_2)$$

and hence (67) and (68) imply that for each $a \leq d_1$, $\pi(a) \leq d_2$ if and only if $\pi'(a) \leq d_2$. ☐

Proof of Theorem 8.4. The equation (63) implies condition (3) of Lemma 8.3 with $d_1 = d_2 = 0$. Hence we conclude condition (1) of Lemma 8.3 in this case, which is just the assertion of Theorem 8.4. ☐

We finish this subsection by showing how to generate non-trivial examples, of Lemma 8.3 and Theorem 8.4, and specify one such example explicitly; this may serve to illustrate how this lemma and this theorem work in practice.

Example 8.1. *Let f be as in Example 2.11. Then*

$$f_{1,2,\mathbf{0}}(a_1,0)=f_{1,3,\mathbf{0}}(a_1,0)$$

for all $a_1\in\mathbb{Z}$. Hence if W,W' are the respective weights of $f_{1,2,\mathbf{0}},f_{1,3,\mathbf{0}}$, then $W\neq W'$, and, in more detail, W,W' are both 4-periodic, and their associated bijections π,π' satisfy

$$\pi(0)=\pi'(0)=0,\ \pi(1)=\pi'(1)=1,$$

and

$$\pi(-1)=\pi'(-2)=2,\ \pi(-2)=\pi'(-1)=3.$$

Hence $\pi'\neq\pi$, but it is nonetheless true that

$$\forall a_1\in\mathbb{Z}_{\le 0},\qquad \pi(a_1)\le 0 \iff \pi'(a_1)\le 0$$

(which moreover holds for all $a_1\in\mathbb{Z}$). One can similarly generate examples of $f_{1,2,\mathbf{d}}$ and $f_{1,3,\mathbf{d}}$ for any $\mathbf{d}\in\mathbb{Z}^4$. One can also generate examples as in Example 2.9 with $n\ge 5$.

Example 8.2. *Let $G=K_n$ be the complete graph on n vertices, and $f=1+r_{\mathrm{BN}}$ the Riemann function associated to the Baker-Norine rank function on G. Then Folinsbee and Friedman [11] show that any two-variable restriction of f has non-negative weight. Hence one can generate further examples of $f_{1,2,\mathbf{d}},f_{1,3,\mathbf{d}}$ for various $\mathbf{d}\in\mathbb{Z}^n$.*

8.4 Generalization of Lemma 8.3

When $\mathcal{M}_{f;i,j,\mathbf{d}}$ and $\mathcal{M}_{f;i,j',\mathbf{d}}$ are virtual k-diagrams, we will need the following generalization of Lemma 8.3. (To prove Theorem 8.2, we need only that (4) implies (1) below.)

Lemma 8.4. *Let $W_1,\cdots,W_s$ and $W_1',\cdots,W_s'$ be two sequences of perfect matchings $\mathbb{Z}^2\to\mathbb{Z}$. Let $\mathbf{d}\in\mathbb{Z}^2$, and for each $a_1\in\mathbb{Z}$ and $j\in[s]$, let $\mathbf{e}_{a_i,j}$ and $\mathbf{e}'_{a_i,j}$ denote, respectively, the standard basis vector $\mathbf{e}_{a_1}\in k^{\mathbb{Z}}$ in, respectively $\mathcal{M}_{W_j,\mathbf{d}}(A_1)$ and $\mathcal{M}_{W_j',\mathbf{d}}(A_1)$. Let*

$$W=W_1+\cdots+W_s,\quad W'=W_1'+\cdots+W_s',$$

and

$$\mathcal{M}_{\mathbf{d}}=\mathcal{M}_{W_1,\mathbf{d}}\oplus\cdots\oplus\mathcal{M}_{W_s,\mathbf{d}},\quad \mathcal{M}'_{\mathbf{d}}=\mathcal{M}_{W_1',\mathbf{d}}\oplus\cdots\oplus\mathcal{M}_{W_s',\mathbf{d}}.$$

Then for any $\mathbf{d}\in\mathbb{Z}^2$, the following are equivalent:

1. *there exists an isomorphism $\phi\colon\mathcal{M}\to\mathcal{M}'$ that for each $a_1\in\mathbb{Z}_{\le d_1}$, $\phi(A_1)$ restricts to a bijection from $\mathbf{e}_{a_1,1},\ldots,\mathbf{e}_{a_1,s}$ to $\mathbf{e}'_{a_1,1},\ldots,\mathbf{e}'_{a_1,s}$ (and, moreover, in this case one can also take from the subset of $\mathbf{e}_{a_1,r}$ with $\pi_r(a_1)\le d_2$ to those $\mathbf{e}'_{a_1,r}$ with $\pi_r'(a_1)\le d_2$ where π_r,π_r' are the bijections associated to W_r,W_r');*

2. *letting π_r,π_r' for $r\in[s]$ denote the bijections associated to W_r,W_r',*

$$\forall a_1\in\mathbb{Z}_{\le d_1},\quad \bigl|\{r\mid\pi_r(a_1)\le d_2\}\bigr|=\bigl|\{r\mid\pi_r'(a_1)\le d_2\}\bigr|;$$

3. $$\forall a_1\in\mathbb{Z}_{\le d_1},\quad \sum_{a_2\le d_2}W(a_1,a_2)=\sum_{a_2\le d_2}W'(a_1,a_2);$$

4. *if $f=\mathfrak{s}W$ and $f'=\mathfrak{s}W'$, then*

$$\forall a_1\in\mathbb{Z}_{\le d_1},\quad f(a_1,d_2)=f'(a_1,d_2).$$

Moreover, if the above conditions hold, then all maps $\phi(A_1)$ are determined as those that for each $a_1\in\mathbb{Z}_{\le d_1}$ restricts to a bijection

$$\{\mathbf{e}_{a_1,r}\mid\pi_r(a_1)\le d_2\}\to\{\mathbf{e}'_{a_1,r}\mid\pi_r'(a_1)\le d_2\}$$

and to a bijection

$$\{\mathbf{e}_{a_1,r}\mid\pi_r(a_1)\ge d_2+1\}\to\{\mathbf{e}'_{a_1,r}\mid\pi_r'(a_1)\ge d_2+1\}.$$

Another way to think of this lemma is to recall that if $W_1,\ldots,W_s$ are perfect matchings, and $W = W_1 + \cdots + W_s$, then $\mathcal{M}_{W_1,\mathbf{d}} \oplus \cdots \oplus \mathcal{M}_{W_s,\mathbf{d}} \simeq \mathcal{I}^W_{\mathbf{d}}$. Hence this lemma shows that $\mathcal{I}^W_{\mathbf{d}} \simeq \mathcal{I}^{W'}_{\mathbf{d}}$.

Proof. (1) $\iff$ (2): this is Lemma 8.2.

(2) $\iff$ (3): this is by definition: if W is any perfect matching, then π is the unique bijection such that $W(a,\pi(a)) = 1$ for all $a \in \mathbb{Z}$. Hence, for any d_2 and a_1 we have

$$\sum_{a_2 \le d_2} W(a_1,a_2) = \bigl|\{r \mid \pi_r(a_1) \le d_2\}\bigr|.$$

Similarly

$$\sum_{a_2 \le d_2} W'(a_1,a_2) = \bigl|\{r \mid \pi'_r(a_1) \le d_2\}\bigr|.$$

So if for some a_1, the number of values of r such that $\pi_r(a_1) \le d_2$ is the number of r' with $\pi_{r'}(a_1) \le d_2$, then for this particular value of a_1,

$$\sum_{a_2 \le d_2} W(a_1,a_2) = \sum_{a_2 \le d_2} W'(a_1,a_2)$$

Now we apply this fact to all $a_1 \le d_1$.

(3) $\iff$ (4): similar to (67), we have for all $a \in \mathbb{Z}$,

$$f(a,d_2) - f(a-1,d_2) = \sum_{a_2 \le d_2} W(a,a_2) = \bigl|\{r \mid \pi_r(a) \le d_2\}\bigr|,$$

and similarly

$$f'(a,d_2) - f'(a-1,d_2) = \sum_{a_2 \le d_2} W'(a,a_2) = \bigl|\{r \mid \pi'_r(a) \le d_2\}\bigr|.$$

The claim after (1)–(4) about $\phi(A_1)$ follows from (65) (and the discussion below it). (This claim is not needed in what follows, but serves to illustrate the way that $\phi(A_1)$—and therefore all of ϕ—is constructed.) ☐

8.5 Proof of Theorem 8.2

To prove Theorem 8.2, we need only the part of Lemma 8.4 that asserts condition (4) there implies condition (1).

Proof of Theorem 8.2. Since $f_{i,j,\mathbf{d}}$ is a Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$, we can write

$$W = \mathfrak{m} f_{i,j,\mathbf{d}} = W_1 + \cdots + W_s - \tilde{W}_1 - \cdots - \tilde{W}_{s-1}$$

where the W_i and $\tilde{W}_i$ are perfect matchings, and we may similarly write

$$W' = \mathfrak{m} f_{i,j',\mathbf{d}} = W'_1 + \cdots + W'_{s'} - \tilde{W}'_1 - \cdots - \tilde{W}'_{s'-1}$$

To show that $\mathcal{M}_{W,\mathbf{0}} \simeq \mathcal{M}_{W',\mathbf{0}}$ it suffices to show that

$$\left(\bigoplus_{i=1}^{s} \mathcal{M}_{W_i,\mathbf{0}}\right) \oplus \left(\bigoplus_{i=1}^{s'-1} \mathcal{M}_{\tilde{W}'_i,\mathbf{0}}\right) \simeq \left(\bigoplus_{i=1}^{s-1} \mathcal{M}_{\tilde{W}_i,\mathbf{0}}\right) \oplus \left(\bigoplus_{i=1}^{s'} \mathcal{M}_{W'_i,\mathbf{0}}\right) \tag{69}$$

Let

$$f_1 = \mathfrak{s}(W_1 + \cdots + W_s), \quad f_2 = \mathfrak{s}(\tilde{W}_1 + \cdots + \tilde{W}_{s-1})$$

so that $f_{i,j,\mathbf{d}} = f_1 - f_2$ and similarly

$$f'_1 = \mathfrak{s}(W'_1 + \cdots + W'_{s'}), \quad f'_2 = \mathfrak{s}(\tilde{W}'_1 + \cdots + \tilde{W}'_{s'-1})$$

and so $f_{i,j',\mathbf{d}} = f'_1 - f'_2$. By definition

$$f_{i,j,\mathbf{d}}(a_1,0) = f(\mathbf{d} + \mathbf{e}_i a_1) = f_{i,j',\mathbf{d}}(a_1,0)$$

for all $a_1 \in \mathbb{Z}$. It follows for all $a_1 \in \mathbb{Z}$ we have

$$(f_1 - f_2)(a_1,0) = (f'_1 - f'_2)(a_1,0),$$

and therefore

$$\forall a_1 \in \mathbb{Z}, \quad (f_1 + f'_2)(a_1,0) = (f_2 + f'_1)(a_1,0).$$

Hence applying Lemma 8.4 with f, f' there replaced with $f_1 + f'_2$ and $f_2 + f'_1$ respectively (and $W_1,\ldots,W_s$ there replaced with $W_1,\ldots,W_s,\tilde{W}'_1,\ldots,\tilde{W}'_{s'-1}$ here, and $W'_1,\ldots,W'_s$ there with $\tilde{W}_1,\ldots,\tilde{W}_{s-1},W'_1,\ldots,W'_{s'}$ here), we have that condition (4) of this lemma holds, and therefore condition (1) holds. Therefore (69) holds, and therefore

$$\mathcal{M}_{W,\mathbf{0}} \simeq \mathcal{M}_{W',\mathbf{0}}$$

as virtual k-diagrams. ☐

9 The First Duality Theorems

In this section, we show that the k-diagram $\underline{k}_{/B_1,B_2}$ is a "dualizing" k-diagram, in that for any k-diagram $\mathcal{F}$ we have that there is an isomorphism

$$H^1(\mathcal{F})^* \to \mathrm{Hom}(\mathcal{F}, \underline{k}_{/B_1,B_2})$$

which is "natural" or "functorial" in $\mathcal{F}$.

We then prove that any for perfect matching, W, and any $\mathbf{K}, \mathbf{L} \in \mathbb{Z}^2$ with $\mathbf{L} = \mathbf{K} + \mathbf{1}$, for any $\mathbf{d}$ there is a isomorphism

$$H^1(\mathcal{M}_{W,\mathbf{d}})^* \to H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}).$$

Since these are finite dimensional vector spaces, by replacing W and $\mathbf{d}$ with, respectively, $W^*_{\mathbf{L}}$ and $\mathbf{K} - \mathbf{d}$, we moreover get isomorphisms

$$H^i(\mathcal{M}_{W,\mathbf{d}})^* \to H^{1-i}(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \quad \text{for } i = 0, 1.$$

We use this to infer that if W is the weight of any Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$, there are equivalences of virtual k-vector spaces

$$H^i\big([\mathcal{M}_{W,\mathbf{d}}]\big)^* \;\sim\; H^{1-i}\big([\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}]\big) \quad \text{for } i = 0, 1,$$

where $[\mathcal{M}_{W,\mathbf{d}}]$ is the equivalence class of virtual k-diagrams, and the dual of a virtual k-vector space is appropriately defined. We use this to infer that for any Riemann function $f\colon \mathbb{Z}^n \to \mathbb{Z}$ and any $\mathbf{d}, \mathbf{K} \in \mathbb{Z}^n$ there are equivalences

$$H^i\big([\mathcal{M}_{f \text{ at } \mathbf{d}}]\big)^* \;\sim\; H^{1-i}\big([\mathcal{M}_{f^\wedge_{\mathbf{K}} \text{ at } \mathbf{K}-\mathbf{d}}]\big) \quad \text{for } i = 0, 1. \tag{70}$$

9.1 Representing $H^1(\mathcal{F})^*$

Theorem 9.1. *For any k-diagram $\mathcal{F}$ there is a* natural *isomorphism*

$$H^1(\mathcal{F})^* \to \mathrm{Hom}(\mathcal{F}, \underline{k}_{/B_1,B_2}),$$

where "natural" means "functorial" in the sense that if $\mu\colon \mathcal{F} \to \mathcal{G}$ is any morphism, then the natural map $H^1(\mathcal{G})^* \to H^1(\mathcal{F})^*$ obtained by dualizing the map μ induces from $H^1(\mathcal{F}) \to H^1(\mathcal{G})$ is the same as the map

$$\mathrm{Hom}(\mathcal{G}, \underline{k}_{/B_1,B_2}) \to \mathrm{Hom}(\mathcal{F}, \underline{k}_{/B_1,B_2}).$$

In modern parlance, the functor $\mathcal{F} \to H^1(\mathcal{F})^*$ is represented by the k-diagram $\underline{k}_{/B_1,B_2}$. In Section 10, we will give a conceptually simple proof of this theorem using standard techniques from homological algebra. Here we content ourselves with proving this theorem directly, which is straightforward, although a bit tedious.

For the proof below, note that if $\mathcal{L}\colon U \to V$ is any linear map of (possibly infinite-dimensional) k-vector spaces, then if V^* denotes the k-dual space of V, i.e., the vector space of maps $V \to k$, then the usual dual map $\mathcal{L}^*\colon V^* \to U^*$ is given by

$$\forall w\colon V \to k, \quad \mathcal{L}^*(w) = w \circ \mathcal{L}; \tag{71}$$

we claim that there is an isomorphism

$$\big(\mathrm{coker}(\mathcal{L})\big)^* \to \ker\big(\mathcal{L}^*\big) \tag{72}$$

constructed as follows: any $w \in \big(\mathrm{coker}(\mathcal{L})\big)^*$ is a map w from $\mathrm{coker}(\mathcal{L}) = (V/\mathrm{Image}(\mathcal{L}))$ to k, and so the quotient map $V \to V/\mathrm{Image}(\mathcal{L})$ followed by w gives a map $\tilde{w}\colon V \to k$, which takes $\mathrm{Image}(\mathcal{L})$ to 0. Hence $\tilde{w} \in V^*$ and satisfies $\mathcal{L}^*\tilde{w} = \tilde{w} \circ \mathcal{L} = 0$; hence $\tilde{w} \in \ker(\mathcal{L}^*)$. Conversely, any $v \in \ker(\mathcal{L}^*)$ is a map $V \to k$ such that the composition

$$U \xrightarrow{\mathcal{L}} V \xrightarrow{v} k$$

is the zero map, i.e., v takes $\mathrm{Image}(\mathcal{L})$ to 0; hence v determines a map $\tilde{v}\colon V/\mathrm{Image}(\mathcal{L}) \to k$, and hence a map $\mathrm{coker}(\mathcal{L}) \to k$. Hence $\tilde{v} \in (\mathrm{coker}(\mathcal{L}))^*$. We easily check that the maps $w \mapsto \tilde{w}$ and $v \mapsto \tilde{v}$ are inverses.

We warn the reader that in Subsection 10.6 we will see that if $\mathcal{L}\colon U \to V$ is any linear map, there is a natural map

$$\mathrm{coker}(\mathcal{L}^*) \to \big(\ker(\mathcal{L})\big)^*,$$

however, in contrast to (72), (to the best of our knowledge) one needs to assume the axiom of choice (or Zorn's lemma) to ensure that this is an isomorphism. Hence it is remarkable that (72) is an isomorphism without assuming the axiom of choice.

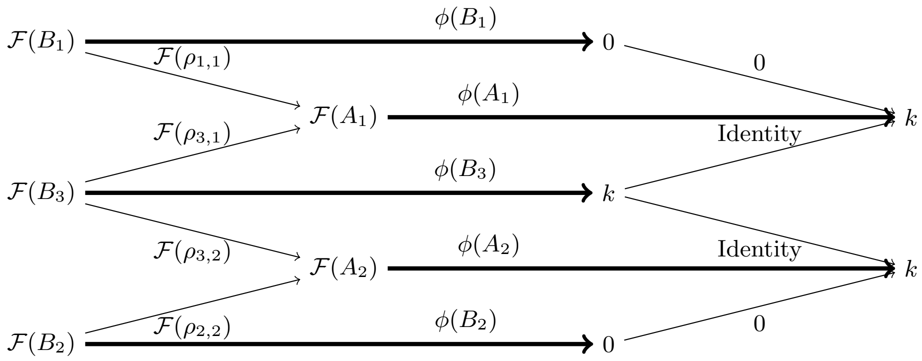


Figure 10: A morphism $\phi\colon \mathcal{F} \to \underline{k}_{/B_1,B_2}$

Proof of Theorem 9.1. Consider a morphism $\phi\colon \mathcal{F} \to \underline{k}_{/B_1,B_2}$; see Figure 10. Hence we have

$$\bigl(\phi(A_1), -\phi(A_2)\bigr) \in \bigl(\mathcal{F}(A_1)\bigr)^* \oplus \bigl(\mathcal{F}(A_2)\bigr)^*$$

(when we view $\phi(A_i)\colon \mathcal{F}(A_i) \to k$ as an element of the dual space of $\mathcal{F}(A_i)$); let us prove that, moreover,

$$\bigl(\phi(A_1), -\phi(A_2)\bigr) \in \ker\Bigl(\bigl(\mathcal{F}(\partial)\bigr)^*\Bigr).$$

To do so, in view of (27), we have $(\mathcal{F}(\partial))^*$ is the map

$$\bigl(\mathcal{F}(\partial)\bigr)^*\colon \bigl(\mathcal{F}(A_1)\bigr)^* \oplus \bigl(\mathcal{F}(A_2)\bigr)^* \to \bigl(\mathcal{F}(B_1)\bigr)^* \oplus \bigl(\mathcal{F}(B_2)\bigr)^* \oplus \bigl(\mathcal{F}(B_3)\bigr)^*$$

given as the map taking (w_1, w_2) with $w_i \in (\mathcal{F}(A_i))^*$ as follows:

$$(w_1, w_2) \mapsto \Bigl(\ \bigl(\mathcal{F}(\rho_{1,1})\bigr)^*(w_1)\ ,\ \bigl(\mathcal{F}(\rho_{2,2})\bigr)^*(w_2)\ ,\ -\bigl(\mathcal{F}(\rho_{3,1})\bigr)^*(w_1) - \bigl(\mathcal{F}(\rho_{3,2})\bigr)^*(w_2)\ \Bigr),$$

which, in view of (71) we may write more simply as the map

$$(w_1, w_2) \mapsto \Bigl(\ w_1\mathcal{F}(\rho_{1,1})\ ,\ w_2\mathcal{F}(\rho_{2,2})\ ,\ -w_1\mathcal{F}(\rho_{3,1}) - w_2\mathcal{F}(\rho_{3,2})\ \Bigr) \tag{73}$$

where, for brevity, we have omitted the composition symbol $\circ$.

So set $w_1 = \phi(A_1)$ and $w_2 = -\phi(A_2)$. Since $\underline{k}_{/B_1,B_2}$ has value 0 at B_1, we have $\phi(A_1)\mathcal{F}(\rho_{1,1})$ must be the zero map. Arguing similarly for B_2, we have

$$\begin{aligned} w_1\mathcal{F}(\rho_{1,1}) &= 0, \\ -w_2\mathcal{F}(\rho_{2,2}) &= 0. \end{aligned} \tag{74}$$

Similarly, since for $i = 1, 2$ the map from the B_3 value of $\underline{k}_{/B_1,B_2}$ to the A_i is the identity map (on k), we have

$$\phi(B_3) = \phi(A_i)\mathcal{F}(\phi_{3,i}),$$

and hence

$$w_1\mathcal{F}(\phi_{3,1}) = -w_2\mathcal{F}(\phi_{3,2}) = \phi(B_3),$$

and hence

$$w_1\mathcal{F}(\phi_{3,1}) + w_2\mathcal{F}(\phi_{3,2}) = 0. \tag{75}$$

In view of (74)–(75) we have $(w_1, w_2) = (\phi(A_1), -\phi(A_2))$ is taken to $(0, 0, 0)$ under the map (73).

Next we claim that, conversely, if $(w_1, w_2) \in \ker(((\mathcal{F}(\partial))^*)$, then there exists some $\phi \in \mathrm{Hom}(\mathcal{F}, \underline{k}_{/B_1,B_2})$ such that $\phi(A_1) = w_1$ and $\phi(A_2) = -w_2$. Namely, this determines ϕ at A_1, A_2; this forces the values of ϕ at the B_i, namely we set $\phi(B_i) = 0$ for $i = 1, 2$ and we set

$$\phi(B_3) = w_1\mathcal{F}(\rho_{3,1}) = \phi(A_1)\mathcal{F}(\rho_{3,1})$$

(so that ϕ intertwines with the restrictions from B_3 to A_1 of $\mathcal{F}$ and $\underline{k}_{/B_1,B_2}$). Now we verify that ϕ is actually a morphism: for example, since $(w_1, w_2) \in \ker(((\mathcal{F}(\partial))^*)$, in view of (73) we have

$$w_1\mathcal{F}(\rho_{3,1}) + w_2\mathcal{F}(\rho_{3,2}) = 0$$

and it follows that

$$\phi(B_3) = -w_2\mathcal{F}(\rho_{3,2}) = \phi(A_2)\mathcal{F}(\rho_{3,2}),$$

and therefore ϕ intertwines with the restrictions from B_3 to A_2. Similarly ϕ intertwines with the restrictions from B_i to A_i for $i = 1, 2$, i.e., $\phi(A_i)\mathcal{F}(\rho_{i,i}) = 0$, in view of (73).

It follows that the map

$$\phi \mapsto \big(\phi(A_1), -\phi(A_2)\big)$$

is gives an isomorphism (80).

To check the desired functoriality, say that $\mu\colon \mathcal{F} \to \mathcal{G}$ is any morphism. The map (80) with $\mathcal{G}$ replacing $\mathcal{F}$ is given by associating to $\phi \in \mathrm{Hom}(\mathcal{G}, k_{/B_1,B_2})$ the element

$$\big(\phi(A_1), -\phi(A_2)\big) \in \ker\Big(\big(\mathcal{G}(\partial)\big)^*\Big). \tag{76}$$

Then to ϕ we associate the element

$$\phi \circ \mu \in \mathrm{Hom}(\mathcal{F}, k_{/B_1,B_2}),$$

and therefore to $\phi \circ \mu$ we associate the element

$$\big(\phi \circ \mu(A_1), -\phi \circ \mu(A_2)\big) \in \ker\Big(\big(\mathcal{F}(\partial)\big)^*\Big). \tag{77}$$

But to an element of $H^1(\mathcal{G})$, namely an element (76), the action of μ taking $H^1(\mathcal{G})$ to $H^1(\mathcal{F})$ is precisely μ applied to each element, which again gives (77). Hence the isomorphism (80) is functorial (or natural). $\square$

9.2 A Duality Theorem for Perfect Matchings

Theorem 9.2. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ be a perfect matching. Then for any $\mathbf{K} \in \mathbb{Z}^2$ and $\mathbf{L} = \mathbf{K} + \mathbf{1}$, for any $\mathbf{d}$ there is an isomorphism*

$$\mathrm{Hom}\big(\mathcal{M}_{W,\mathbf{d}}, \underline{k}_{/B_1,B_2}\big) \to \mathrm{Hom}\big(\underline{k}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}\big) \tag{78}$$

induced by decomposing the above diagrams into indicator diagrams and taking the isomorphism

$$\mathrm{Hom}(\underline{k}_{/B_1,B_2}, \underline{k}_{/B_1,B_2}) \to \mathrm{Hom}(\underline{k}, \underline{k}) \tag{79}$$

which takes the identity morphism of $k_{/B_1,B_2}$ to the identity morphism of $\underline{k}$ (both Hom *sets are isomorphic to k, in view of Example 5.3). Moreover,* (79) *gives us an isomorphism*

$$H^1(\mathcal{M}_{W,\mathbf{d}})^* \to \mathrm{Hom}(\underline{k}, \mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}), \tag{80}$$

which gives us isomorphisms

$$H^i(\mathcal{M}_{W,\mathbf{d}})^* \to H^{1-i}(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \quad \text{for } i = 0, 1. \tag{81}$$

Before giving the proof, let us make a remark regarding this theorem.

We remark that if $f = \mathfrak{s}W$ in the above theorem, then Theorem 4.2 and (13) imply that

$$b^1(\mathcal{M}_{W,\mathbf{d}}) = f^\wedge_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = b^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}});$$

applying this equation with $W, \mathbf{d}$ respectively replaced with $W^*_{\mathbf{L}}$ and $\mathbf{K} - \mathbf{d}$, we therefore get

$$b^i(\mathcal{M}_{W,\mathbf{d}}) = b^{1-i}(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}) \quad \text{for } i = 0, 1. \tag{82}$$

Hence (81) strengthens this formula, by giving the isomorphism of vector spaces (81) that upon taking dimensions implies (82).

Proof. We remark that for any $\mathbf{a}, \mathbf{d} \in \mathbb{Z}^2$, $\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}$ is one of our four basic diagrams, and equals $\underline{k}_{/B_1,B_2}$ if and only if $\mathbf{a} \geq \mathbf{d} + \mathbf{1}$ (recall Definition 6.1); hence

$$\mathrm{Hom}(\mathcal{I}_{\mathbf{d}\geq\mathbf{a}}, \underline{k}_{/B_1,B_2})$$

is 0 unless $\mathbf{a} \geq \mathbf{d} + \mathbf{1}$, in which case it equals

$$\mathrm{Hom}(\underline{k}_{/B_1,B_2}, \underline{k}_{/B_1,B_2}).$$

(recall Example 5.2). Since

$$\mathcal{M}_{W,\mathbf{d}} = \bigoplus_{W(\mathbf{a})=1} \mathcal{I}_{\mathbf{d}\geq\mathbf{a}},$$

we have

$$\mathrm{Hom}(\mathcal{M}_{W,\mathbf{d}},\underline{k}_{/B_1,B_2})\simeq\prod_{W(\mathbf{a})=1}\mathrm{Hom}(\mathcal{I}_{\mathbf{d}\geq\mathbf{a}},\underline{k}_{/B_1,B_2})=\bigoplus_{W(\mathbf{a})=1,\ \mathbf{a}\geq\mathbf{d}+\mathbf{1}}\mathrm{Hom}(\underline{k}_{/B_1,B_2},\underline{k}_{/B_1,B_2}).$$

On the other hand, we similarly have

$$\mathrm{Hom}(\underline{k},\mathcal{I}_{\mathbf{d}\geq\mathbf{a}})$$

is 0 unless $\mathbf{a}\leq\mathbf{d}$, in which case it equals $\mathrm{Hom}(\underline{k},\underline{k})$. Hence

$$\begin{aligned}\mathrm{Hom}(\underline{k},\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})&=\bigoplus_{W^*_{\mathbf{L}}(\mathbf{a})=1,\ \mathbf{a}\leq\mathbf{K}-\mathbf{d}}\mathrm{Hom}(\underline{k},\underline{k})\\&=\bigoplus_{W(\mathbf{L}-\mathbf{a})=1,\ \mathbf{a}\leq\mathbf{K}-\mathbf{d}}\mathrm{Hom}(\underline{k},\underline{k})\end{aligned}$$

which, upon substituting $\mathbf{a}'=\mathbf{L}-\mathbf{a}$ in the sum,

$$=\bigoplus_{W(\mathbf{a}')=1,\ \mathbf{L}-\mathbf{a}'\leq\mathbf{K}-\mathbf{d}}\mathrm{Hom}(\underline{k},\underline{k})=\bigoplus_{W(\mathbf{a}')=1,\ \mathbf{a}'\geq\mathbf{1}+\mathbf{d}}\mathrm{Hom}(\underline{k},\underline{k}).$$

Hence in (78), the left-hand-side has one copy of $\mathrm{Hom}(\underline{k}_{/B_1,B_2},\underline{k}_{/B_1,B_2})$ for each $\mathbf{a}$ with $W(\mathbf{a})=1$ and $\mathbf{a}\geq\mathbf{d}+\mathbf{1}$, and the right-hand-side one copy of $\mathrm{Hom}(\underline{k},\underline{k})$ for each such $\mathbf{a}$. Hence, we have an isomorphism (78).

Composing the isomorphism in Theorem 9.1 with (78) gives us an isomorphism (80), which also proves the $i=1$ case of (81). Since these are finite-dimensional vector spaces, this also gives an isomorphism

$$H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})^*\to H^1(\mathcal{M}_{W,\mathbf{d}}).$$

If we apply this isomorphism when we replace all occurrences of $W,\mathbf{d}$ with, respectively, $W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}$ (so that $W^*_{\mathbf{L}}$ is therefore replaced with W, and $\mathbf{K}-\mathbf{d}$ with $\mathbf{d}$), we have an isomorphism

$$H^0(\mathcal{M}_{W,\mathbf{d}})^*\to H^1(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}).$$

This proves the $i=0$ case of (81). Hence, both the $i=0$ and $i=1$ cases of (81) hold. □

9.3 The Dual of a Virtual k-Vector Space

In this subsection, we define the dual of a virtual k-vector space and make some brief remarks about this definition. In Appendix A we discuss some more foundational aspects about virtual vector spaces that partially justify our definition.

In Theorem 9.2, we have an isomorphism from the dual of $H^1(\mathcal{M}_{W,\mathbf{d}})$ to $H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})$. In our more general duality theorems (namely Theorem 9.3 and Theorem 9.4 below), $H^1(\mathcal{M}_{W,\mathbf{d}})$ and will be a virtual k-vector space (V_1,V_2), $H^0(\mathcal{M}_{W,\mathbf{d}})$ a virtual vector space (V_3,V_4); Theorem 9.2 then provides an isomorphism $V_1^*\to V_3$ and $V_2^*\to V_4$. For this reason, we will be pretty much forced to make the following definition.

Definition 9.1. *Let (V_1,V_2) be a virtual k-vector space (which, by convention, means V_1,V_2 are finite-dimensional). We define the* dual *of (V_1,V_2) to be the virtual k-vector space (V_1^*,V_2^*).*

Proposition 9.1. *Let $(V_1,V_2)\sim(V_3,V_4)$ be equivalent (finite dimensional) virtual k-vector spaces. Then any equivalence, i.e., any isomorphism*

$$\mu\colon V_0\oplus V_1\oplus V_4\to V_0\oplus V_2\oplus V_3$$

for some V_0, gives rise to an equivalence $(V_1,V_2)^*\sim(V_3,V_4)^*$.

Proof. Since μ is an isomorphism, the dual map of μ,

$$\mu^*\colon(V_0\oplus V_2\oplus V_3)^*\to(V_0\oplus V_1\oplus V_4)^*$$

yields an isomorphism

$$V_0^*\oplus V_2^*\oplus V_3^*\to V_0^*\oplus V_1^*\oplus V_4^*;$$

since all the V_i are finite-dimensional, so are all the V_i^*. □

We also remark that in Theorem 9.3 one could avoid references to the dual space of a virtual vector space provided that one has a good notion of "pairing" and "perfect pairing" of virtual vector spaces. See Appendix A for further remarks about pairings, as well as other remarks that motivate Definition 9.1 as possibly (i.e., in future work) fitting into a broader notion of "morphisms" of virtual vector spaces, rather than just being a definition of necessity.

9.4 Duality for Weights of Riemann Functions $\mathbb{Z}^2 \to \mathbb{Z}$

Theorem 9.3. *Let W be the weight of an arbitrary Riemann function $\mathbb{Z}^2 \to \mathbb{Z}$. Then for any $\mathbf{d}, \mathbf{K}, \mathbf{L} \in \mathbb{Z}^2$ there are equivalences of virtual k-vector spaces*

$$H^i([\mathcal{M}_{W,\mathbf{d}}])^* \ \sim\ H^{1-i}([\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}]) \quad \text{for } i = 0, 1. \tag{83}$$

The proof of this theorem is mostly a matter of unwinding the various definitions and applying Theorem 9.2.

Remark 9.1. *There is a chance that one could view the above theorem, specifically* (83)*, as giving isomorphisms*

$$H^i([\mathcal{M}_{W,\mathbf{d}}])^* \to H^{1-i}([\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}]) \quad \text{for } i = 0, 1,$$

provided that we are willing to "categorize" our virtual k-vector spaces, by "localizing," i.e., formally inverting all equivalence maps. This might turn our virtual k-vector spaces as a union of disjoint groupoids, one groupoid for each dimension (i.e., each element of $\mathbb{Z}$). However, this hardly seems worth it here; but see more comments in Appendix A.

Proof of Theorem 9.3. Write W as a difference of sums of perfect matchings

$$W = (W_1 + \cdots + W_s) - (\tilde{W}_1 + \cdots + \tilde{W}_{s-1}), \tag{84}$$

whereupon $H^i([\mathcal{M}_{W,\mathbf{d}}])$ refers to the kernel and cokernel of the virtual Fredholm map

$$\big(\mathcal{M}_{W_1,\mathbf{d}}(\partial) \oplus \cdots \oplus \mathcal{M}_{W_s,\mathbf{d}}(\partial)\big) \ominus \big(\mathcal{M}_{\tilde{W}_1,\mathbf{d}}(\partial) \oplus \cdots \oplus \mathcal{M}_{\tilde{W}_{s-1},\mathbf{d}}(\partial)\big).$$

In view of (84) we have

$$W^*_{\mathbf{L}} = \big((W_1)^*_{\mathbf{L}} + \cdots + (W_s)^*_{\mathbf{L}}\big) - \big((\tilde{W}_1)^*_{\mathbf{L}} + \cdots + (\tilde{W}_{s-1})^*_{\mathbf{L}}\big),$$

and for any $\mathbf{d} \in \mathbb{Z}^2$ we have that $H^i(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})$ refers to the equivalence class of the kernel and cokernel of

$$\big(\mathcal{M}_{(W_1)^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}(\partial) \oplus \cdots \oplus \mathcal{M}_{(W_s)^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}(\partial)\big) \ominus \big(\mathcal{M}_{(\tilde{W}_1),\mathbf{K}-\mathbf{d}}(\partial) \oplus \cdots \oplus \mathcal{M}_{(\tilde{W}_{s-1})^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}(\partial)\big).$$

In view of (81) we have for each $\mathbf{d} \in \mathbb{Z}^2$,

$$H^1(\mathcal{M}_{W_i,\mathbf{d}})^* \to H^0(\mathcal{M}_{(W_i)^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}})$$

for all i, i.e., an isomorphism

$$\Big(\operatorname{coker}\big(\mathcal{M}_{W_i,\mathbf{d}}(\partial)\big)\Big)^* \to \ker\big(\mathcal{M}_{(W_i)^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}(\partial)\big),$$

and similarly with $\widetilde{W_i}$ replacing W_i. Hence, in view of Definition 9.1, we get an equivalence of virtual vector spaces

$$H^1(\mathcal{M}_{W,\mathbf{d}})^* \ \sim\ H^0(\mathcal{M}_{W^*_{\mathbf{L}},\mathbf{K}-\mathbf{d}}).$$

This proves the $i = 1$ case of (83). Replacing $W, \mathbf{d}$ everywhere in the above equation with $W^*_{\mathbf{L}}, \mathbf{K} - \mathbf{d}$ and taking duals gives the $i = 0$ case of (83). Hence, both the $i = 0$ and $i = 1$ cases of (83) hold. ☐

9.5 Duality for All Riemann Functions

In this subsection, we establish (70) for an arbitrary Riemann function $f \colon \mathbb{Z}^n \to \mathbb{Z}$. The first lemma below is the essential insight; roughly speaking, it says that the two-variable restriction of a generalized Riemann-Roch formula yields a generalized Riemann-Roch formula for the two-variable restriction. In other words, if $f \colon \mathbb{Z}^n \to \mathbb{Z}$ is a Riemann function, then for all $\mathbf{d}, \mathbf{K} \in \mathbb{Z}^n$ we have

$$f(\mathbf{d}) - f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = \deg(\mathbf{d}) + C_f \tag{85}$$

where C_f is the offset of f. Now consider $\mathbf{K}$ and $d_3, \ldots, d_n$ to be fixed, and (d_1, d_2) varying over all of $\mathbb{Z}^2$; then

$$g(d_1, d_2) = f(\mathbf{d})$$

is a Riemann function of two variables d_1, d_2, and g therefore satisfies

$$g(d_1, d_2) - g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2) = d_1 + d_2 + C_g \tag{86}$$

where C_g is the offset of g. The first lemma shows (easily) that the right-hand-sides of (86) and (85) are equal, and therefore the negative term of the left-hand-sides of (86) and (85) are equal, i.e.,

$$f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2)$$

where d_1, d_2 are varying. The first lemma expresses this equality of functions in a way that is useful to establish (70), namely as (89) below.

Lemma 9.1. *Let $n \geq 2$, $f \colon \mathbb{Z}^n \to \mathbb{Z}$ be a Riemann function with offset C_f, let $\mathbf{d}, \mathbf{K} \in \mathbb{Z}^n$, and let $(\tilde{K}_1, \tilde{K}_2) \in \mathbb{Z}^2$. Set*

$$\mathbf{d}' = \mathbf{d} - d_1\mathbf{e}_1 - d_2\mathbf{e}_2 = (0, 0, d_3, \ldots, d_n),$$

and let $g = f_{1,2,\mathbf{d}'}$. Then

1. *for all $a_1, a_2 \in \mathbb{Z}$,*
$$g(a_1, a_2) = f_{1,2,\mathbf{d}'}(a_1, a_2) = f(a_1, a_2, d_3, \ldots, d_n);$$
2. *the offset of $g = f_{1,2,\mathbf{d}'}$ is*
$$C_g = d_3 + \cdots + d_n + C_f;$$
3. $$f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = g^{\wedge}_{\tilde{K}_1, \tilde{K}_2}(\tilde{K}_1 - d_1, \tilde{K}_2 - d_2); \tag{87}$$
and
4. *setting $\mathbf{K}' = (0, 0, K_3, \ldots, K_n)$, we have*
$$\forall a_1, a_2 \in \mathbb{Z}, \quad (f^{\wedge}_{\mathbf{K}})_{1,2,\mathbf{K}'-\mathbf{d}'}(a_1, a_2) = g^{\wedge}_{(K_1,K_2)}(a_1, a_2) = (f_{1,2,\mathbf{d}'})^{\wedge}_{(K_1,K_2)}(a_1, a_2). \tag{88}$$

Moreover, for any $\mathbf{d}, \mathbf{K}$, and $\mathbf{d}', \mathbf{K}'$ as above, we have an equality of functions

$$(f^{\wedge}_{\mathbf{K}})_{1,2,\mathbf{K}'-\mathbf{d}'} = (f_{1,2,\mathbf{d}'})^{\wedge}_{(K_1,K_2)}. \tag{89}$$

Proof. (1) is immediate. (2) follows from the fact that with $d_3, \ldots, d_n$ fixed and $a_1 + a_2$ sufficiently large we have

$$f(a_1, a_2, d_3, \ldots, d_n) = a_1 + a_2 + d_3 + \cdots + d_n + C_f,$$

which by (1) equals

$$g(a_1, a_2) = a_1 + a_2 + C_g$$

for $a_1 + a_2$ sufficiently large. Hence $C_g = d_3 + \cdots + d_n + C_f$.

To prove (3), in view of (12) we have

$$f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = f(\mathbf{d}) - \deg(\mathbf{d}) - C_f, \tag{90}$$

and similarly

$$g^{\wedge}_{(\tilde{K}_1,\tilde{K}_2)}(\tilde{K}_1 - d_1, \tilde{K}_2 - d_2) = g(d_1, d_2) - (d_1 + d_2) - C_g$$

which, in view of (1) and (2),

$$= f(\mathbf{d}) - (d_1 + d_2) - (d_3 + \cdots + d_n + C_f) = f(\mathbf{d}) - \deg(\mathbf{d}) - C_f$$

which is just the right-hand-side of (90); this yields (87).

To prove (4), taking $\tilde{K}_i = K_i$ for $i = 1, 2$, (87) gives

$$f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2).$$

Hence we have

$$(f^{\wedge}_{\mathbf{K}})_{1,2,\mathbf{K}'-\mathbf{d}'}(K_1 - d_1, K_2 - d_2) = f^{\wedge}_{\mathbf{K}}(\mathbf{K} - \mathbf{d}) = g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2),$$

and therefore

$$(f^{\wedge}_{\mathbf{K}})_{1,2,\mathbf{K}'-\mathbf{d}'}(K_1 - d_1, K_2 - d_2) = g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2) = (f_{1,2,\mathbf{d}'})^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2).$$

But the functions

$$(f^{\wedge}_{\mathbf{K}})_{1,2,\mathbf{K}'-\mathbf{d}'}, \quad g = f_{1,2,\mathbf{d}'}, \quad \text{and} \quad g^{\wedge}_{(K_1,K_2)} = (f_{1,2,\mathbf{d}'})^{\wedge}_{(K_1,K_2)}$$

are, as functions $\mathbb{Z}^2 \to \mathbb{Z}$, independent of d_1, d_2, since $\mathbf{d}'$ discards the components d_1, d_2. Hence, for fixed $f, \mathbf{K}$ and $d_3, \ldots, d_n$, we have

$$\forall d_1, d_2 \in \mathbb{Z}, \quad (f^{\wedge}_{\mathbf{K}})_{1,2,\mathbf{K}'-\mathbf{d}'}(K_1 - d_1, K_2 - d_2) = g^{\wedge}_{(K_1,K_2)}(K_1 - d_1, K_2 - d_2),$$

which upon setting $a_i = K_i - d_i$ for $i = 1, 2$ proves (88).

Finally (89) follows from (4) and the value of g. ☐

The next lemma is straightforward but needs to be stated formally.

Lemma 9.2. *Let $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ be the weight of a Riemann function, $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, and for some $\mathbf{t} \in \mathbb{Z}^2$ let g be the translation of f by $\mathbf{t}$, i.e., given by*

$$g(\mathbf{d}) = f(\mathbf{d} + \mathbf{t}).$$

Then $W' = \mathfrak{m}g$ satisfies

$$\forall \mathbf{d} \in \mathbb{Z}^2, \quad [\mathcal{M}_{W',\mathbf{d}}] = [\mathcal{M}_{W,\mathbf{d}+\mathbf{t}}].$$

Proof. We leave the proof as an exercise[8]. □

Theorem 9.4. *Let $f\colon \mathbb{Z}^n \to \mathbb{Z}$ be an arbitrary Riemann function. Then for any $\mathbf{d}, \mathbf{K} \in \mathbb{Z}^n$ there are equivalences*

$$H^i\big([\mathcal{M}_{f \text{ at } \mathbf{d}}]\big)^* \;\sim\; H^{1-i}\big([\mathcal{M}_{f^\wedge_{\mathbf{K}} \text{ at } \mathbf{K}-\mathbf{d}}]\big) \quad \text{for } i = 0, 1, \tag{91}$$

which specifically arises by setting $\mathbf{d}' = (0, 0, d_3, d_4, \ldots, d_n)$ taking $W = \mathfrak{m}f_{1,2,\mathbf{d}'}$, and applying (83).

Proof. By Definition 8.2 we have

$$[\mathcal{M}_{f \text{ at } \mathbf{d}}] = [\mathcal{M}_{\mathfrak{m}f_{1,2,\mathbf{d}},(0,0)}].$$

But for all $\mathbf{a} \in \mathbb{Z}^2$ we have

$$f_{1,2,\mathbf{d}}(\mathbf{a}) = f_{1,2,\mathbf{d}'}\big((d_1, d_2) + \mathbf{a}\big)$$

since both sides equal $f(\mathbf{d} + a_1\mathbf{e}_1 + a_2\mathbf{e}_2)$. Hence $f_{1,2,\mathbf{d}}$ is a translation of $f_{1,2,\mathbf{d}'}$ by (d_1, d_2), and Lemma 9.2 implies that

$$[\mathcal{M}_{\mathfrak{m}f_{1,2,\mathbf{d}},(0,0)}] = [\mathcal{M}_{W,(d_1,d_2)}].$$

In view of (83) we have an isomorphism

$$H^i\big([\mathcal{M}_{f \text{ at } \mathbf{d}}]\big)^* \to H^{1-i}\big([\mathcal{M}_{W^*_{K_1+1,K_2+1},(K_1-d_1,K_2-d_2)}]\big). \tag{92}$$

Since $W = \mathfrak{m}g$ where $g = f_{1,2,\mathbf{d}'}$, (89) implies that

$$W^*_{K_1+1,K_2+1} = \mathfrak{m}\big(g^\wedge_{(K_1,K_2)}\big) = \mathfrak{m}\big((f^\wedge_K)_{1,2,\mathbf{K}'-\mathbf{d}'}\big)$$

with $\mathbf{K}' = (0, 0, K_3, \ldots, K_n)$. Lemma 9.2 similarly implies that

$$\Big[\mathcal{M}_{\mathfrak{m}\big((f^\wedge_K)_{1,2,\mathbf{K}'-\mathbf{d}'}\big),(K_1-d_1,K_2-d_2)}\Big] = \Big[\mathcal{M}_{\mathfrak{m}\big((f^\wedge_K)_{1,2,\mathbf{K}-\mathbf{d}}\big),(0,0)}\Big]$$

which, by Definition 8.2,

$$= \big[\mathcal{M}_{f^\wedge_K \text{ at } \mathbf{K}-\mathbf{d}}\big].$$

Hence, this equality and (92) yields (91). □

10 Stronger Duality Properties and Further Remarks

In this section, we will prove some stronger duality properties of $\underline{k}_{/B_1,B_2}$. To do so, we will develop some foundations of the homological algebra of k-diagrams, including a discussion of *skyscraper k-diagrams* and *coskyscraper k-diagrams*, based on the unpublished results in [12]. We also discuss the connection of k-diagrams to Grothendieck's sheaf theory and classical sheaf theory. This discussion will tie up a number of loose ends: for example, we will show that our definition of the cohomology groups of k-diagrams agree with the usual definition of of these groups, both in the context of sheaf theory for Grothendieck topologies and for classical topological spaces. We will also comment on periodic Riemann functions and possible future work.

Let us summarize the main results of this section.

The first stronger duality property of $\underline{k}_{/B_1,B_2}$ is that for any k-diagram, $\mathcal{F}$, we have

$$H^0(\mathcal{F})^* \simeq \mathrm{Ext}^1(\mathcal{F}, \underline{k}_{/B_1,B_2}), \tag{93}$$

where Ext^1 is the "first Ext group;" in order to define this group, and to make some basic computations, we will need some results from homological algebra.

When we define Ext groups, we will see that $\mathrm{Ext}^0(\mathcal{F}, \mathcal{G})$ is isomorphic to $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$, and $\mathrm{Ext}^i(\underline{k}, \mathcal{F})$ is isomorphic to $H^i(\mathcal{F})$ for all i; hence combining (93) with Theorem 9.1 yields the following theorem.

[8] Hint: write W as a difference of sums of perfect matchings

$$W = W_1 + \cdots + W_s - \tilde W_1 - \cdots - \tilde W_{s-1}.$$

Then setting W_i' to be the translation by $\mathbf{t}$ of W_i, and similarly for $\tilde W_i'$, we have

$$W' = W_1' + \cdots + W_s' - \tilde W_1' - \cdots - \tilde W_{s-1}'.$$

Since W_i' is translation by $\mathbf{t}$ of W_i, we have $\mathcal{M}_{W_i',\mathbf{d}} = \mathcal{M}_{W_i,\mathbf{d}+\mathbf{t}}$, and similarly for $\tilde W_i'$ and $\tilde W_i$. Now take direct sums.

Theorem 10.1. *For any k-diagram, $\mathcal{F}$, and for $i = 0, 1$, there are isomorphisms*

$$H^i(\mathcal{F})^* \to \mathrm{Ext}^{1-i}(\mathcal{F}, \underline{k}_{/B_1,B_2}) \tag{94}$$

that are functorial (i.e., natural) in $\mathcal{F}$. In other words, $\mathrm{Hom}(\mathcal{F}, \underline{k}_{/B_1,B_2})$ *and* $\mathrm{Ext}^1(\mathcal{F}, \underline{k}_{/B_1,B_2})$ *are isomorphic to the kernel and cokernel, respectively, of the dual map*

$$\mathcal{F}(\partial)^* \colon \mathcal{F}(A)^* \to \mathcal{F}(B)^*$$

defined in Definition 4.1.

The proof is given in Subsection 10.7. This theorem shows that $\underline{k}_{/B_1,B_2}$ plays the role of the canonical sheaf in the statement of Serre duality (e.g., Section III.7 of [16]).

The second duality result, proven at the end of this section, gives a method for computing the *Serre functor*, S, on a k-diagram, and we show that $S(\underline{k}) \simeq \underline{k}_{/B_1,B_2}[1]$; the proof is an immediate consequence of the proofs of Theorems 10.1 and 10.2 below.

To prove these results, we will develop the notion of skyscraper and coskyscraper k-diagrams. The reader familiar with sheaf theory in topology or algebraic geometry (e.g., [16]) will be able to check that skyscraper k-diagrams are the analog of skyscraper sheaves in topological sheaf theory (coskyscraper k-diagrams do not generally exist in topological sheaf theory). We stress that all k-diagrams have a two-term injective resolution with skyscraper k-diagrams, and a two-term projective resolution with coskyscraper k-diagrams. This makes working with k-diagrams much simpler than with sheaves over general topological spaces.

In this section, we will also show that the definition of the cohomology groups of a k-diagram, Definition 4.1, agrees with the usual definition of cohomology groups. Namely, we will prove the following theorem.

Theorem 10.2. *Let $\mathcal{F}$ be a k-diagram. Then the cohomology groups, $H^i(\mathcal{F})$, of $\mathcal{F}$ defined in Definition 4.1 are isomorphic to the groups* $\mathrm{Ext}^i(\underline{k}, \mathcal{F})$ *defined by viewing the category of k-diagrams as an abelian category.*

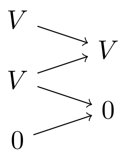
We will briefly review some facts about homological algebra, and refer the reader to the textbook [17] or [16], Section III.1 for details.

We remark that the modern foundations of homological algebra implicitly assume that Zorn's lemma holds, and it follows that for any two k-vector spaces $A \subset B$, there is a $B' \subset B$ such that B is the direct sum of A and B'. Hence, we assume this here (see Subsection 10.6 for further discussion).

10.1 Skyscraper and Co-Skyscraper k-Diagrams

To compute Ext groups below, we will use *skyscraper* and *coskyscraper* diagrams that we now describe.

For any k-vector space, V, consider the k-diagram, denoted $\mathrm{Sky}_{A_1}(V)$, depicted below:



$\mathrm{Sky}_{A_1}(V)$

where all maps $V \to V$ are identity maps; we call this the *skyscraper diagram of V at A_1*. We easily check that for any $\mathcal{F}$, and any

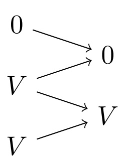
$$\phi \in \mathrm{Hom}(\mathcal{F}, \mathrm{Sky}_{A_1}(V)),$$

the value of $\phi(A_1) \colon \mathcal{F}(A_1) \to V$ determines all of ϕ: indeed, the map $\phi(B_1) \colon \mathcal{F}(B_1) \to V$ must equal $\phi(A_1) \circ \mathcal{F}(\rho_{1,1})$; conversely, any map $\mathcal{F}(A_1) \to V$ extends to such a morphism ϕ. Hence the map $\phi \mapsto \phi(A_1)$ sets up an isomorphism

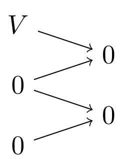
$$\mathrm{Hom}(\mathcal{F}, \mathrm{Sky}_{A_1}(V)) \simeq \mathrm{Hom}_k(\mathcal{F}(A_1), V),$$

where Hom_k denotes the morphisms as k-vector spaces.

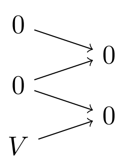
One similarly defines $\mathrm{Sky}_{A_2}(V)$, and for $j = 1, 2, 3$ one defines $\mathrm{Sky}_{B_j}(V)$ to be the k-diagram whose only nonzero value is V, at B_j. We depict these k-diagrams below.



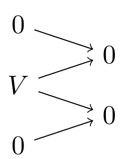
$\mathrm{Sky}_{A_2}(V)$



$\mathrm{Sky}_{B_1}(V)$



$\mathrm{Sky}_{B_2}(V)$



$\mathrm{Sky}_{B_3}(V)$

This gives for any $P = A_1, A_2, B_1, B_2, B_3$ and any k-vector space, V, a diagram, $\mathrm{Sky}_P(V)$ with an isomorphism

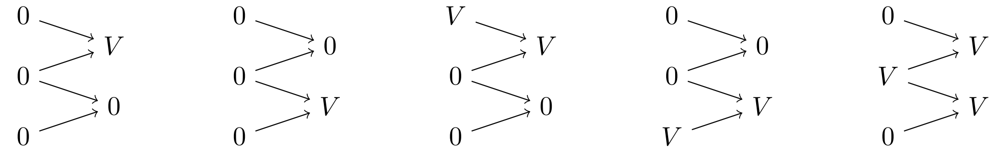
$$\mathrm{Hom}(\mathcal{F}, \mathrm{Sky}_P(V)) \simeq \mathrm{Hom}_k(\mathcal{F}(P), V),$$

given by

$$\text{for } \phi \in \mathrm{Hom}(\mathcal{F}, \mathrm{Sky}_P(V)), \qquad \phi \mapsto \phi(P).$$

Skyscrapers are particularly useful because one can prove that they are *injective* k-diagrams (see Proposition 10.2 below).

Similarly one defines for a k-vector sapce V the *coskyscraper k-diagram at A_i or at B_j* to be the diagrams depicted below:



$\mathrm{CoSky}_{A_1}(V)$ $\mathrm{CoSky}_{A_2}(V)$ $\mathrm{CoSky}_{B_1}(V)$ $\mathrm{CoSky}_{B_2}(V)$ $\mathrm{CoSky}_{B_3}(V)$

One verifies that for $P = A_1, A_2, B_1, B_2, B_3$ and any k-vector space, V, there is an isomorphism

$$\mathrm{Hom}(\mathrm{CoSky}_P(V), \mathcal{G}) \simeq \mathrm{Hom}_k(V, \mathcal{G}(P)) \tag{95}$$

taking ϕ to $\phi(P)$. Coskyscrapers are particularly useful because one can prove that they are *projective* k-diagrams (see Proposition 10.2 below).

10.2 A Minimal Introduction to Homological Algebra and Value-by-Value Evaluation

Here we will briefly review some facts about homological algebra, and refer the reader to the textbook [17] or [16], Section III.1 for details.

In order to use apply homological algebra, we need to know that k-diagrams and their morphisms form an *abelian category* (see [17], Definition 1.2.2 or [16], Section III.1); in computations, we will need to know how to compute the kernel, image, and cokernel in the sense defined for abelian categories.

Definition 10.1. *If $\phi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of k-diagrams, then the* kernel *(respectively* image *and* cokernel*) of ϕ is the k-diagram whose value at $P = A_1, A_2, B_1, B_2, B_3$ equals* $\ker(\phi(P))$ *(respectively,* $\mathrm{Image}(\phi(P))$ *and* $\mathrm{coker}(\phi(P))$*), and whose restriction maps are induced by those of $\mathcal{F}$ (respectively, those of $\mathcal{G}$ in both cases).*

In other words, we define *kernel*, *image*, and *cokernel* by evaluating them "value-by-value."

The next proposition is well known.

Proposition 10.1. *The category of k-diagrams is an abelian category. The definitions of kernel, image, and cokernel in Definition 10.1 agree with those notions when viewing the category of k-diagrams as an abelian category.*

The reader can prove this proposition directly; however, in Subsection 10.8 we will give two other proofs: one by a direct appeal to [1], and another by appealing to the special nature of sheaf theory over finite topological spaces. Of course, the reader may prefer to just assume this proposition and carry out the computations below.

A doubly-infinite sequence of morphisms of k-diagrams

$$\cdots \to \mathcal{F}^{-1} \xrightarrow{d^{-1}} \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \cdots$$

is *exact in position i* if $\mathrm{Image}(d^{i-1}) = \ker(d^i)$, i.e., for all $P \in \{A_1, A_2, B_1, B_2, B_3\}$ we have $\mathrm{Image}(d^{i-1}(P)) = \ker(d^i(P))$. A sequence of morphisms is *exact* if it is exact at each position. The analogous definition holds for a finite sequence of morphisms, or a one-sided infinite sequence of morphisms.

A *short exact sequence* is an exact sequence

$$0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \mathcal{F}^3 \to 0. \tag{96}$$

For such a sequence, and any k-diagram, $\mathcal{G}$, the morphisms in the above sequence determine, via composition, a sequence

$$0 \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}^1) \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}^2) \to \mathrm{Hom}(\mathcal{G}, \mathcal{F}^3) \to 0 \tag{97}$$

and a sequence

$$0 \to \mathrm{Hom}(\mathcal{F}^3, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}^2, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}^1, \mathcal{G}) \to 0. \tag{98}$$

We say that a k-diagram, $\mathcal{G}$ is *injective* (respectively, *projective*), if for any short exact sequence (96), the resulting sequence (97) (respectively, (98)) is exact. Any finite direct sum of injectives is injective, and any of projectives is projective.

In contrast with a lot of commonly used abelian categories—such as the category of sheaves of abelian groups or of k-vector spaces on a topological space—each k-diagram has simple projective and injective resolutions.

Proposition 10.2. *Any skyscraper (respectively, coskyscraper) k-diagram is injective (respectively, projective). Any k-diagram $\mathcal{F}$ fits into an exact sequence*

$$0 \to \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{F} \to 0 \tag{99}$$

where $\mathcal{P}_1, \mathcal{P}_0$ are projective—actually direct sums of coskyscraper diagrams; we call (99) *a* two-term projective resolution of $\mathcal{F}$. *Moreover, if the values of $\mathcal{F}$ are finite dimensional k-vector spaces, then the coskyscraper sheaves in the projective resolution can be taken to be are of the form* $\mathrm{CoSky}_P(V)$ *where the V are finite dimensional k-vector spaces. Similarly, any $\mathcal{G}$ fits into an exact sequence,*

$$0 \to \mathcal{G} \to \mathcal{I}^0 \to \mathcal{I}^1 \to 0, \tag{100}$$

where $\mathcal{I}^0, \mathcal{I}^1$ are injective; we call (100) *a* two-term injective resolution of $\mathcal{G}$. *Moreover, if the values of $\mathcal{G}$ are finite dimensional k-vector spaces, then the injective resolution is a sum of k-diagrams* $\mathrm{Sky}_P(V)$ *where the V are finite dimensional k-vector spaces.*

We will prove this proposition in Subsection 10.4.

The fact that any k-diagram has a two-term projective resolution and a two-term injective resolution makes the definition of Ext groups especially simple. (For the general definition of Ext groups, see [16, 17].)

For any k-diagrams, $\mathcal{F}, \mathcal{G}$, we take a projective resolution (99) and define the group $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ for $i = 0, 1$, respectively, as the kernel and cokernel of the resulting maps

$$\mathrm{Hom}(\mathcal{P}_0, \mathcal{G}) \to \mathrm{Hom}(\mathcal{P}_1, \mathcal{G}),$$

which are independent (up to isomorphism) of the projective k-diagrams $\mathcal{P}_1, \mathcal{P}_0$ and morphism $\mathcal{P}_1 \to \mathcal{P}_0$ yielding an exact sequence; $\mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. Furthermore, taking any injective resolution (100), the resulting kernel and cokernel of the map

$$\mathrm{Hom}(\mathcal{F}, \mathcal{I}^0) \to \mathrm{Hom}(\mathcal{F}, \mathcal{I}^1)$$

are also isomorphic to $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ for, respectively, $i = 0, 1$ (see, e.g., Theorem 2.7.6[9] (page 63) of [17]).

Consider any short exact sequence of k-diagrams

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

(i.e., for each $P = A_1, A_2, B_1, B_2, B_3$, the sequence

$$0 \to \mathcal{F}_1(P) \to \mathcal{F}_2(P) \to \mathcal{F}_3(P) \to 0$$

is exact). In this case for any k-diagram, $\mathcal{G}$, there is a long exact sequence

$$0 \to \mathrm{Ext}^0(\mathcal{F}_3, \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}_2, \mathcal{G}) \to \mathrm{Ext}^0(\mathcal{F}_1, \mathcal{G}) \to \mathrm{Ext}^1(\mathcal{F}_3, \mathcal{G}) \to \cdots$$

and this sequence is "natural" or "functorial" in $\mathcal{G}$, i.e., if $\mathcal{G}_1 \to \mathcal{G}_2$ is a morphism, then there are morphisms $\mathrm{Ext}^i(\mathcal{F}_j, \mathcal{G}_1) \to \mathrm{Ext}^i(\mathcal{F}_j, \mathcal{G}_2)$ that commute with the maps in the two resulting long exact sequences. Similarly, to any short exact sequence $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$, there is a long exact sequence

$$0 \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}_1) \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}_2) \to \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}_3) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}_1) \to \cdots \tag{101}$$

that is functorial in $\mathcal{F}$ (see, for example, the discussion of *universality*, since k-diagrams have enough injectives (see [16], Corollary 1.4, Section III.1).

[9] One is also using the Freyd-Mitchell Embedding Theorem, namely Theorem 1.6.1 of [17] (page 25), so that working with R-modules implies the same results over any small abelian category.

10.3 Working with Ext Groups

Our discussion of skyscraper and coskyscraper k-diagrams have some easy consequences that we will need when we make computations; we state these results in the two propositions below, and we leave both proofs to the reader.

Proposition 10.3. *Let P be one of A_1, A_2, B_1, B_2, B_3. For each k-diagram, $\mathcal{G}$, and each $u \in \mathcal{G}(P)$, the isomorphisms*

$$\mathrm{Hom}(\mathrm{CoSky}_P(k), \mathcal{G}) \simeq \mathrm{Hom}_k(k, \mathcal{G}(P)) \simeq \mathcal{G}(P)$$

determine a unique

$$\iota_{P,u} \in \mathrm{Hom}\bigl(\mathrm{CoSky}_P(k), \mathcal{G}\bigr)$$

such that $\iota_{P,u}(P)$ takes 1 to u. For each k-diagram, $\mathcal{F}$, and each $w \in \mathcal{F}(P)^*$, the isomorphisms

$$\mathrm{Hom}(\mathcal{F}, \mathrm{Sky}_P(k)) \simeq \mathrm{Hom}_k(\mathcal{F}(P), k) \simeq \mathcal{F}(P)^*$$

determine a unique

$$\iota^{P,w} \in \mathrm{Hom}\bigl(\mathcal{F}, \mathrm{Sky}_P(k)\bigr)$$

such that $\iota^{P,w}(P)$ takes each $u \in \mathcal{F}(P)$ to $w(u) \in k$.

In computing Ext groups and the Serre functor, the following observations will be helpful.

Definition 10.2. *For $P, P' \in \{A_1, A_2, B_1, B_2, B_3\}$, we say that P' is* a specialization of P *if $P' = P$, or if $P = A_i$ for some $i = 1, 2$ and $P' = B_i, B_3$; we also write $P' \leq P$, which gives a partial order on the set $\{A_1, A_2, B_1, B_2, B_3\}$.*

Proposition 10.4. *For P, P' each equal one of A_1, A_2, B_1, B_2, B_3. We have*

$$\mathrm{Hom}(\mathrm{CoSky}_P(k), \mathrm{CoSky}_{P'}(k)) \simeq \bigl(\mathrm{CoSky}_{P'}(k)\bigr)(P)$$

which equals k or 0 according to whether or not P' is a specialization of P. Furthermore, let P' be a specialization of P, and consider the map

$$\mu \in \mathrm{Hom}(\mathrm{CoSky}_P(k), \mathrm{CoSky}_{P'}(k))$$

given by $\mu(P)$ takes $1 \in \mathrm{Hom}(\mathrm{CoSky}_{P,k}(P)$ to $\alpha \in k = \mathrm{CoSky}_{P',k}(P)$. Then for any k-diagram, $\mathcal{F}$, the map that μ induces by composition

$$\mathrm{Hom}(\mathrm{CoSky}_{P'}(k), \mathcal{F}) \to \mathrm{Hom}(\mathrm{CoSky}_P(k), \mathcal{F}),$$

when equivalently viewed as a map

$$\mathcal{F}(P') \to \mathcal{F}(P),$$

is the restriction map in $\mathcal{F}$ from $\mathcal{F}(P') \to \mathcal{F}(P)$ multiplied by α. Similarly, for P, P' each equal one of A_1, A_2, B_1, B_2, B_3, we have

$$\mathrm{Hom}(\mathrm{Sky}_P(k), \mathrm{Sky}_{P'}(k)) \simeq \bigl(\mathrm{Sky}_P(k)\bigr)(P')$$

which equals k or 0 according to whether or not P' is a specialization of P; if

$$\mu \in \mathrm{Hom}(\mathrm{Sky}_P(k), \mathrm{Sky}_{P'}(k))$$

satisfies $(\mu(P'))1 = \alpha$, then the map that μ induces by composition

$$\mathrm{Hom}(\mathcal{F}, \mathrm{Sky}_{P'}(k)) \to \mathrm{Hom}(\mathcal{F}, \mathrm{Sky}_P(k)),$$

when equivalently viewed as a map

$$\mathcal{F}(P')^* \to \mathcal{F}(P)^*,$$

is the dual of restriction map in $\mathcal{F}$ from $\mathcal{F}(P')^* \to \mathcal{F}(P)^*$ multiplied by α.

The proof is a straightforward checking of the various cases of P and P' in $\{A_1, A_2, B_1, B_2, B_3\}$, which we leave to the reader. [The analogous result holds (and is similarly easy to check) for presheaves (in the sense of [1]) over any semitopological category; see [12].]

10.4 Two-Term Injective and Projective Resolutions

In this section we prove Proposition 10.2. The proof will introduce some useful tools to construct some slightly different resolutions of $\underline{k}$ and $\underline{k}_{/B_1,B_2}$.

Proof of Proposition 10.2. For any short exact sequence (96), and any $P = A_1, A_2, B_1, B_2, B_3$,

$$0 \to \mathcal{F}^1(P) \to \mathcal{F}^2(P) \to \mathcal{F}^3(P) \to 0 \tag{102}$$

is a short exact sequence of k-vector spaces. For any k-vector space, V, setting $\mathcal{G} = \mathrm{CoSky}_{P,V}$, (97) is equivalent to the sequence of k-vector spaces

$$0 \to \mathrm{Hom}_k\bigl(V, \mathcal{F}^1(P)\bigr) \to \mathrm{Hom}_k\bigl(V, \mathcal{F}^2(P)\bigr) \to \mathrm{Hom}_k\bigl(V, \mathcal{F}^3(P)\bigr) \to 0,$$

which we easily verify is exact by choosing a basis for V, which reduces this to the case $V = k$, which is equivalent to (102). Hence, any coskyscraper k-diagrams is projective.

Similarly for $\mathcal{G} = \mathrm{Sky}_{P,V}$, (98) is equivalent to the sequence

$$0 \to \mathrm{Hom}_k\bigl(\mathcal{F}^3(P), V\bigr) \to \mathrm{Hom}_k\bigl(\mathcal{F}^2(P), V\bigr) \to \mathrm{Hom}_k\bigl(\mathcal{F}^1(P), V\bigr) \to 0$$

which we see is exact by choosing a basis, X, for $\mathrm{Image}(\mathcal{F}^1(P))$ in $\mathcal{F}^2(P)$, and then extending this to a basis $X \cup X'$ for all of $\mathcal{F}^2(P)$. We then see that each element of $\mathrm{Hom}_k\bigl(\mathcal{F}^2(P), V\bigr)$ is determined by one of each of $\mathrm{Hom}_k\bigl(\mathcal{F}^i(P), V\bigr)$ for $i = 1, 3$. Hence, any skyscraper k-diagram is injective.

For the projective resolution of a k-diagram, $\mathcal{F}$, we note that for each $P = A_1, A_2, B_1, B_2, B_3$, the isomorphism

$$\mathrm{Hom}(\mathrm{CoSky}_P(\mathcal{F}(P)), \mathcal{F}) \simeq \mathrm{Hom}_k\bigl(\mathcal{F}(P), \mathcal{F}(P)\bigr)$$

(taking ϕ to $\phi(P)$) determines a unique map

$$\phi_{P,\mathcal{F}} \colon \mathrm{CoSky}_P\bigl(\mathcal{F}(P)\bigr) \to \mathcal{F} \tag{103}$$

that corresponds to the identity map of $\mathrm{Hom}_k\bigl(\mathcal{F}(P), \mathcal{F}(P)\bigr)$, i.e., the identity map $\phi_P(P)$ is the identity map $\mathcal{F}(P) \to \mathcal{F}(P)$. This sets up a morphism

$$\phi \colon \mathcal{P}_0 \to \mathcal{F},$$

where

$$\mathcal{P}_0 = \bigoplus_P \mathrm{CoSky}_P\bigl(\mathcal{F}(P)\bigr), \quad \phi = \bigoplus_P \phi_P.$$

We see that $\ker(\phi)$ is 0 on B_1, B_2, B_3, and hence $\ker(\phi)$ is a sum of coskyscrapers at A_1, A_2. Hence we get a resolution

$$\ker(\phi) = \mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{F},$$

where $\mathcal{P}_1, \mathcal{P}_0$ are direct sums of coskyscraper k-diagrams.

Similarly the maps

$$\phi^{P,\mathcal{F}} \colon \mathcal{F} \to \mathrm{Sky}_P\bigl(\mathcal{F}(P)\bigr)$$

such that $\phi^P(P)$ is the identity, yield an injective map $\mathcal{F} \to \mathcal{I}^0$ whose cokernel at A_1, A_2 is 0, and hence we get a two-term injective resolution $\mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1$ where $\mathcal{I}^0 = \oplus_P \mathrm{Sky}_P\bigl(\mathcal{F}(P)\bigr)$. □

[Similar remarks hold for the category of presheaves on any finite categories that are *semitopological* in the sense of [12]; see [13] for examples of this general principle in another special case, namely the semitopological categories used to define a *sheaf on a graph.*]

10.5 Proof of Theorem 10.2

To prove Theorem 10.2 we rely on the following straightforward calculation.

Lemma 10.1. *The k-diagram $\underline{k}$ has a projective resolution*

$$0 \to \bigoplus_{i=1}^{2} \mathrm{CoSky}_{A_i}(k) \xrightarrow{\mu_1} \bigoplus_{j=1}^{3} \mathrm{CoSky}_{B_j}(k) \xrightarrow{\mu_0} \underline{k} \to 0, \tag{104}$$

where

$$\mu_0 = \bigoplus_{j=1}^{3} \phi_{B_j,\underline{k}} \tag{105}$$

with ϕ_{B_j} *as in* (103)*, and the map* μ_1 *is the as follows: the restriction to* μ_1 *of* $\mathrm{CoSky}_{A_1}(k)$ *is determined by the element* $(1,0,-1)$ *in*

$$\left(\bigoplus_{j=1}^{3}\mathrm{CoSky}_{B_j}(k)\right)(A_1)=\bigoplus_{j=1}^{3}\bigl(\mathrm{CoSky}_{B_j}(k)\bigr)(A_1)=k\oplus\{0\}\oplus k,$$

i.e., equals the element $\iota_{A_1,(1,0,-1)}$ *as in Proposition 10.3, and similarly* μ_1 *restricted to* $\mathrm{CoSky}(A_2)$ *is* $\iota_{A_2,(0,1,-1)}$.

Proof. Setting

$$\mathcal{P}_0=\bigoplus_{j=1}^{3}\mathrm{CoSky}_{B_j}(k)$$

we have $\mathcal{P}_0(B_j)=k$ for $j=1,2,3$, and μ_0 in (105) has $\mu_0(B_j)\colon k\to k$ being the identity map. Furthermore,

$$\mathcal{P}_0(A_1)=\bigoplus_{j=1}^{3}\bigl(\mathrm{CoSky}_{B_j}(k)\bigr)(A_1)=k\oplus\{0\}\oplus k$$

and $\mu_0(A_1)$ takes $k\oplus\{0\}\oplus k\to\underline{k}(A_1)=k$ by the map taking $(u_1,0,u_3)$ to u_1+u_3. Similarly $\mu(A_2)$ takes $\mathcal{P}(A_2)=\{0\}\oplus k\oplus k$ to $\underline{k}(A_2)$ by the map taking $(0,u_2,u_3)$ to u_2+u_3. It follows μ_0 is surjective, and that $\mathcal{P}_1=\ker(\mu_0)$ has $\mathcal{P}_1(B_j)=0$ for all j, and

$$\mathcal{P}_1(A_1)=\ker\bigl((u_1,0,u_3)\mapsto u_1+u_3\bigr),\quad \mathcal{P}_1(A_2)=\ker\bigl((0,u_2,0,u_3)\mapsto u_2+u_3\bigr).$$

Hence $\mathcal{P}_1(A_1)$ is a one dimensional k-vector space, spanned by $(1,0,-1)\in k\oplus\{0\}\oplus k$, and $\mathcal{P}(A_2)$ similarly, spanned by $(1,0,-1)$. Since for $\mathrm{CoSky}_{A_i,k}$ has value k at A_i and everywhere else $\{0\}$, $\mathcal{P}_1$ is the sum of one-dimensional coskyscrapers with the value k and the maps $\mathcal{P}_1\to\mathcal{P}_0$ is as in the statement of the lemma. ☐

We remark that in the above lemma, $\mathcal{P}_1(A_1)$ is a the one-dimensional kernel of the map $(u_1,0,u_3)\mapsto u_1+u_3$; hence this kernel is spanned both by $\pm(1,0,-1)$; the choice of one over the other is arbitrary, and similarly, the choice of $\mu_1(A_1)$ could take 1 to $(1,0,-1)$ or $(-1,0,1)$. Similarly for $\mathcal{P}_1(A_2)$. [The opposite category of $\mathcal{C}$ above is category in [13] associated to a graph with three vertices, corresponding to B_1,B_2,B_3 and two edges corresponding to A_1,A_2, and the choice between $\pm(1,0,-1)$ is analogous to a choice of orientation of the edge corresponding to A_1; similarly for A_2.]

Proof of Theorem 10.2. We can compute $\mathrm{Ext}^i(\underline{k},\mathcal{F})$ using the projective resolution (104), which is therefore the kernel and cokernel of the map that μ_1 induces on

$$\mathrm{Hom}\left(\bigoplus_{j=1}^{3}\mathrm{CoSky}_{B_j}(k),\mathcal{F}\right)\to\mathrm{Hom}\left(\bigoplus_{i=1}^{2}\mathrm{CoSky}_{A_i}(k),\mathcal{F}\right),$$

which is equivalent to a map

$$\nu\colon\bigoplus_{j=1}^{3}\mathcal{F}(B_j)\to\bigoplus_{i=1}^{2}\mathcal{F}(A_i).\tag{106}$$

For $i=1,2$ and $j=1,2,3$, let $\alpha_{i,j}\in k$ be given as

$$\alpha_{1,1}=\alpha_{2,2}=1,\quad \alpha_{1,3}=\alpha_{2,3}=-1,\quad \alpha_{1,2}=\alpha_{2,1}=0.\tag{107}$$

According to Proposition 10.4, since μ_1 takes $\mathrm{CoSky}_{A_i,k}$ to $\mathrm{CoSky}_{B_j,k}$ by multiplication by $\alpha_{i,j}$, it follows that ν maps $\mathcal{F}(B_j)$ to $\mathcal{F}(A_i)$ by multiplication by $\alpha_{i,j}$. But this is precisely the map $\mathcal{F}(\partial)$. Hence $\mathrm{Ext}^i(\underline{k},\mathcal{F})$ for $i=0,1$ are, respectively, the kernel and cokernel of $\mathcal{F}(\partial)$. ☐

10.6 A Subtlety of Linear Algebra and Zorn's Lemma

Let $\mathcal{L}\colon U\to V$ be any linear map of (possibly infinite-dimensional) k-vector spaces. Then we get a map:

$$\mathrm{coker}(\mathcal{L}^*)\to\bigl(\ker(\mathcal{L})\bigr)^*,\tag{108}$$

as follows: any $\ell\in U^*$ gives, by restriction, a map $\ker(\mathcal{L})\to k$, i.e., an element of $(\ker(\mathcal{L}))^*$, and if $\ell-\ell'\in\mathrm{Image}(\mathcal{L}^*)$, then ℓ,ℓ' agree on $\ker(\mathcal{L})$. To know that this map is an isomorphism, one needs to know that any linear map $\ell\colon\ker(\mathcal{L})\to k$ extends to a map $U\to k$. This is true if we assume the axiom of choice or Zorn's Lemma, for then using Zorn's lemma, we can find a subspace $U'\subset U$ such that U splits as a direct sum of

$\ker(\mathcal{L})$ and $U' \in U$, and this allows us to extend ℓ to all of U (defining ℓ to be zero on U', which defines ℓ on all of U).

[Even if one does not assume the axiom of choice or Zorn's lemma, (108) will be an isomorphism in certain situations: for example, if U has a basis and $\ker(\mathcal{L})$ is finite dimensional, then we can perform a finite basis exchange to have a basis for $\ker(\mathcal{L})$ that extends to a basis for U. Note that this holds for $L = \mathcal{M}_{W,\mathbf{d}}(\partial)$ for any perfect matching W, the values of $\mathcal{M}_{W,\mathbf{d}}$ have a basis (indexed by $\mathbb{Z}$ or $\mathbb{Z}_{\le d_i}$ for $i = 1, 2$) and the kernel of L is finite dimensional.]

On the other hand, the foundations of homological algebra assume Zorn's lemma, since one uses Baer's Criterion to show that k is an injective k-module, which assumes Zorn's lemma (see, e.g., [17], proof of Bear's Criterion 2.3.1, page 39). In fact, to say that k is injective is precisely to say that if $A \to B$ is any injection, then any map $A \to k$ is the composition of the map $A \to B$ with some map $B \to k$, which implies that any linear functional on a subspace of B has an extension to all of B. Hence, we will assume Zorn's lemma for the rest of this section.

10.7 Proof of Theorem 10.1

Proof of Theorem 10.1. Similar to (104), let us prove that $\underline{k}_{/B_1,B_2}$ has the injective resolution

$$0 \to \underline{k}_{/B_1,B_2} \xrightarrow{\mu^0} \bigoplus_{i=1}^{2} \mathrm{Sky}_{A_i}(k) \xrightarrow{\mu^1} \bigoplus_{j=1}^{3} \mathrm{Sky}_{B_j}(k) \to 0. \tag{109}$$

First, we take μ^0 to be $\iota^{A_1,1} \oplus \iota^{A_2,1}$ with notation as in Proposition 10.3, which yields an injection

$$\mu^0 \colon \underline{k}_{/B_1,B_2} \to \mathcal{I}^0, \quad \text{where} \quad \mathcal{I}^0 = \bigoplus_{i=1}^{2} \mathrm{Sky}_{A_i}(k),$$

where setting $\mathcal{I}^1 = \mathcal{I}^0/\mu^0(\underline{k})$, we have $\mathcal{I}^1(A_i) = 0$ for $i, 1, 2$, and

$$\mathcal{I}^1(B_1) = \mathcal{I}^1(B_2) = k/\{0\} = k, \quad \mathcal{I}^1(B_3) = k \oplus k/\mathrm{diag},$$

where diag is the diagonal, i.e., the span of $(1, 1)$ in $k \oplus k$. Hence identifying $k \oplus k/\mathrm{diag}$ with the class of $(-1, 0)$, we get

$$\mathcal{I}^1 \simeq \bigoplus_{j=1}^{3} \mathrm{Sky}_{B_j}(k),$$

with the map $\mu^1 \colon \mathcal{I}^0 \to \mathcal{I}^1$ being the map taking $\mathrm{Sky}_{A_i,k}$ to $\mathrm{Sky}_{B_j,k}$ being multiplication by α_{ij}, where $\alpha_{i,j}$ are given in (107). Hence for any $\mathcal{F}$ we have an exact sequence

$$0 \leftarrow \mathrm{Ext}^0(\mathcal{F}, \underline{k}_{/B_1,B_2}) \leftarrow \bigoplus_{i=1}^{2} \mathrm{Hom}_k(\mathcal{F}(A_i), k) \leftarrow \bigoplus_{j=1}^{3} \mathrm{Hom}_k(\mathcal{F}(B_i), k) \leftarrow \mathrm{Ext}^1(\mathcal{F}, \underline{k}_{/B_1,B_2}) \leftarrow 0,$$

i.e.,

$$0 \leftarrow \mathrm{Ext}^0(\mathcal{F}, \underline{k}_{/B_1,B_2}) \leftarrow \bigoplus_{i=1}^{2} \mathcal{F}(A_i)^* \leftarrow \bigoplus_{j=1}^{3} \mathcal{F}(B_i)^* \leftarrow \mathrm{Ext}^1(\mathcal{F}, \underline{k}_{/B_1,B_2}) \leftarrow 0.$$

But in view of the values of the $\alpha_{i,j}$, the map

$$\bigoplus_{i=1}^{2} \mathcal{F}(A_i)^* \leftarrow \bigoplus_{j=1}^{3} \mathcal{F}(B_i)^* \tag{110}$$

above is precisely the dual map of ν in (106). Hence, in view of the isomorphisms (72) and (108), the duals of kernel and cokernel of ν are the cokernel and kernel (respectively) of the map in (110). This establishes (93) and hence (94).

The functoriality in $\mathcal{F}$ can be verified directly or by appealing to the functoriality of (101). □

10.8 Homological Algebra of k-Diagrams, Value-By-Value Evaluation, and Sheaf Theory

To apply the machinery of homological algebra, one has to verify that k-diagrams form an *abelian category*, and to prove Proposition 10.1. As mentioned there, the reader who is so inclined can verify Proposition 10.1 from scratch. However, it is simpler to point out that (1) this proposition is well known, and (2) the reader familiar with sheaf theory on topological spaces can view k-diagrams as equivalent to sheaves of k-vector spaces on a certain topological space. Let us explain both these points.

First, we remark that, more generally, many convenient notions regarding k-diagrams and their morphisms can be evaluated "value-by-value," just as the notions of kernel, image, and cokernel of a morphism in Proposition 10.1. Let us give some further examples of these "value-by-value" evaluations.

10.8.1 Examples of Value-By-Value Evaluation

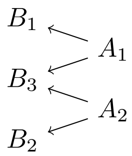
If $\mathcal{F}$ is a k-diagram, then a *subdiagram* of $\mathcal{F}$ refers to any k-diagram, $\mathcal{F}'$ such that at each $P = A_1, A_2, B_1, B_2, B_3$, $\mathcal{F}'(P) \subset \mathcal{F}(P)$ is a subspace, and such that each restriction map, $\mathcal{F}'(\rho_{ij})$, of $\mathcal{F}'$ is the restriction of $\mathcal{F}(\rho_{ij})$ to the subspace $\mathcal{F}(B_i)$. In this case one easily verifies that the value-by-value inclusion of $\mathcal{F}'$ into $\mathcal{F}$ gives a morphism $\phi\colon \mathcal{F}' \to \mathcal{F}$. In this case there is a *quotient* $\mathcal{F}/\mathcal{F}'$, whose values are $\mathcal{F}(P)/\mathcal{F}'(P)$.

The reader familiar with topological sheaf theory will notice that if we take a value-by-value quotient $\mathcal{F}/\mathcal{F}'$ as above, the result is not generally a sheaf (see, e.g., [16], Section II.1, top of page 65); to get the usual notion of a quotient sheaf one has to take the extra step of *sheafifying* the result, i.e., taking the sheaf associated to the presheaf given by $U \to \mathcal{F}(U)/\mathcal{F}'(U)$ for open subsets U. For k-diagrams we never need this extra step.

Similarly, for a morphism of k-diagrams $\phi\colon \mathcal{F} \to \mathcal{G}$, the *kernel*, *image*, and *cokernel* of ϕ are defined to be the diagrams whose values at each $P = A_1, A_2, B_1, B_2, B_3$ are the kernel, image, and cokernels of $\phi(P)$. Again, in general topological sheaf theory, the additional step of *sheafifying* is needed for images and cokernels (see, e.g., [16], Section II.1, top of page 64, just above Proposition/Definition 1.2).

10.8.2 Why Value-By-Value Evaluation Works

To see why "value-by-value" evaluation works, we appeal to [1], Exposé I, Section 3, Proposition 3.1 and Corollaire 3.2: consider the category $\mathcal{C}$, whose objects and non-identity morphisms are depicted below.



$\mathcal{C}$ (111)

Then a k-diagram is the same thing as a contravariant functor from $\mathcal{C}$ to the category of k-vector spaces, which is called, in [1], Exposé I, a *presheaf* of k-vector spaces on $\mathcal{C}$. Then Proposition 3.1 and Corollaire 3.2 of [1], Exposé I, Section 3 implies that for any category, $\mathcal{C}$, the notions of subdiagram, kernel, image, etc., agree with the value-by-value evaluation above (we also refer the reader to [13] and [12] for similar observations with $\mathcal{C}$ replaced with categories arising from graphs).

10.8.3 k-diagrams and Sheaf Theory

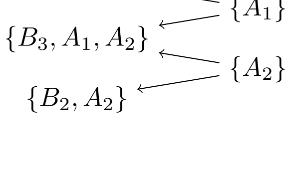
To see that "value-by-value" evaluation works in k-diagrams, one can alternatively appeal to sheaf theory (e.g., [16], Section II.1) on one particular topological space. Let us explain how.

If X is a topological space on a finite set, say that an open set $U \subset X$ is *irreducible* if U is non-empty and cannot be written as the union of proper open subsets of X. The set of irreducible open subset, $\mathrm{Irred}(X)$, of X becomes a category under inclusion. Each sheaf, $\mathcal{F}$, of k-vector spaces on X restricts to a presheaf (in the sense of Grothendieck, i.e., a contravariant functor from $\mathrm{Irred}(X)$ to the category of k-vector spaces) of vector spaces on $\mathrm{Irred}(X)$. It is not hard to verify (see, e.g., [12]) that this functor from the category of sheaves of k-vector spaces on X to presheaves of k-vector spaces on $\mathrm{Irred}(X)$ is an equivalence of categories; in other words, one can reconstruct—up to isomorphism—a sheaf on X by knowing its restriction to $\mathrm{Irred}(X)$, and any presheaf on $\mathrm{Irred}(X)$ arises as the restriction of a sheaf on X.

Let X be the finite topological space on the points A_1, A_2, B_1, B_2, B_3 with a basis of open subsets

$$\{A_1\},\ \{A_2\},\ \{B_1, A_1\},\ \{B_2, A_2\},\ \{B_3, A_1, A_2\}.$$

Then one easily sees that $\mathrm{Irred}(X)$ is precisely the category



(so, for example, B_1, B_2, B_3 are closed points, and $\{A_1\}, \{A_2\}$ are open subsets, B_1 and B_3 are specializations of A_1, etc.). Hence Irred(X) in this case is the same thing as the category (111), which is how we build our diagrams.

We remark that finite topological spaces have the property that each point, P, of the space has a minimal open subset, U_P, that contains it. It follows that the "stalk at P" of a sheaf is simply its value at U_P. This is an alternative way to see that "value-by-value evaluation" works in all finite topological spaces.

We also remark that if P, P' are points in a topological space, we say that P' is a *specialization* of P when the closure of P contains P' (see [1], Subsection IV.4.2.2 or [16], Exercise II.3.17). Hence, this notion agrees with the notion in Definition 10.2.

10.9 The Usual Skyscraper k-Diagrams the Riemann-Roch Theorem

If $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is a perfect matching, and $\mathbf{d} \in \mathbb{Z}^2$, then we easily check that there is an exact sequence

$$0 \to \mathcal{M}_{W,\mathbf{d}} \to \mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i} \to \mathrm{Sky}_{A_i}(k) \to 0. \tag{112}$$

Furthermore, b^1 of any skyscraper k-diagram is 0 since any skyscraper k-diagram is injective, and we easily check that

$$b^0\big(\mathrm{Sky}_{A_i}(k)\big) = 1.$$

This therefore mimics the usual short exact sequence in the modern formulation of the Riemann-Roch theorem, e.g., [16], the proof of the Riemann-Roch theorem, page 296, the sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D+P) \to k(P) \to 0.$$

Hence

$$\chi\big(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}\big) = \chi\big(\mathcal{M}_{W,\mathbf{d}}\big) + \chi\big(\mathrm{Sky}_{A_i}(k)\big) = \chi\big(\mathcal{M}_{W,\mathbf{d}}\big) + 1,$$

which is another way of deriving (30). Furthermore, the full strength of Lemma 4.1 as it applies to the Betti numbers of $\mathcal{M}_{W,\mathbf{d}}$ and $\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}$ (as in Corollary 4.1) can also be seen by considering the long exact sequence that arises from (112), namely

$$0 \to H^0\big(\mathcal{M}_{W,\mathbf{d}}\big) \to H^0\big(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}\big) \to k \to H^1\big(\mathcal{M}_{W,\mathbf{d}}\big) \to H^1\big(\mathcal{M}_{W,\mathbf{d}+\mathbf{e}_i}\big) \to 0.$$

10.10 $\mathcal{O}$-Modules and Periodic k-Diagrams

In this section we explain why Serre duality cannot hold in the context of k-diagrams for sheaves the form $\mathcal{M}_{W,\mathbf{d}}$. This also explains why we work both with sheaves of the form $\mathcal{M}_{W,\mathbf{d}}$ — which most closely mimic Serre duality — and the simpler k-diagrams $\mathcal{I}_{\mathbf{d}}^{\oplus W}$. This also motivates future work of ours [9].

The sheaf theory of algebraic geometry works with sheaves of $\mathcal{O}$-modules, where $(X, \mathcal{O})$ is a locally ringed space (see, e.g., [16], Sections II.2, page 72 and III.2) We remark that k-diagrams are nothing more than $\mathcal{O}$-modules over the ringed space $(X, \mathcal{O})$ where X is the five-point topological space described in the previous subsection, and $\mathcal{O} = \underline{k}$.

If $W\colon \mathbb{Z}^2 \to \mathbb{Z}$ is a perfect matching that is also r periodic—which is the case in the Baker-Norine rank and related rank functions [2, 5]—then the k-diagrams $\mathcal{M}_{W,\mathbf{d}}$ are $\mathcal{O}$-modules for a much larger sheaf of rings (i.e., k-diagram of rings) over the same space, namely the diagram $\mathcal{O} = \mathcal{O}_r = \mathcal{O}_{k,r}$ whose values are for $i = 1, 2$:

$$\mathcal{O}(A_i) = k[x_i, 1/x_i],\ \mathcal{O}(B_i) = k[y_i],\ \mathcal{O}(B_3) = k[v, 1/v], \tag{113}$$

whose restriction maps take v to x_1^r, x_2^{-r} and take y_i to $1/x_i$, i.e.,

$$\mathcal{O}(\rho_{i,i})(y_i) = 1/x_i \text{ (for } i = 1, 2), \quad \mathcal{O}(\rho_{3,1})(v) = x_1^r, \quad \mathcal{O}(\rho_{3,2})(v) = x_2^{-r}. \tag{114}$$

We depict this k-diagram of rings in Figure 11. For $r = 1$, $(X, \mathcal{O})$ is the Riemann sphere and the $\mathcal{M}_{W,\mathbf{d}}$ are line bundles, but for $r \geq 2$ this is a much more mysterious space.[10] [This is not an orbifold; perhaps it is a cover of an orbifold reflecting a Čech cohomology computation or something related.] We believe there is a duality theorem akin to Serre duality, involving the k-diagrams $\mathcal{M}_{W,\mathbf{d}}$ as $\mathcal{O}$-modules. We plan to address this in a future work. This may shed more light on the $\mathcal{M}_{W,\mathbf{d}}$. It is also related to our discussion of Serre functors in the next subsection.

Note that our four basic k-diagrams (Figure 6) are not $\mathcal{O}_{k,r}$ modules. Hence in works like [9], we cannot write $\mathcal{M}_{W,\mathbf{d}}$ (which is a $\mathcal{O}_{k,r}$-module if W is r-periodic) as a sum of our four basic k-diagrams.

[10] We thank Ehud de Shalit for a discussion of $\mathcal{O}_{k,r}$, and Luc Illusie for questions regarding $\mathcal{M}_{W,\mathbf{d}}$ and $\mathcal{O}_{k,r}$-modules.

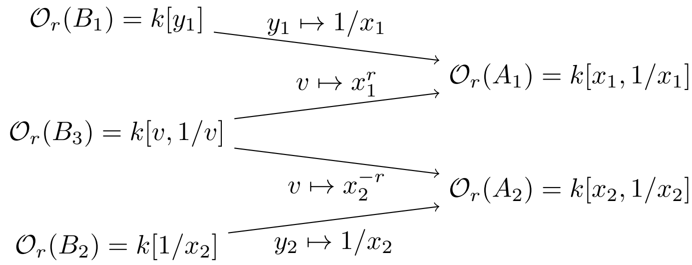


Figure 11: The Diagram of Rings, $\mathcal{O}_r = \mathcal{O}_{k,r}$: this k-diagram has more structure: its values are rings, and restriction maps are also morphisms of rings. Hence $\mathcal{O}_r = \mathcal{O}_{k,r}$ is much larger and more structured; hence for any $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is much smaller than $\mathrm{Hom}_{\underline{k}}(\mathcal{F}, \mathcal{G})$. This smallness is crucial if we want to get a stronger form of Serre duality.

10.10.1 Serre Duality, $\underline{k}$-modules, and Some "Bad News"

Let us give a bit more detail regarding $\underline{k}$-modules and $\mathcal{O}_{k,r}$. The usual statement of Serre duality (e.g., [16] (e.g., Theorems 7.1 and 7.6, Chapter III) gives isomorphisms:

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \to H^{n-i}(X, \mathcal{F})^*, \quad \text{for } 0 \le i \le n,$$

for an n-dimensional projective scheme, X, and a coherent sheaf, $\mathcal{F}$ (the subscript $\mathcal{O} = \mathcal{O}_X$ is understood in [16], but here we add this for emphasis). In particular, in the case of curves, and $i = 0$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \to H^1(X, \mathcal{F})^* \tag{115}$$

for coherent sheaves, $\mathcal{F}$.

Now notice that if $(X, \mathcal{O})$ is any *ringed space*, and $\mathcal{F}$ is any sheaf of $\mathcal{O}$-modules, then the group

$$H^1(X, \mathcal{F})$$

is the same whether we view $\mathcal{F}$ as (1) a sheaf of abelian groups, or (2) a $\mathcal{O}$-module (Proposition 2.6, Chapter III). We emphasize that this holds in the generality of *ringed spaces*, not merely the locally ringed spaces of algebraic geometry. In particular, $H^1(X, \mathcal{F})$ is the same in when computed in the category of $\mathcal{O}$-modules with $\mathcal{O} = \underline{k}$ or $\mathcal{O}$ as in (113) and (114).

The above leads to some "good news" and some "bad news" if we work with $\mathcal{O}$-modules with $\mathcal{O} = \underline{k}$, i.e., sheaves of k-vector spaces, i.e., k-diagrams: indeed, ω_X in (115) is uniquely determined up to isomorphism and determined by

$$H^1(\mathcal{F})^* \simeq \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \omega_X) \tag{116}$$

plus the usual "functoriality" that one insists in Serre duality (this is a special case of "Yoneda's lemma," i.e., the uniqueness of an object representing a functor). To concretely compute ω_X, assuming that this is to hold all $\mathcal{F}$'s that are k-diagrams with whose values are finite dimensional vector spaces, then this must hold for the k-diagrams $\mathrm{CoSky}_P(k)$ for $P = A_1, A_2, B_1, B_2, B_3$; if so, using (95) we have

$$\omega_X(P) \simeq \mathrm{Hom}_k\bigl(k, \omega_X(P)\bigr) \simeq \mathrm{Hom}_{\underline{k}}\bigl(\mathrm{CoSky}_P(k), \omega_X\bigr) \simeq H^1(\mathrm{CoSky}_P(k))^*,$$

which determines the dimensions of $\omega_X(P)$; the functoriality determines the restriction maps of $\omega_X(P)$, and we easily see that

$$\omega_X = \underline{k}_{/B_1,B_2}.$$

And, indeed, we easily check that (116) does hold for $\omega_X = \underline{k}_{/B_1,B_2}$ for all large class of k-diagrams, $\mathcal{F}$, including our four basic k-diagrams, and therefore on the $\mathcal{M}_{W,\mathbf{d}}$ and sums thereof.

Hence, the "good news" is that there is a "dualizing sheaf" $\underline{k}_{/B_1,B_2}$, and we have

$$H^1(\mathcal{F})^* \simeq \mathrm{Hom}_{\underline{k}}\bigl(\mathcal{F}, \underline{k}_{/B_1,B_2}\bigr).$$

for a very general class of k-diagrams, including the $\mathcal{M}_{W,\mathbf{d}}$.

At the same time, here is some "bad news."

First, the dualizing sheaf $\underline{k}_{/B_1,B_2}$ reflects the geometry of $\underline{k}$ and k-diagrams, and nothing more interesting. It is likely that the dualizing sheaf ω_X in (116) for $\mathcal{O} = \mathcal{O}_{k,r}$ may reflect more interesting "geometry." Moreover, if we are primarily interested in the $\mathcal{M}_{W,\mathbf{d}}$, which are not $\mathcal{O}_{k,r}$ line bundles and don't seem to be even coherent $\mathcal{O}_{k,r}$-modules, then the appropriate dualizing sheaf might reflect the geometry of $\mathcal{M}_{W,\mathbf{d}}$.

Second, stronger forms of duality doesn't quite work; this is explained in [10]; see, for example, Theorem 11.2 the remark beneath it. The point is that one might expect a stronger form of Serre duality, whose verification is — at least in principle — easier, since it involves only "local" computations in the derived category.

We intend to make some remarks on the above in [9] and further research.

10.11 The Serre Functor on Chains of k-Diagrams

In this section, we briefly discuss so-called *Serre functor*, S, and show that $S(\underline{k}) = \underline{k}_{/B_1,B_2}[1]$, which explains Theorem 10.1.

To motivate our discussion of the Serre functor, notice that Theorem 10.1 yields an isomorphism

$$\forall i \in \mathbb{Z}, \quad \mathrm{Ext}^i(\underline{k}, \mathcal{F})^* \simeq H^i(\mathcal{F})^* \to \mathrm{Ext}^{1-i}(\mathcal{F}, \underline{k}_{/B_1,B_2})$$

and for any $P = A_1, A_2, B_1, B_2, B_3$ we have an isomorphism

$$\forall i \in \mathbb{Z}, \quad \mathrm{Ext}^i\big(\mathrm{CoSky}_P(k), \mathcal{F}\big)^* \to \mathrm{Ext}^{1-i}\big(\mathcal{F}, \mathrm{Sky}_P(k)\big).$$

It follows that if $\mathcal{G}$ is a sum $\underline{k} \oplus \mathrm{CoSky}_P(k)$, there is no duality theorem that has a single k-diagram associated to $\mathcal{G}$ in a duality theorem involving

$$\mathrm{Ext}^i(\mathcal{G}, \mathcal{F})^*.$$

So even if one seeks a duality theorem valid only for k-diagrams, one is pretty much forced to express this by working in a larger context. The derived category is such a context, and it is a common tool for expressing duality theorems.

Hence, we assume that the reader is familiar with the derived category (see [17], Chapter 10, or [15], Chapter I), and we will describe the so-called *Serre functor* in these terms. Since each k-diagram has a two-term injective and a two-term projective resolution, it is simplest to work in $\mathcal{D} = \mathcal{D}^{\mathrm{b}}$ of bounded chains of k-diagrams, each of whose values are finite-dimensional k-vector spaces[11]. The *Serre functor*, S, (see, e.g., [6,7] or [12], Section 2.12 and the references therein) is the functor that takes an object $\mathcal{G} \in \mathcal{D}$ to the functor $S(\mathcal{G}) \colon \mathcal{D} \to \mathcal{V}$, where $\mathcal{V}$ is the category of finite dimensional k-vector spaces and $\mathcal{G} \mapsto S(\mathcal{G})$ is the functor,

$$\mathcal{F} \mapsto \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}, \mathcal{F})^*$$

(all Hom sets are assumed to be finite-dimensional k-vector spaces); if $S(\mathcal{G})$ is representable, then there is a $\mathcal{G}' \in \mathcal{D}$ such that

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}, \mathcal{F})^* \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}, \mathcal{G}'),$$

and we write $S(\mathcal{G}) = \mathcal{G}'$; if so then $\mathcal{G}'$ is uniquely determined up to unique isomorphism. The recipe to compute the Serre functor in [12] (written "left-to-right" or $! \to *$), in our terms, is to observe that

$$S\big(\mathrm{CoSky}_P(k)\big) = \mathrm{Sky}_P(k),$$

since if $\mathcal{F}$ equals $\cdots \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$ is any bounded chain of k-diagrams, then since $\mathrm{CoSky}_P(k)$ is projective, viewing $\mathrm{CoSky}_P(k)$ as a complex in degree 0 we have

$$\mathrm{Hom}_{\mathcal{D}}\big(\mathrm{CoSky}_P(k), \mathcal{F}\big) = \mathrm{Hom}_{\mathcal{K}}\big(\mathrm{CoSky}_P(k), \mathcal{F}\big) = H^0\big(\mathcal{F}(P)\big),$$

where $\mathrm{Hom}_{\mathcal{K}}$ denotes chain maps modulo homotopy, and $H^0(\mathcal{F}(P))$ denotes the 0-th cohomology group of the exact sequence

$$\cdots \to \mathcal{F}^{-1}(P) \to \mathcal{F}^0(P) \to \mathcal{F}^1(P) \to \cdots$$

Similarly, since $\mathrm{Sky}_P(k)$ is injective, we have

$$\mathrm{Hom}_{\mathcal{D}}\big(\mathcal{F}, \mathrm{Sky}_P(k)\big) = \mathrm{Hom}_{\mathcal{K}}\big(\mathcal{F}, \mathrm{Sky}_P(k)\big) = H^0\big(\mathcal{F}(P)\big)^*.$$

It follows that if $\mathcal{F}$ is any chain that is 0 outside of degree 0, and is a sum of k-diagrams of the form $\mathrm{CoSky}_P(k)$ in degree 0, then $S(\mathcal{F})$ is represented by exchanging each CoSky with a Sky. It then follows by induction that if $\mathcal{G}$ is a chain of sums of coskyscraper k-diagrams (each of finite dimension), then the recipe of exchanging the coskyscrapers with skyscrapers computes the Serre functor: for the inductive step, one can use Lemma I.7.2 of [15], which states that for any complex $\mathcal{G}$ equal $\cdots \to \mathcal{G}^{-1} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots$ and $n \in \mathbb{Z}$ there is a distinguished triangle

$$\tau_{\geq n}(\mathcal{G}) \to \mathcal{G}^n \to \tau_{>n}(\mathcal{G}) \to \tau_{\geq n}(\mathcal{G})[1],$$

[11] As mentioned in [12], since our k-diagrams involve only a finite number of values and restriction maps, to work in homological algebra we need to take only finite limits of our diagrams, which are again diagrams whose values are finite dimensional. See also the last paragraph of this section.

and to apply the five-lemma.

Lemma 10.1 shows that $\underline{k}$ is isomorphic to the chain

$$\bigoplus_{i=1}^{2} \mathrm{CoSky}_{A_i}(k) \xrightarrow{\mu_1} \bigoplus_{j=1}^{3} \mathrm{CoSky}_{B_j}(k)$$

with the A_i coskyscrapers in degree 0. Hence $S(\underline{k})$ is given by

$$\bigoplus_{i=1}^{2} \mathrm{Sky}_{A_i}(k) \xrightarrow{\tilde{\mu}_1} \bigoplus_{j=1}^{3} \mathrm{Sky}_{B_j}(k)$$

with the A_i skyscrapers in degree 0, and where $\tilde{\mu}_1$ is μ_1 with each morphism of coskyscrapers replaced by the corresponding one of skyscrapers. The proof of Theorem 10.1 shows that μ^1 of (109) is the map with the coefficients (107) which are the same as for μ_1; hence $\tilde{\mu}_1 = \mu^1$. It follows that $S(\underline{k})$ is isomorphic to the element of $\mathcal{D}$ which consists of the single non-zero k-diagram $\underline{k}_{/B_1,B_2}$ in degree -1, i.e., the element $\underline{k}_{/B_1,B_2}[1]$. Hence

$$S(\underline{k}) \simeq \underline{k}_{/B_1,B_2}[1].$$

We remark that—as mentioned in [12]—because k-diagrams are built on a diagram with finitely many values and morphisms, there is no problem in working with k-diagrams whose values are finite-dimensional vector spaces. The problem is that this does not allow us to work with the $\mathcal{M}_{W,\mathbf{d}}$, and the Hom sets involving the $\mathcal{M}_{W,\mathbf{d}}$ are not finite dimensional. One might still be able to work in this larger context and infer that $S(\underline{k}) = \underline{k}_{/B_1,B_2}[1]$ in such a context, as this equality is essentially the content of Theorem 10.1. Finally, we remark that the study of $\mathcal{O}$-modules with $\mathcal{O} = \mathcal{O}_{k,r}$ as in the previous subsection, in the case where W is r-periodic, may allow for a calculation of the Serre functor in some sense.

A Remarks Related to the Definition of a Dual Virtual Vector Space

In this appendix, we make some remarks to indicate why Definition 9.1 may be part of a larger notion of a "morphism" from one virtual k-vector space to another. However, our remarks do not yield a satisfactory notion of a morphism that is compatible with our notion of equivalence of two virtual k-vector spaces. This appendix is independent of the rest of this paper.

While we limit our discussions to virtual vector spaces, ideally one would have similar remarks for virtual objects of more general categories (such as the category of k-Fredholm maps and of k-diagrams).

We hope that future work will give a better justification for Definition 9.1. Ideally, future work would construct a category (or bicategory) of virtual vector spaces such that a morphism from $V_1 \ominus V_2$ to $k = k \ominus 0$ is precisely $V_1^* \ominus V_2^*$.

Note that if V_1, V_3 are finite dimensional k-vector spaces, then $V_1 \otimes_k V_3$ and $\mathrm{Hom}_k(V_1, V_3)$ are k-vector spaces, both of dimension $\dim(V_1)\dim(V_3)$. Hence, the following definition maintains this dimension under equivalence.

Definition A.1. *If V_1, V_2, V_3, V_4 are four finiite dimensional k-vector spaces, then we define*

$$\mathrm{Hom}\big(V_1 \ominus V_2, V_3 \ominus V_4\big) = \mathrm{Hom}(V_1, V_3) \oplus \mathrm{Hom}(V_2, V_4) \ominus \mathrm{Hom}(V_1, V_4) \ominus \mathrm{Hom}(V_2, V_3)$$

and

$$(V_1, V_2) \otimes (V_3, V_4) = (V_1 \otimes V_3) \oplus (V_2 \otimes V_4) \ominus (V_1 \otimes V_4) \ominus (V_2 \otimes V_3).$$

The problem with the above definition is that if $V_1 \ominus V_2$ is equivalent to $V_1' \ominus V_2'$, as virtual vector spaces, and similarly with $V_3 \ominus V_4$ and $V_3' \ominus V_4'$, it is not clear how to relate the two sets

$$\mathrm{Hom}\big(V_1 \ominus V_2, V_3 \ominus V_4\big),\ \mathrm{Hom}\big(V_1' \ominus V_2', V_3' \ominus V_4'\big).$$

To make Definition A.1 compatible with equivalence, one might be able to change the Hom sets to get a category, analogous to how one constructs the derived category (i.e., where one works with Homs of chain maps by first taking chain maps modulo homotopy equivalence, and then one "localizes" the category so that quasi-isomorphisms become isomorphisms).

However, if we ignore the above difficulties, and merely work with Definition A.1 as is, this definition gives

$$\mathrm{Hom}(V_1 \ominus V_2, k) = \mathrm{Hom}(V_1 \ominus V_2, k \ominus 0) = (V_1^*, V_2^*),$$

which therefore gives that Definition 9.1 agrees with the usual definition. We also note that an "element" of $V_1 \ominus V_2$ would be an element of

$$\mathrm{Hom}(k, V_1 \ominus V_2) = \mathrm{Hom}(k \ominus 0, V_1 \ominus V_2) = \mathrm{Hom}(k, V_1) \ominus \mathrm{Hom}(k, V_2)$$

which suggests that an element of $V_1 \ominus V_2$—if such a thing makes sense—would be a formal difference of an element of V_1 "minus" an element of V_2.

We also note that for Hom in Definition A.1, there is a natural composition of

$$\mu \in \mathrm{Hom}(V_1 \ominus V_2, V_3 \ominus V_4), \quad \nu \in \mathrm{Hom}(V_3 \ominus V_4, V_5 \ominus V_6),$$

namely if $\mu_{ij} \colon \mathrm{Hom}(V_i, V_j)$ with $i = 1, 2$ and $j = 3, 4$, and $\nu_{j\ell} \colon \mathrm{Hom}(V_j, V_\ell)$ with $j = 3, 4$ and $\ell = 5, 6$, then we could set $\nu \circ \mu$ to be the maps $(\nu \circ \mu)_{i\ell}$ given by

$$(\nu \circ \mu)_{i\ell} = \sum_{j=3,4} \nu_{j\ell} \circ \mu_{ij}.$$

Under this composition law, Hom is associative, and has an identity morphism for $V_1 \ominus V_2$, namely $\mathrm{Id}_{V_1} \oplus \mathrm{Id}_{V_2}$.

A specific problem with the above definitions is that the virtual vector $k \ominus k$ is equivalent to 0, and yet there is no isomorphism between these virtual vector spaces, since any morphism in $k \ominus k$ that factors through 0 must be the zero map, and hence cannot equal the identity map of $k \ominus k$.

We remark that an isomorphism of k-vector spaces $V_1^* \to V_3$ is the same as a perfect (or non-degenerate) pairing, $V_1 \times V_3 \to k$, and such a pairing extends to a map $V_1 \otimes V_3 \to k$. Hence, as an alternative to working with "dual spaces" and making sense of what constitutes an "isomorphism"

$$(V_1 \ominus V_2)^* \to V_3 \ominus V_4, \tag{117}$$

one could try to define a reasonable notion of a "pairing" and "perfect pairing"

$$(V_1 \ominus V_2) \times (V_3 \ominus V_4) \to k. \tag{118}$$

In fact, the isomorphism (117) in Theorem 9.3 is really based on a pairing $V_1 \times V_3$ and one $V_2 \times V_4$. Therefore, if one accepts that two such pairings should give rise to a pairing (118), then one can avoid any reference to dual spaces and isomorphism (117). Again, the link between pairings and tensor products is that in an ideal setting, a pairing (118) would "extend" to a map

$$(V_1 \ominus V_2) \otimes (V_3 \ominus V_4) \to k.$$

Hence, one starting point for finding an ideal setting is to search for a notion of the tensor product of virtual vector spaces, which may be related to Definition A.1 above.

Another approach to making the formal differences of k-vector spaces (or of k-Fredholm maps, or of k-diagrams, etc.) is to declare $\mathrm{Hom}(V_1 \ominus V_2, V_3 \ominus V_4)$ to equal any morphism

$$f \colon V_0 \oplus V_1 \oplus V_4 \to V_0 \oplus V_2 \oplus V_3,$$

since equivalence of virtual k-vector spaces is the case when a map as above is an isomorphism. This notion of morphism does have a composition law, although it seems that the composition law is not associative (hence we may get some type of 2-category or bicategory): namely, if $g \in \mathrm{Hom}(V_3 \ominus V_4, V_5 \ominus V_6)$, i.e.,

$$g \colon V_0' \oplus V_3 \oplus V_6 \to V_0' \oplus V_4 \oplus V_5,$$

one could add to f the identity map $\mathrm{id}_{V_0' \oplus V_6}$ and get a composition

$$(V_0' \oplus V_6) \oplus (V_0 \oplus V_1 \oplus V_4) \xrightarrow{\mathrm{id}_{V_0' \oplus V_6} \oplus f} (V_0' \oplus V_6) \oplus (V_0 \oplus V_2 \oplus V_3)$$

which can be composed with the morphism

$$(V_0 \oplus V_2) \oplus (V_0' \oplus V_3 \oplus V_6) \xrightarrow{\mathrm{id}_{V_0 \oplus V_2} \oplus g} (V_0 \oplus V_2) \oplus (V_0' \oplus V_4 \oplus V_5)$$

by rearranging the direct sums of the domain of $\mathrm{id}_{V_0 \oplus V_2} \oplus g$, which upon rearranging direct sums is a map

$$V_0'' \oplus V_1 \oplus V_6 \to V_0'' \oplus V_2 \oplus V_5, \quad \text{where } V_0'' = V_0 \oplus V_0' \oplus V_4.$$

However, it is not clear to us that this notion makes the Hom set from a virtual k-vector space to another having dimension equal to the product of the two vector spaces, which seems like a desirable property.

Acknowledgements

We are grateful to a referee for the paper who made numerous corrections and many suggestions to improve the exposition.

References

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269.

[2] O. Amini and M. Manjunath, *Riemann-Roch for sub-lattices of the root lattice* A_n, Electron. J. Combin. 17(1) (2010), Research Paper 124.

[3] Y. An, M. Baker, G. Kuperberg, and F. Shokrieh, *Canonical representatives for divisor classes on tropical curves and the matrix-tree theorem*, Forum Math. Sigma 2 (2014), Paper No. e24.

[4] S. Backman, *Riemann-Roch theory for graph orientations*, Adv. Math. 309 (2017), 655–691.

[5] M. Baker and S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Adv. Math. 215(2) (2007), 766–788.

[6] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. 53(6) (1989), 1183–1205.

[7] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. 125(3) (2001), 327–344.

[8] R. Cori and Y.L. Borgne, *The Riemann-Roch theorem for graphs and the rank in complete graphs*, available at `https://arxiv.org/abs/1308.5325`, 2013.

[9] N. Folinsbee and J. Friedman, *Periodic Riemann functions and duality*, to appear.

[10] N. Folinsbee and J. Friedman, *Sheaves and duality in the two-vertex graph riemann-roch theorem*, available at `https://arxiv.org/abs/1712.08695`, 2017.

[11] N. Folinsbee and J. Friedman, *Generalized Riemann functions, their weights and their evaluation on complete graphs*, Electron. J. Combin. 30(2) (2023), Paper No. 2.33.

[12] J. Friedman, *Cohomology of Grothendieck topologies and lower bounds in Boolean complexity*, available at `http://arxiv.org/abs/cs/0512008` and `http://www.math.ubc.ca/~jf/pubs/web_stuff/groth1.pdf`, 2005.

[13] J. Friedman, *Sheaves on graphs, their homological invariants, and a proof of the Hanna Neumann conjecture: with an appendix by Warren Dicks*, Mem. Amer. Math. Soc. 233(1100) (2015), xii+106. With an appendix by Warren Dicks.

[14] M. Hall, Jr, *Combinatorial theory*, Wiley Classics Library. John Wiley & Sons, Inc., New York, second edition, 1998. A Wiley-Interscience Publication.

[15] R. Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.

[16] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[17] C. A. Weibel, *An introduction to homological algebra*, volume 38 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1994.