Associative-commutative Spectra for Some Varieties of Groupoids

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**Abstract:** The associative spectrum of a groupoid (i.e., a set with a binary operation) measures its nonassociativity while the associative-commutative spectrum measures both nonassociativity and noncommutativity of the groupoid. The two spectra are also the coefficients of the Hilbert series of certain operads. We establish upper bounds for the two spectra of various varieties of groupoids defined by different sets of identities and provide examples (often groupoids with three elements) for which the upper bounds are achieved. Our results have connections to many interesting combinatorial objects and integer sequences and naturally lead to some questions for future studies.

**Keywords:** Associative-commutative spectrum; Associative spectrum; Binary operation; Tree; 3-element groupoid

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1. Introduction

A groupoid \((G, \ast)\) is a basic algebraic structure that consists of a set \(G\) together with a binary operation \(\ast\) defined on \(G\). Associativity and commutativity are common properties that could be satisfied by a groupoid. Csinkány and Waldhauser [3] defined the associative spectrum (also called the subassociativity type by Breit and Silberger [2]) to measure the failure of a groupoid to be associative, and we introduced the associative-commutative spectrum, or simply ac-spectrum, to measure both nonassociativity and noncommutativity of a groupoid in earlier work [6]; see the definition below.

**Definition 1.1.** Fix a countable list of distinct variables \(x_1, x_2, \ldots\). Let \(B_n\) denote the set of all bracketings of \(x_1, x_2, \ldots, x_n\), which are terms in the language of groupoids obtained by inserting pairs of parentheses into the word \(x_1 x_2 \cdots x_n\) in all valid ways. Let \(F_n\) denote the set of full linear terms over \(x_1, x_2, \ldots, x_n\), which are obtained by permuting the variables in the bracketings of \(x_1, x_2, \ldots, x_n\). We can view \(B_n\) as a subset of \(F_n\). Every term \(t \in F_n\) induces an \(n\)-ary operation \(t^*\) on a groupoid \((G, \ast)\). It is often convenient to think about the terms in \(F_n\) or the \(n\)-ary operations induced by them in terms of the corresponding (ordered, full) binary trees with \(n\) labeled leaves; see the example below for \(B_4\), which can give \(F_4\) if the variables are permuted in all possible ways.

\[
\begin{align*}
(\ast x_1 x_2 (x_3 \ast x_4)) & \quad (x_1 \ast x_2) (\ast x_3 x_4) & \quad (x_1 \ast (x_2 \ast x_3)) \ast x_4 & \quad x_1 \ast ((x_2 \ast x_3) \ast x_4) & \quad x_1 \ast ((x_2 \ast (x_3 \ast x_4))
\end{align*}
\]

The associative spectrum (resp., ac-spectrum) of a groupoid \((G, \ast)\), or of its binary operation \(\ast\), is a sequence whose \(n\)th term is \(s_n^\ast(\ast) := |P_n(\ast)|\) (resp., \(s_n^{ac}(\ast) := |\mathcal{P}_n(\ast)|\)), where \(P_n(\ast) := \{t^* : t \in B_n\}\) (resp., \(\mathcal{P}_n(\ast) := \{t^* : t \in F_n\}\)), for \(n = 1, 2, \ldots\). It turns out that \(\{P_n(\ast)\}_{n \geq 1}\) (resp., \(\{\mathcal{P}_n(\ast)\}_{n \geq 1}\)) together with a composition function becomes a nonsymmetric operad (resp., symmetric operad) that satisfies certain coherence axioms [11], and the Hilbert series of this operad is the generating function (resp., exponential generating function) of the associative spectrum (resp., ac-spectrum) of \((G, \ast)\).

By the above definition, we have (1) \(s_1^\ast(\ast) = 1 = s_1^{ac}(\ast)\) for \(n = 1, 2\), (2) \(s_3^\ast(\ast) = 1\), and (3) \(s_3^{ac}(\ast)\) is either 1 or 2, depending on whether \(\ast\) is commutative. Thus we may assume \(n \geq 3\) when necessary. It is easy to see that...
isomorphic or anti-isomorphic groupoids have the same associative spectrum and the same ac-spectrum, where two groupoids \((G, \ast)\) and \((H, \otimes)\) are said to be anti-isomorphic, denoted by \(G \simeq H^{\text{op}}\), if there is a bijection \(f : G \to H\) such that \(f(a \ast b) = f(a) \otimes f(b)\) for all \(a, b \in G\).

It is clear that \(s^a_n(\ast) = 1\) for all \(n \in \mathbb{N}\) if and only if \(\ast\) is associative and that \(s^a_m(\ast) = 1\) for all \(n \in \mathbb{N}\) if and only if \(\ast\) is associative and commutative, where \(\mathbb{N} := \{1, 2, \ldots\}\). On the other hand, we have \(s^a_n(\ast) \leq C_{n-1}\), where \(C_n := \frac{1}{n+1}\binom{2n}{n}\) is the ubiquitous Catalan number, and thus \(s^a_n(\ast) \leq n!C_{n-1}\). We showed in previous work [6] that a commutative groupoid \((G, \ast)\) must have \(s^a_{nc}(\ast) \leq D_{n-1}\), where \(D_n := \frac{(2n)!}{(2^n)n!}\) is the solution to Schröder’s third problem [13, A001147], and that an associative groupoid \((G, \ast)\) must have \(s^a_{nc}(\ast) \leq n!\), which holds as an equality if the groupoid is noncommutative and has an identity element (see Theorem 7.1 for a generalization).

In addition, the precise values of the associative spectrum and ac-spectrum have been determined for various groupoids [3–6, 8, 9], including 2-element groupoids, generalizations of addition and subtraction, exponentiation, arithmetic/geometric/harmonic mean, cross product, Lie algebras with an \(sl_2\)-triple, graph algebras, and so on. The results show connections with interesting combinatorial objects, avoided patterns, and integer sequences. However, the ac-spectra of 3-element groupoids are largely undetermined.

\[
\begin{array}{c|c|c|c}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 \\
\end{array}
\end{array}
\]

Table 1: Some 3-element groupoids

According to the Siena Catalog [1], there are 3330 non-isomorphic 3-element groupoids, which are indexed from 1 to 3330. Each of these groupoids is determined by a binary operation \(\ast\) defined on the set \(\{0, 1, 2\}\). We write them as SC1, SC2, \ldots, SC3330. There are 729 idempotent 3-element groupoids, which can be labeled in a different way: ID0, ID1, \ldots, ID728. Csákány and Waldhauser [3] showed the following (see Table 1).

- Both ID35 = SC271 \(\simeq SC1610^{\text{op}}\) and ID68 = SC356 \(\simeq SC2032^{\text{op}}\) have associative spectrum \(s^a_n(\ast) = 2^{n-2}\) for \(n \geq 2\).
- Both SC1066 and SC10 \(\simeq SC367^{\text{op}}\) have associative spectrum \(s^a_n(\ast) = n - 1\) for \(n \geq 1\).
- Both SC405 and SC3242 \(\simeq SC3302^{\text{op}}\) have associative spectrum \(s^a_n(\ast) = 3\) for \(n \geq 3\) (it is easy to check that \(s^a_1(\ast) = 1\) for \(n = 1, 2\) and \(s^a_1(\ast) = 2\) for \(n = 3\)).
- The groupoid SC79 has associative spectrum \(s^a_n(\ast) = F_{n+1} - 1\) for \(n \geq 2\), where \(F_{n+1}\) is the Fibonacci number defined by \(F_{n+1} := F_n + F_{n-1}\) for \(n \geq 1\) and \(F_1 = i\) for \(i = 0, 1\).

Our original motivation for this work was to determine the ac-spectra of the above 3-element groupoids, whose Cayley tables are given in Table 1. However, we are able to establish more general results on various varieties of groupoids, where a variety of groupoids axiomatized by a set \(\Sigma\) of identities is the family of all groupoids satisfying the identities in \(\Sigma\). For each variety of groupoids considered in this paper, we establish an upper bound for the associative spectra and an upper bound for the ac-spectra of the groupoids belonging to this variety; if the latter upper bound is reached by a member of the variety, so is the former. Moreover, we show that both upper bounds are attained by at least one 3-element groupoid.

For example, we showed in earlier work [6] that a commutative groupoid must have \(s^a_{nc}(\ast) \leq D_{n-1}\) and if the equality in this upper bound holds, so does the equality in the upper bound \(s^a_n(\ast) \leq C_{n-1}\). In the same paper, we showed that \(s^a_{nc}(\ast) = D_{n-1}\) for a 3-element groupoid called the rock-paper-scissors groupoid, which turns out to be isomorphic to SC1108, and the proof is also valid for SC2407 and SC3093.

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Therefore, we have the following result.

**Theorem 1.1** ([6]). A groupoid \((G, *)\) satisfying the identity \(xy \approx yx\) must have \(s_n^a(*) \leq C_{n-1}^a\) and \(s_n^{ac}(*) \leq D_{n-1}\) for \(n = 1, 2, \ldots\), where the first inequality holds as an equality whenever the second does and both equalities hold for the 3-element groupoids \(SC_{1108}\), \(SC_{2407}\), and \(SC_{3903}\).

In this paper, we provide a series of results that are similar to the above one. A summary of our results is given by Table 2, where we use the well-known Bell number \(B_n\) counting partitions of the set \(\{1, 2, \ldots, n\}\) into unordered nonempty blocks, the restricted Bell number \(B_{n,m}\) counting partitions of \(\{1, 2, \ldots, n\}\) into unordered nonempty blocks of size at most \(m\) [12], and the ordered Bell number or Fubini number \(B'_n\) counting partitions of \(\{1, 2, \ldots, n\}\) into ordered nonempty blocks [13, A000670]. The “\(n \geq m\)” column in Table 2 gives the smallest values of \(n\) for which the upper bounds of \(s_n^a(*)\) and \(s_n^{ac}(*)\) are valid and sharp. Note that different varieties of groupoids in the table may have the same associative spectrum upper bound but different ac-spectrum upper bounds (the upper bounds for \(s_n^{ac}(*)\) in Prop. 3.2 and Prop. 3.3 are different when \(n = 3\)). Therefore, the ac-spectrum may offer a finer distinction between groupoids than the associative spectrum.

<table>
<thead>
<tr>
<th>Identities satisfied by ((G, *))</th>
<th>(n \geq m)</th>
<th>(s_n^a(*) \leq )</th>
<th>(s_n^{ac}(*) \leq )</th>
<th>Examples for =</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1, 1</td>
<td>(n)</td>
<td>(n)</td>
<td>SC275 (≈ SC20929(^{pp}))</td>
<td>Prop. 3.1</td>
</tr>
<tr>
<td>(2), (7), (15)</td>
<td>4, 4</td>
<td>(n + 1)</td>
<td>(n + 1)</td>
<td>SC405</td>
<td>Prop. 3.3</td>
</tr>
<tr>
<td>(3), (7), (11), (13), (17), (18)</td>
<td>4, 4</td>
<td>(2n^2)</td>
<td>(2n^2)</td>
<td>SC3162 (≈ SC24671(^{pp}))</td>
<td>Thm. 3.1</td>
</tr>
<tr>
<td>(2), (7)</td>
<td>2, 2</td>
<td>(2^{n-1} - 1)</td>
<td>(2^{n-1} - 1)</td>
<td>SC1066</td>
<td>Prop. 4.1</td>
</tr>
<tr>
<td>(4), (5), (7)</td>
<td>2, 2</td>
<td>(2^{n-2})</td>
<td>(2^n - 2)</td>
<td>SC2302 (≈ SC2155(^{pp}))</td>
<td>Prop. 4.3</td>
</tr>
<tr>
<td>(2), (11)</td>
<td>2, 2</td>
<td>(F_{n+1} - 1)</td>
<td>(B_{n-1} - 2)</td>
<td>SC79, SC1701</td>
<td>Prop. 5.1</td>
</tr>
<tr>
<td>(3), (5)</td>
<td>2, 2</td>
<td>(2^{n-2})</td>
<td>(nB_{n-1})</td>
<td>SC41 (≈ SC388(^{pp}))</td>
<td>Thm. 5.1</td>
</tr>
<tr>
<td>(5), (7)</td>
<td>2, 2</td>
<td>(nB_{n-1})</td>
<td>(nB_{n-1})</td>
<td>SC62 (≈ SC144(^{pp}))</td>
<td>Thm. 5.2</td>
</tr>
</tbody>
</table>

\((1) \, xy \approx x\) \((2) \, xy \approx yx\) \((3) \,(xy)z \approx (xz)y\) \((4) \, x(yz) \approx y(xz)\) \((5) \, x(yz) \approx x(yz)\) \((6) \, x(yz) \approx z(yx)\)
\((7) \, w(xyz) \approx w(xy)z\) \((8) \, (wx)yz \approx (wxy)z\) \((9) \, (wxy)z \approx (wxy)z\)
\((10) \, ((wx)y)z \approx ((wx)y)z\) \((11) \, ((wx)yz) \approx ((wx)yz)\) \((12) \, (wz)yz \approx (wxy)z\)
\((13) \, (wxy)z \approx (wxyz)\) \((14) \, w(xyz) \approx (wxyz)\) \((15) \, (wxy)(yz) \approx (wv)(xyz)\)
\((16) \, (vwx)(yz) \approx (vwx)(xyz)\)

Table 2: Summary of results

It is sometimes convenient to use not only identities but other conditions to describe a family of groupoids satisfying certain upper bounds for their spectra. Recall that every term \(t \in \mathcal{L}_n\) corresponds to a binary tree with \(n\) leaves labeled by \(1, \ldots, n\). Each leaf \(i\) has its depth \(d_i(t)\) (resp. left depth \(h_l(t)\) or right depth \(h_r(t)\)) defined as the number of edges (resp., left/right edges) in the unique path to the root of \(t\). By abuse of notation, we also speak of these three kinds of depths for the variables in \(t\). Previous work [4, 6] used the congruence modulo \(m\) relation on depths to study the associative spectra and ac-spectra of certain groupoids, and some of the results can be rephrased to include Proposition 4.3 as a special case. We can also generalize Proposition 3.4 and Proposition 3.5 in a similar way.

The paper is structured as follows. We give some basic definitions and properties on the associative spectrum and ac-spectrum in Section 2. We establish some polynomial upper bounds and exponential upper bounds in...
Section 3 and Section 4, respectively. We provide more upper bounds related to set partitions in Section 5. We use congruence on leaf depths in binary trees to provide generalizations of some of our results in Section 6. Finally, we make some remarks and pose some questions for future research in Section 7.

2. Preliminaries

We first give some notation and terminology. A term $t$ over a set of variables $X$ (we often use $X_n := \{x_1, \ldots, x_n\}$) is a bracketing of a word $x_{i_1} \cdots x_{i_k}$, where $x_{i_1}, \ldots, x_{i_k} \in X$; let $\text{var}(t)$ denote the set of all variables in $t$. If $i_1, \ldots, i_k$ are distinct, then $t$ is called a linear term with $|t| := k$. Define the leftmost bracketing $[t_1, \ldots, t_k]$ of terms $t_1, \ldots, t_k$ recursively by $[t_1] := t_1$ and $[t_1, \ldots, t_n] := ([t_1, \ldots, t_{n-1}][t_n])$ for $n \geq 1$. Similarly, define the rightmost bracketing $\langle t_1, \ldots, t_k \rangle$ recursively by $\langle t_1 \rangle := t_1$ and $\langle t_1, \ldots, t_n \rangle := (\langle t_1 \rangle \langle t_2, \ldots, t_{n+1} \rangle)$ for $n \geq 1$. We can write every term as $t = [t_0, t_1, \ldots, t_m]$ with $|t_0| = 1$ for some $m \in \mathbb{N}$; this is known as the leftmost decomposition [6, Definition 6.1.2], which can also be obtained by writing $t = (t_1t_R)$ if $t$ is not a variable, then further writing $t_R = (t_1', t_R')$ if the left subterm $t_1$ is not a variable, and continuing in this way to decompose left subterms until we reach one that is a single variable.

Terms can be evaluated in a groupoid $G, \ast$ as follows. Given an assignment $h : X \to G$ of values from $G$ for the variables in $X$, we can extend $h$ to a map $\bar{h}$ defined on the set of all terms over $X$ with the following recursive definition. We have $\bar{h}(x) := h(x)$ for every variable $x \in X$ (because $\bar{h}$ extends $h$), and if $t = (t_1t_2)$ for subterms $t_1$ and $t_2$, then we define $\bar{h}(t) := \bar{h}(t_1) \ast \bar{h}(t_2)$. In this way, every term $t$ over $X_n$ induces an $n$-ary operation $t^*$ on $(G, \ast)$ (called a term function): $t^*(a_1, \ldots, a_n) := \bar{h}(t)$, where $\bar{h}$ is the extension of the assignment $h$; $X_n \to G$ that maps $x_i$ to $a_i$ for all $i \in \{1, \ldots, n\}$. For notational simplicity, we will denote the extension $\bar{h}$ of an assignment $h$ also by $h$.

An identity is a pair of terms, usually written as $s \equiv t$. A groupoid $(G, \ast)$ satisfies an identity $s \equiv t$ if $s^* = t^*$. (Here we have assumed that $s$ and $t$ are terms over $X_n$ for some $n \in \mathbb{N}$ – this can always be done.)

In the subsequent sections, we will prove several results, each of which provides upper bounds for the asc-spectrum and the associative spectrum of a variety of groupoids axiomatized by a set $\Sigma$ of identities, i.e., the family of all groupoids satisfying the identities in $\Sigma$. We will employ the following proof technique. We assume that a groupoid $(G, \ast)$ satisfies certain identities. Using these identities, we transform each full linear term $t$ into an equivalent term $t'$ that is in “standard form” (terms $t$ and $t'$ are equivalent if $(G, \ast)$ satisfies $t \equiv t'$, i.e., $t^* = (t')^*$). It thus follows that $s_G^m(\ast)$, i.e., the number of term functions induced by full linear terms with $m$ variables on $(G, \ast)$, is bounded above by the number of terms in standard form, so it is then a matter of counting the standard forms. Similarly, finding $s_G^m(\ast)$ amounts to counting the standard forms that can be obtained from bracketings.

Let $t$ be a linear term. Assume that $\text{var}(t) = \{x_{i_1}, \ldots, x_{i_m}\}$ and that $x_{i_k}$ occurs to the left from $x_{i_\ell}$ in $t$ if and only if $k < \ell$. Assume further that $\{j_1, \ldots, j_m\} = \{1, \ldots, m\}$ and $j_1 < j_2 < \cdots < j_m$. Let

$$t^L := [x_{i_1}, \ldots, x_{i_m}], \quad t^L < := [x_{i_1}, \ldots, x_{i_m}],$$

$$t^R := \langle x_{i_1}, \ldots, x_{i_m} \rangle, \quad t^R < := \langle x_{i_1}, \ldots, x_{i_m} \rangle,$$

i.e., $t^L$ and $t^L<$ ($t^R$ and $t^R<$, resp.) are leftmost (rightmost, resp.) bracketings of the variables of $t$; in the former, the variables occur in the same order as in $t$, while in the latter, the variables occur in the increasing order of the indices.

The next lemma will be frequently used to establish our main results.

**Lemma 2.1.** (1) Let $(G, \ast)$ be a groupoid, and write an arbitrary term in $F_\infty$ as $t = [t_0, t_1, \ldots, t_m]$ with $|t_0| = 1$ (leftmost decomposition).

(i) If $(G, \ast)$ satisfies the identity $w(xy)z \approx w(xy)z$, then $(G, \ast)$ also satisfies the identities $t \approx [t_0, t^L_1, \ldots, t^L_m]$ and $t \approx [t_0, t^R_1, \ldots, t^R_m]$.

(ii) If $(G, \ast)$ satisfies the identities $w(xy)z \approx w(xy)z$ and either $x(yz) \approx x(zy)$ or $xy \approx yx$, then $(G, \ast)$ also satisfies the identities $t \approx [t_0, t^L_1, \ldots, t^L_m]$ and $t \approx [t_0, t^R_1, \ldots, t^R_m]$.

(iii) If $(G, \ast)$ satisfies the identity $(xy)z \approx (zx)y$, then $(G, \ast)$ also satisfies the identity $t \approx [t_0, t_{\sigma(1)}, \ldots, t_{\sigma(m)}]$ for every permutation $\sigma \in S_m$.

(iv) If $(G, \ast)$ satisfies the identities $x(yz) \approx x(zy)$ and $(xy)z \approx (zx)y$, then $(G, \ast)$ also satisfies the identity $t \approx [t_0, t^L_1, \ldots, t^L_m]$.

*More specifically, we are speaking about terms in the language of groupoids, i.e., terms of type (2). Since our language contains only one operation symbol, which is binary, we may simply omit it from terms. Variables and brackets are sufficient for writing down terms unambiguously in this language.
Proof. (i) We can use the identity \( w(x(yz)) \approx w((xy)z) \) repeatedly to transform each \( t_i \) to the form \( x_j \), where \( x_j \) is the leftmost variable of \( t_i \), and then apply the same procedure to \( s \) to eventually transform \( t_i \) into \( t_R^i \). A similar argument shows that each \( t_i \) can be transformed into \( t_R^i \).

(ii) By (i), \((G, \ast)\) satisfies \( t \approx [t_0, t_R^1, \ldots, t_R^m] \). We may arbitrarily permute the variables in each \( t_R^i \), \( i \in \{1, \ldots, m\} \), thanks to the identities

\[
\begin{align*}
w(x(yz)) & \approx w(x(yz)) \approx w((xy)z) \approx w(z(xy)) \approx w((yx)z) \approx w(y(xz)) \approx w(y(yz)).
\end{align*}
\]

Thus \((G, \ast)\) satisfies \( t \approx [t_0, t_R^1, \ldots, t_R^m] \). A similar argument shows that \((G, \ast)\) satisfies \( t \approx [t_0, t_R^1, \ldots, t_R^m] \) for any \( i \in \{1, \ldots, m - 1\} \). Since the adjacent transpositions generate \( S_m \), it follows that \((G, \ast)\) satisfies \( t \approx [t_0, t_R^1, \ldots, t_R^m] \) for every \( \sigma \in S_m \).

(iii) We may use the identity \((xy)z \approx (xz)y\) to swap the subterms \( t_i \) and \( t_{i+1} \) in \([t_0, t_1, \ldots, t_m]\), for any \( i \in \{1, \ldots, m - 1\} \).

(iv) By (iii), we can permute the subterms \( t_1, \ldots, t_m \), so it suffices to prove that \((G, \ast)\) satisfies \( x \ast x \approx x^{(3^{b-\ell})} \) for any linear term \( s \) with \( x \not\in \text{var}(s) \). We prove this by induction on \(|s|\). This is trivial when \(|s| = 1 \) and this holds for \(|s| = 2 \) by the identity \( x(yz) \approx x(yz) \).

By the inductive hypothesis and (iii), we may assume that \( s_j = s^{(k-\ell)}_0 \) for all \( j \in \{1, \ldots, \ell\} \). Consequently, \((G, \ast)\) satisfies \( x \ast x \approx x^{(s^{(k-\ell)}_0 \ast \ast \ell)} \) for every \( \ast \ast \ell \). We may swap \( s^{(k-\ell)}_0 \ast \ast \ell \) if necessary to obtain a term of the form \([x_1, \ldots, x_i]([x_1, \ldots, x_i]) \), where \( i_1 < \cdots < i_k \) and \( i_k+1 < \cdots < i_m \) and \( i_1 < i_k+1 \). Using the identities \((x(yz)) \approx x(yz) \) and \((x(yz)) \approx x(yz) \), we obtain

\[
x \ast x \approx [x, [x_{i_1}, \ldots, x_{i_m}]]([x_{i_1}, \ldots, x_{i_m}])] = [x, [x_{i_1}, \ldots, x_{i_m}]]([x_{i_1}, \ldots, x_{i_m}]) = \cdots \approx [x, [x_{i_1}, \ldots, x_{i_m}]]([x_{i_1}, \ldots, x_{i_m}]) = \cdots \approx [x, [x_{i_1}, \ldots, x_{i_m}]]([x_{i_1}, \ldots, x_{i_m}]).
\]

Since \( i_1 \) is the smallest of the indices \( i_1, \ldots, i_m \) we can then apply the identity \((xy)z \approx (xz)y\) and (iii) to sort the variables in the subterm \([x_{i_1}, \ldots, x_{i_k}, \ldots, x_{i_m}] \) in the increasing order of indices, and we obtain \( x \ast x \approx x^{(s^{(k-\ell)}_0 \ast \ast \ell)} \), as desired.

\[\square\]

3. Polynomial upper bounds

In this section, we establish some polynomial upper bounds for the \( a \)-spectra of groupoids belonging to certain varieties of groupoids; in contrast, their associative spectra all have constant upper bounds.

For our first variety of groupoids, we can actually determine their associative spectrum and ac-spectrum.

**Proposition 3.1.** A groupoid \((G, \ast)\) with at least two elements satisfying the identity \( xy \approx x \) must have \( s_n^a(\ast) = 1 \) and \( s_n^a(\ast) = n \) for \( n \geq 1 \). In particular, the above two equalities hold for the 2-element groupoid \((\{0, 1\}, \ast)\) defined by \( x \ast y := x \) for all \( x, y \in \{0, 1\} \) and the 3-element groupoids SC275 and SC2029.

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| 1 | 0 | 1 | 2 |
| 2 | 0 | 1 | 2 |

**SC275** SC2029

\[\square\]

Proof. If \((G, \ast)\) is a groupoid with at least two elements satisfying the identity \( xy \approx x \), then \( s_n^a(\ast) = 1 \) and \( s_n^a(\ast) = n \) for all \( n \geq 1 \) since the \( n \)-ary operation \( t^\ast \) induced by every term \( t \in \mathcal{F}_n \) is determined by the leftmost variable of \( t \) and distinct variables induce distinct operations.

In earlier work [6, Example 4.1.2], we showed that the 2-element groupoid \((\{0, 1\}, \ast)\) with \( x \ast y := x \) for all \( x, y \in \{0, 1\} \) has \( s^a_n(\ast) = 1 \) and \( s^a_n(\ast) = n \) for \( n \geq 1 \). One can check that SC275 satisfies the identity \( xy \approx x \) and that SC2029 is anti-isomorphic to SC275. Thus their associative spectrum and ac-spectrum are also given as above.

The upper bounds in the next result are achieved by the 3-element groupoids SC7 and SC28, which are anti-isomorphic to SC4 (by swapping 1 and 2) and SC5, respectively.
Proof. Let $t$ be an arbitrary term in $F_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $t_0 = x_a$ for some $a \in \{1, 2, \ldots, n\}$. By Lemma 2.1, we may assume that $t_i = t_i^{R<}$ for all $i \in \{1, \ldots, m\}$ and $|t_1| \leq \cdots \leq |t_m|$. We then distinguish the two cases below.

- If $|t_m| > 1$, then $t_m = \langle x_{b_1}, \ldots, x_{b_n} \rangle$ and we can apply (ii) to swap the leftmost variable of $t_m$ and $[x_{a_1}, t_{m-1} \ldots, x_{a_n}]$. The resulting term $x_{b_1} \langle [x_{a_1}, t_{m-1} \ldots, x_{a_n}] \rangle$ can be transformed to

\[
\langle x_{b_1}, x_{b_2}, \ldots, x_{b_n} \rangle
\]

by Lemma 2.1. Then we can apply (ii) to swap $x_{b_1}$ with $x_1$, and finally we can turn the term into

\[
\langle x_1, \ldots, x_n \rangle.
\]

- If $|t_m| = 1$, then $t = [x_{a_1}, x_{b_1}, \ldots, x_{b_{n-1}}]$, and we can apply (i) to make sure $b_1 < \cdots < b_{n-1}$.

It follows that $s_{ac}^n(\ast) \leq n + 1$ since there are $n$ possibilities for $a$ in the second case. If the variables $x_1, \ldots, x_n$ are ordered increasingly in $t$, then we must have $a = 1$ in the second case. Thus $s_{ac}^n(\ast) \leq 2$. If $s_{ac}^n(\ast) = n + 1$, then the two cases above cannot yield identical $n$-ary operations on $(G, \ast)$, and thus $s_{ac}^n(\ast) = 2$.

Now we determine $s_{ac}^n(\ast)$ and $s_{ac}^{n+1}(\ast)$ for SC7. It is routine to check that SC7 satisfies the identities (i), (ii), and (iii). Let $t$ be an arbitrary term in $F_n$. We may assume that $t = \langle x_1, \ldots, x_n \rangle$ or $t = [x_1, x_{b_1}, \ldots, x_{b_{n-1}}]$, with $b_1 < \cdots < b_{n-1}$ by the above argument. For the former, we have $h(t) = 0$ for all $h : X_n \to \{0, 1, 2\}$. For the latter, we have $h(t) = 2$ if $h(x_1) = 2$ and $h(x_{b_1}) = \cdots = h(x_{b_{n-1}}) = 1$ or $h(t) = 0$ otherwise. Therefore $s_{ac}^n(\ast) = n + 1$, which implies $s_{ac}^{n+1}(\ast) = 2$.

In a similar way, we can determine $s_{ac}^n(\ast)$ and $s_{ac}^{n+1}(\ast)$ for SC28, which also satisfies the identities (i), (ii), and (iii). If $t = \langle x_1, \ldots, x_n \rangle$, then $h(t) = 0$ for all $h : X_n \to \{0, 1, 2\}$. If $t = [x_1, x_{b_1}, \ldots, x_{b_{n-1}}]$, then $h(t) = 1$ if $h(x_1) \in \{1, 2\}$ and $h(x_{b_1}) = 2$ for $i = 1, \ldots, n - 1$, or $h(t) = 0$ otherwise. It follows that $s_{ac}^n(\ast) = n + 1$, which implies $s_{ac}^{n+1}(\ast) = 2$.

The upper bounds in the next result are very close to but not the same as those in Proposition 3.2.

**Proposition 3.3.** A groupoid $(G, \ast)$ satisfying the identities below must have $s_{ac}^n(\ast) \leq 2$ and $s_{ac}^{n+1}(\ast) \leq 3$ for $n = 3$ and $s_{ac}^n(\ast) \leq 3$ and $s_{ac}^{n+1}(\ast) \leq n + 1$ for $n = 4, 5, \ldots$

(i) $xy \approx yx$, (ii) $w(xyz) \approx w(yzx)$, (iii) $v(w(xyz)) \approx (vwy)(xyz)$

If $s_{ac}^n(\ast)$ reaches its upper bound, so does $s_{ac}^{n+1}(\ast)$, and both upper bounds are reached by SC405 (see Table 1).

**Proof.** Let $t$ be an arbitrary term in $F_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $|t_0| = 1$. By (i), (ii), and Lemma 2.1, we can assume that $t_i = t_i^{R<}$ for all $i \in \{1, \ldots, m\}$. We can use (i) to swap $u := [t_0, t_1, \ldots, t_{m-1}]$ and $t_m$. By Lemma 2.1, the resulting term $t_m u$ is equivalent to $t_m u^{R<}$. Because $t_m = t_m^{R<}$, we can apply (i) again to transform this into

\[
u^{R<} = \langle x_{i_1}, \ldots, x_{i_k} | x_{i_{k+1}}, \ldots, x_{i_n} \rangle,
\]

where $\{x_1, \ldots, x_k \}$ is $\var{t_0, t_1, \ldots, t_{m-1}}$ and $\{x_{i_{k+1}}, \ldots, x_{i_n} \} = \var{t_m}$ with $i_1 < \cdots < i_k$ and $i_{k+1} < \cdots < i_n$. Note that $i_j = j$ for $j = 1, \ldots, n$ if $t \in B_n$. If $k = 2$, then we can show that

\[
\langle x_{i_1}, x_{i_2} | x_{i_3}, \ldots, x_{i_n} \rangle = \langle x_{i_1}, x_{i_2} \rangle \langle x_{i_3}, \ldots, x_{i_n} \rangle.
\]

We have either $i_1 = 1$ or $i_2 = 1$. If $i_1 = 1$, then we can do the following transformations to make the leftmost index 1.

\[
\langle i_1, i_2 | i_3, \ldots, i_n \rangle \xrightarrow{\text{(iii)}} \langle i_1, i_2, i_3 | i_4, \ldots, i_n \rangle \xrightarrow{\text{(i)}} \langle i_4, \ldots, i_5 | i_1, i_2, i_3 \rangle
\]

\[
\xrightarrow{\text{Lemma 2.1}} \langle i_4, \ldots, i_5 | i_3, i_1, i_2 \rangle \xrightarrow{\text{(i)}} \langle i_3, i_1, i_2 | i_4, \ldots, i_5 \rangle \xrightarrow{\text{(iii)}} \langle i_3, i_1, i_2 | i_5, i_4, \ldots, i_7 \rangle.
\]
Here we drop $x$ for ease of notation and represent an application of an identity by an arrow with the label of the identity above it. Similarly, we can make the second leftmost index 2 and then make the rest 3, ..., $n$. If $3 \leq k \leq n - 2$, then we have

$$
\langle i_1, \ldots, i_k | (i_{k+1}, \ldots, i_n) \rangle \to \langle i_1, i_2 | (i_{3}, \ldots, i_k | (i_{k+1}, \ldots, i_n) \rangle \text{ (Lemma 2.1)} \langle i_1, i_2 | (i_{3}, \ldots, i_n) \rangle.
$$

Here the application of (iii) uses $v = x_{i_1}$, $w = x_{i_2}$, $x = (x_{i_3}, \ldots, x_{i_k})$, $y = x_{i_{k+1}}$, and $z = (x_{i_{k+2}}, \ldots, x_{i_n})$. Thus $t$ induces the same $n$-ary operation on $(G, \ast)$ as one of the following “standard” terms

$$
x_{i_1}(x_{i_2}, \ldots, x_{i_n}), \quad (x_{i_1}, \ldots, x_{i_{n-1}})(x_{i_n}), \quad (x_1, x_2)(x_3, \ldots, x_n).
$$

The first standard term is determined by $i_1$ since $i_2 < \cdots < i_n$ and the second is determined by $i_n$ since $i_1 < \cdots < i_{n-1}$. Moreover, $x_{i_1}(x_{i_2}, \ldots, x_{i_n})$ and $(x_{i_1}, \ldots, x_{i_{n-1}})(x_{i_n})$ induce the same $n$-ary operation on $(G, \ast)$ if $i_1 = i_n$ by (i). Thus there are $n$ possibilities in total for the first two standard terms. On the other hand, the last standard term $(x_1, x_2)(x_3, \ldots, x_n)$ does not occur when $n = 3$. Thus $s_n^{sc}(\ast) \leq 3$ when $n = 3$ and $s_n^{sc}(\ast) \leq 3$ for $n \geq 4$.

If $t \in B_n$ is a bracketing of $x_1, \ldots, x_n$, then by the above argument, it induces the same $n$-ary operation on $(G, \ast)$ as one of $x_1(x_2, \ldots, x_n)$, $(x_1, \ldots, x_{n-1})x_n$, or $(x_1, x_2)(x_3, \ldots, x_n)$. Thus $s_n^{sc}(\ast) \leq 2$ for $n = 3$ and $s_n^{sc}(\ast) \leq 3$ for $n \geq 4$. It is easy to see that the equality holds in the upper bound for $s_n^{sc}(\ast)$ when the equality holds in the upper bound for $s_n^{ac}(\ast)$.

Now we consider SC405. Write an arbitrary term $t \in F_n$ as $t = (t_L)(t_R)$, where $t_L$ and $t_R$ are linear terms. Also view $t$ as a bracketing of $x_1, \ldots, x_n$. We distinguish the following cases on $|t_L|$ and $|t_R|$.

(i) If $|t_L| = 1 < |t_R|$ then $t^{\ast}$ evaluates to 0 or 1, so $t^{\ast}(a_1, \ldots, a_n) = [a_{i_1}/2]$.

(ii) If $|t_L| > 1 \geq |t_R|$ then $t^{\ast}(a_1, \ldots, a_n) = [a_{i_n}/2]$ for the same reason as above.

(iii) If $|t_L| \geq 2$ and $|t_R| \geq 2$ then $t^{\ast}_L$ and $t^{\ast}_R$ both evaluate to 0 or 1, so $t^{\ast}$ is always zero.

For $n = 3$ we must have (i) or (ii), so $t^{\ast}(a_1, a_2, a_3) = [a_i/2]$, where $i$ varies in $\{1, 2, 3\}$. Thus $s_n^{ac}(\ast) = 3$ for $n = 3$. For $n \geq 4$, we have $t^{\ast}(a_1, \ldots, a_n) = [a_i/2]$, where $i$ varies in $\{1, 2, \ldots, n\}$, or $t^{\ast} = 0$. Thus $s_n^{ac}(\ast) = n + 1$ for $n \geq 4$.

The next result involves the 3-element groupoid SC189, which is anti-isomorphic to SC170.

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SC170 SC189

**Proposition 3.4.** A groupoid $(G, \ast)$ satisfying the identities below must have $s_n^{sc}(\ast) \leq 2$ and $s_n^{ac}(\ast) \leq 2n$ for $n = 3, 4, \ldots$, where the first inequality holds as an equality whenever the second does and both hold for the 2-element groupoid $\langle 0, 1, \ast \rangle$ defined by $x \ast y := x + 1 \mod 2$ for all $x, y \in \{0, 1\}$ and the 3-element groupoids SC170 and SC189.

(i) $x(yz) \approx x(yz)$, (ii) $(xy)z \approx (xz)y$, (iii) $w(x(yz)) \approx w((xy)z)$, (iv) $(wx)(yz) \approx (w(x)y)z$, (v) $w(x(yz)) \approx ((wx)y)z$

**Proof.** Let $t$ be an arbitrary term in $F_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $|t_0| = 1$. By (i), (iii), and Lemma 2.1, we may assume that $t_i = t^{R}_i$ for all $i \in \{1, \ldots, m\}$. By (v), we may assume $m \leq 2$. If $m = 1$ then $t^{\ast} = \langle x_{i_1}, \ldots, x_{i_n} \rangle^\ast$ with $i_2 < \cdots < i_n$. If $m = 2$ then we can further use (iv) to obtain $t^{\ast} = \langle x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_n} \rangle^\ast$ and make sure $i_2 < \cdots < i_n$ by (ii) and Lemma 2.1. Thus $s_n^{ac}(\ast) \leq 2n$.

If $t$ is a bracketing of $x_1, \ldots, x_n$, then $t^{\ast} = \langle x_1, \ldots, x_n \rangle^\ast$ or $t^{\ast} = \langle x_1, x_2, (x_3, \ldots, x_n) \rangle^\ast$ by a similar argument. Thus $s_n^{sc}(\ast) \leq 2$, and the equality must hold if $s_n^{ac}(\ast) = 2n$.

It is routine to check that the 2-element groupoid $\langle 0, 1, \ast \rangle$ defined by $x \ast y := x + 1 \mod 2$ for all $x, y \in \{0, 1\}$ satisfies the identities (i)–(v). It has $s_n^{ac}(\ast) = 2$ for $n \geq 2$ by Csákány and Waldhauser [3, §4.1] and $s_n^{ac}(\ast) = 2n$ for $n \geq 3$ by our earlier work [6, Example 4.1.2]. It is easy to see that SC189 is obtained from this 2-element groupoid by adding an absorbing element; hence the term operations behave in essentially the same ways in both groupoids.

Our next result is similar to Proposition 3.4, and we will use leaf depths to generalize them in Section 6. The result here involves two anti-isomorphic groupoids:
Therefore, SCs

A groupoid \((G, \star)\) satisfying the identities below must have

\[
\begin{align*}
s_n^\star(\star) &\leq \begin{cases} 
1 & n = 1, 2 \\
2 & n = 3 \\
3 & n = 4, 5, \ldots
\end{cases} \\
\text{and} \quad s_n^{ac}(\star) &\leq \begin{cases} 
2n & n = 1, 2 \\
3n & n = 4, 5, \ldots
\end{cases}
\end{align*}
\]

where the first inequality holds as an equality if and only if it holds for SC3242 and the anti-isomorphic SC3302.

(i) \(x(yz) \approx x(zy)\),  
(ii) \(w(x(yz)) \approx w((xy)z)\),  
(iii) \((xy)z \approx (xz)y\),  
(iv) \((wx)(yz) \approx (w(yz)x)\),  
(v) \(((vw)x)yz \approx v(w(xyz))\)

**Proof.** The result is trivial when \(n = 1, 2\); assume \(n \geq 3\) below. Let \(t\) be an arbitrary term in \(T_n\) with leftmost decomposition \(t = [t_0, t_1, \ldots, t_m]\), where \(|t_0| = 1\). By (i), (ii), and Lemma 2.1, we may assume that \(t_i = t_{i-}\) for all \(i \in \{1, \ldots, m\}\). By (iii) and Lemma 2.1, we may assume that \(|t_1| \leq \cdots \leq |t_m|\). If \(|t_i| > 1\) for some \(i \in \{1, \ldots, m-1\}\), then we apply (iv) to make sure \(|t_i| = 1\). Thus we may assume that \(|t_1| = \cdots = |t_{m-1}| = 1\). Therefore, \(t\) induces the same \(n\)-ary operation on \((G, \star)\) as \([x_{i_1}, \ldots, x_{i_n}, \langle x_{i_{k+1}}, \ldots, x_{i_m} \rangle]\), where we may further assume that \(x_{i_1} < \cdots < x_{i_n}\) by (iii) and (iv) and that \(k \in \{1, 2, 3\}\) by (v). It follows that \(s_n^{ac}(\star) \leq 2n\) for \(n = 3\) in this case \(k \in \{1, 2\}\) and \(s_n^{ac}(\star) \leq 3n\) for \(n = 4, 5, \ldots\).

If \(t \in T_n\) is a bracketing of \(x_1x_2 \cdots x_n\), then we must have \(i_1 = 1\) since the above argument does not alter the leftmost variable. Thus \(s_n^\star(\star) \leq 2\) for \(n = 3\) and \(s_n^{ac}(\star) \leq 3\) for \(n = 4, 5, \ldots\). It is clear that if the upper bound of \(s_n^{ac}(\star)\) is reached, so is the upper bound of \(s_n^\star(\star)\).

For SC3242, we have \(t^\star(a_1, \ldots, a_n) = (a_1 + d) \mod 3\) whenever the binary tree corresponding to \(t \in T_n\) has leftmost leaf \(i_1\) of left depth \(d\). The number of possibilities for \(i_1\) is \(n\), and the number of possibilities for \(d\) (mod 3) is 1 when \(n \in \{1, 2\}\), 2 when \(n = 3\), and 3 when \(n = 4, 5, \ldots\). The proof is now complete. \(\square\)

We next present a family of groupoids whose associative spectrum and ac-spectrum are bounded above by \(1, 1, 2, 4, 4, 4, \ldots\) and \(1, 2, 9, 32, 50, 72, 98, \ldots\) and show that both upper bounds are reached by the 3-element groupoid SC3162, which is anti-isomorphic to SC2467.

\[
\begin{array}{ccc|ccc}
  * & 0 & 1 & 2 \\
  0 & 1 & 1 & 0 \ \\
  1 & 0 & 0 & 1 \ \\
  2 & 1 & 0 & 1 \\
\end{array}
\]

**Theorem 3.1.** A groupoid \((G, \star)\) satisfying the identities below must have \(s_n^\star(\star) \leq 2\) and \(s_n^{ac}(\star) \leq 2n^2\) for \(n = 3\) and \(s_n^{ac}(\star) \leq 4\) and \(s_n^{ac}(\star) \leq 2n^2\) for \(n = 4, 5, \ldots\), where the upper bound for \(s_n^\star(\star)\) is reached if the upper bound for \(s_n^{ac}(\star)\) is reached and both upper bounds are reached by SC3162 and the anti-isomorphic SC2467.

(i) \(x(yz) \approx x(zy)\),  
(ii) \(w(x(yz)) \approx w((xy)z)\),  
(iii) \((wx)yz \approx (w(yz)x)\),  
(iv) \((w(x)yz) \approx (w(x)yz)\),  
(v) \(((w(x)yz) \approx (((w(x)yz)\)

**Proof.** Let \(t\) be an arbitrary term in \(T_n\) with leftmost decomposition \(t = [x_{a_1}, t_1, t_2, \ldots, t_m]\). By (i), (ii), and Lemma 2.1, we may assume that \(t_i = t_{i-}\) for all \(i \in \{1, \ldots, m\}\). By (vi), we may assume that \(m \leq 3\). Consequently, \(t\) induces the same \(n\)-ary operation on \((G, \star)\) as one of the following four types of standard terms.

**Type 1:** \(m = 1\). Then \(t^\star = (x_{a_1}x_{b_{a_1}}, \ldots, x_{b_{a_1-1}})\), where \(b_1 < \cdots < b_{a_1-1}\).

**Type 2:** \(m = 2\) and \(|t_1| = 1\). Then \(t^\star = (x_{a_1}, x_{b_1}x_{c_1}, \ldots, x_{c_{a_2}}x_{c_{a_2-1}})\), where \(c_1 < \cdots < c_{a_2-1}\).

**Type 3:** \(m = 2\) and \(|t_1| > 1\). Then \(t^\star = (x_{a_1}x_{b_{a_1}}, \ldots, x_{b_{a_1-1}, x_{b_{a_1-1}}})\), where \(b_1 < \cdots < b_{a_1-1}, b_{a_1} < \cdots < b_{a_1-1}\), thanks to (iv).

**Type 4:** \(m = 3\). We may assume that \(|t_1| = 1\) by the identity (v) and that \(|t_2| \geq |t_3|\) by the identity (iii). If \(|t_2| > |t_3| > 1\), then we can write \(t_2 = t'_{x_2}\) for any variable \(x \in \var(t_2)\) and switch \(x\) with \(t_3\) by (iv). Thus we may also assume \(|t_3| = 1\). It follows that \(t^\star = (x_{a_1}, x_{b_1}x_{c_1}, \ldots, x_{c_{a_2-1}}, x_{c_{a_2-1}})\), where \(c_1 < \cdots < c_{a_2-1}\).
Summing up the possibilities for the above four types of standard terms, we obtain that
\[ s_n^\text{st}(\ast) \leq \begin{cases} n + n(n - 1) = n^2 & \text{if } n = 3 \\
 + n(n - 1) + n(n - 1) = 2n^2 & \text{if } n = 4, 5, \ldots \end{cases} \]

If \( t \in B_n \) is a bracketing of \( x_1x_2\cdots x_n \), then there is only one possibility in each of the above four (two when \( n = 3 \)) cases. This shows that \( s_n^\text{st}(\ast) \leq 2 \) for \( n = 3 \) and \( s_n^\text{st}(\ast) \leq 4 \) for \( n = 4, 5, \ldots \). If \( s_n^\text{st}(\ast) = 2n^2 \) then the above four cases must induce distinct terms on \((G, \ast)\), and thus \( s_n^\text{ac}(\ast) = 4 \).

It is routine to check that SC3162 satisfies the identities (i)–(vi). It remains to show that any two distinct standard terms \( t \) and \( t' \) in \( F_n \) must induce distinct \( n \)-ary operations on SC3162. Assume that \( t \) is one of the following, where \( b_1 < \cdots < b_{n-1} \) and \( c_1 < \cdots < c_{n-2} \).

\[ x_a[x_{b_1}, \ldots, x_{b_{n-1}}], [x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-2}}]], [x_a, [x_{b_1}, \ldots, x_{b_{n-2}}], x_{b_{n-1}}], [x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-3}}], x_{c_{n-2}}] \]

Similarly, assume that \( t' \) is one of the following, where \( b'_1 < \cdots < b'_{n-1} \) and \( c'_1 < \cdots < c'_{n-2} \).

\[ x_{a'}[x_{b'_1}, \ldots, x_{b'_{n-1}}], [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-2}}]], [x_{a'}, [x_{b'_1}, \ldots, x_{b'_{n-2}}], x_{b'_{n-1}}], [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-3}}], x_{c'_{n-2}}] \]

It is clear that [0, s1, \ldots, s] gives 0 if \( \ell \) is even or 1 if \( \ell \) is odd, no matter what \( s_1, \ldots, s\ell \) are. Therefore, we only need to consider the following cases.

Case 1: \( t = x_a[x_{b_1}, \ldots, x_{b_{n-1}}] \) and \( t' = x_{a'}[x_{b'_1}, \ldots, x_{b'_{n-1}}] \), where \( a \neq a' \). We have \( h(t) = 0 \neq 1 = h(t') \), where \( h(x_a) = 1 \) and \( h(x) = 0 \) for all \( x \neq x_a \).

Case 2: \( t = x_a[x_{b_1}, \ldots, x_{b_{n-1}}] \) and \( t' = [x_{a'}, x_{b'}, x_{c'_1}, \ldots, x_{c'_{n-3}}], x_{c'_{n-2}}] \). We have \( h(t) = 0 \neq 1 = h(t') \), where \( h(x_a) = h(x_{b'}) = 2 \) and \( h(x) = 0 \) for all \( x \neq [x_a, x_{b'}] \). Here \( a \) may coincide with \( a' \) or \( b' \).

Case 3: \( t = x_a[x_{b_1}, [x_{c_1}, \ldots, x_{c_{n-2}}]] \) and \( t' = [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-2}}]] \), where \( (a, b) \neq (a', b') \).

If \( a \neq a' \) then \( h(t) = 0 \neq 1 = h(t') \), where \( h(x_a) = 0 \) and \( h(x) = 1 \) for all \( x \neq x_a \).

If \( a = a' \) then \( b \neq b' \) and \( h(t) = 0 \neq 1 = h(t') \), where \( h(x_a) = h(x_{b'}) = 2 \) and \( h(x) = 0 \) for all \( x \neq [x_a, x_{b'}] \).

Case 4: \( t = [x_a, x_{b_1}, [x_{c_1}, \ldots, x_{c_{n-2}}]] \) and \( t' = [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-2}}]], x_{c'_{n-1}}] \).

If \( a \neq a' \) then \( h(t) = 0 \neq 1 = h(t') \), where \( h(x_a) = 0 \) and \( h(x) = 1 \) for all \( x \neq x_a \).

If \( a = a' \) then \( h(t) = 0 \neq 1 = h(t') \), where \( h(x_a) = h(x_{b'}) = 2 \) and \( h(x) = 0 \) for all \( x \neq [x_a, x_{b'}] \).

Case 5: \( t = [x_a, x_{b_1}, \ldots, x_{b_{n-2}}], x_{b_{n-1}}] \) and \( t' = [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-2}}]], x_{c'_{n-1}}] \), where \( a \neq a' \). We have \( h(t) = 0 \neq 1 = h(t') \), where \( h(x_a) = 0 \) and \( h(x) = 1 \) for all \( x \neq x_a \).

Case 6: \( t = [x_a, x_{b_1}, [x_{c_1}, \ldots, x_{c_{n-2}}]], x_{c_{n-1}}] \) and \( t' = [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-2}}]], x_{c'_{n-1}}] \), where \( (a, b) \neq (a', b') \).

If \( a \neq a' \) then \( h(t) = 1 \neq 0 = h(t') \), where \( h(x_a) = 0 \) and \( h(x) = 1 \) for all \( x \neq x_a \).

If \( a = a' \) then \( b \neq b' \) and \( h(t) = 1 \neq 0 = h(t') \), where \( h(x_a) = h(x_{b'}) = 2 \) and \( h(x) = 0 \) for all \( x \neq [x_a, x_{b'}] \).

The proof is now complete.

4. Exponential upper bounds

In this section, we establish some exponential upper bounds for the ac-spectra for a few varieties of groupoids; the respective associative spectra may have linear or exponential upper bounds.

**Proposition 4.1.** Every groupoid \((G, \ast)\) satisfying the identities below must have \( s_n^\text{st}(\ast) \leq n - 1 \) and \( s_n^\text{ac}(\ast) \leq 2^{n-1} - 1 \) for \( n = 2, 3, \ldots \) where the first inequality holds as an equality whenever the second does and both equalities hold for SC1066 (see Table 1).

(i) \( xy \approx yx \), (ii) \( w(xyz) \approx w(xy)z \)

**Proof.** Let \( t \) be an arbitrary term in \( F_n \) with least decomposition \( t = [x_a, t_1, t_2, \ldots, t_m] \). By Lemma 2.1, we may assume that \( t_i = t_i^\leq \) for all \( i \in \{1, \ldots, m\} \). Next, we use (i) to swap \([x_a, t_1, \ldots, t_{m-1}] \) and \( t_m \). Then we transform \([x_a, t_1, \ldots, t_{m-1}] \) to a leftmost bracketing again by Lemma 2.1. It follows that \( t \) induces the same \( n \)-ary operation on \((G, \ast)\) as \([x_j, \ldots, x_j][x_{j+1}, \ldots, x_{j+n}]\), where \( \{x_j, \ldots, x_j\} = \text{var}(t_m) \) and \( \{x_{j+1}, \ldots, x_{j+n}\} = X_n \setminus \text{var}(t_m) \). The order of the elements of each set of variables does not affect \( \ast \) by the above, nor does the order of the two sets by (i). Thus \( s_n^\text{ac}(\ast) \) is bounded above by \((2^n - 2)/2 = 2^{n-1} - 1\), the number of partitions of \( \{1, \ldots, n\} \) into two unordered nonempty blocks.

Restricting the above argument to bracketings of \( x_1x_2\cdots x_n \) in \( B_n \) instead of full linear terms in \( F_n \), we have the variables in var\((t_m)\) indexed by larger numbers than the other variables. Thus the partitions of \( \{1, \ldots, n\} \)
associated with these bracketings have two blocks \([1, \ldots, k]\) and \([k + 1, \ldots, n]\) for some \(k \in \{1, \ldots, n - 1\}\). It follows that \(s_{n}^{m}(\ast) \leq n - 1\).

If \(s_{n}^{m}(\ast) = 2^{n-1} - 1\), then distinct partitions of \([1, \ldots, n]\) into two unordered nonempty blocks correspond to distinct \(n\)-ary operations on \((G, \ast)\), and we can restrict this to partitions with two blocks \([1, \ldots, k]\) and \([k + 1, \ldots, n]\) to conclude that \(s_{n}^{m}(\ast) = n - 1\).

It remains to consider SC1066. Every full linear term \(t \in \mathcal{F}_{n}\) can be written as \(t = t_{L}t_{R}\). Let \(h : X_{n} \to \{0, 1, 2\}\) be an assignment. We have that \(h(t) = 1\) if and only if \(h(t_{L}) = h(t_{R}) = 2\) and that \(h(t) = 0\) if and only if \(h(t_{L}) \neq 2\) and \(h(t_{R}) \neq 2\). As observed by Csákány and Waldhauser [3], one can show by induction that \(h(t) = 2\) if and only if \(h\) assigns 2 to an odd number of variables. Thus \(h(t)\) is completely determined by how many variables in \(t_{L}\) and \(t_{R}\) take the value 2. In particular, if \(s = [x_{i_{1}}, \ldots, x_{i_{k}}][x_{i_{k+1}}, \ldots, x_{i_{n}}]\) and \(t = [x_{j_{1}}, \ldots, x_{j_{k}}][x_{j_{k+1}}, \ldots, x_{j_{n}}]\) with \(i_{1}, \ldots, i_{k}\) \(\in\) \([1, \ldots, i_{k}]\) and \(j_{1}, \ldots, i_{k}\) \(\in\) \([1, \ldots, j_{k}]\), then \(s \neq t\)

Since \(h(s) = 0 \neq 1 = h(t)\), where \(h(x_{i}) = h(x_{j}) := 2\) and \(h(x) := 0\) for all \(x \notin \{x_{i_{1}}, \ldots, x_{i_{n}}\}\). This implies that \(s_{n}^{m}(\ast) = 2^{n-1} - 1\), which in turn implies \(s_{n}^{m}(\ast) = n - 1\). \(\square\)

We study another variety of groupoids, for which the associative spectra have the same upper bound \(n - 1\) as in Proposition 4.1 but the ac-spectra have a different upper bound 1, 2, 7, 29, 146, \ldots [13, A185109]. We show that both upper bounds are reached by SC377, which is anti-isomorphic to SC10.

\[
\begin{array}{c|cccc}
   & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 0 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 & 2 \\
\end{array}
\]

Proposition 4.2. A groupoid \((G, \ast)\) must have \(s_{n}^{m}(\ast) \leq n - 1\) for \(n = 2, 3, \ldots\) and

\[
s_{n}^{m}(\ast) \leq n! + \sum_{k=0}^{n-3} n(n-1) \cdots (n-k+1) + n! + \sum_{k=0}^{n-3} k! \binom{n}{k}
\]

for \(n = 1, 2, \ldots\) if it satisfies the identities below, where the first inequality holds as an equality whenever the second does and both equalities hold for SC377 and the anti-isomorphic SC10.

(i) \(x(yz) \approx x(zy) \approx y(xz)\), \quad (ii) \(w(xyz) \approx w(xy)z\)

Proof. We show that we can transform an arbitrary term \(t \in \mathcal{F}_{n}\), whose leftmost decomposition is \(t = [t_{0}, t_{1}, \ldots, t_{m}]\) with \(|t_{0}| = 1\), to a ‘standard’ term of the form \([x_{i_{1}}, \ldots, x_{i_{k}}], x_{i_{k+1}}, \ldots, x_{i_{n}}]\), where \(\ell \in \{0, 3, 4, \ldots, n\}\) and \(i_{1} < \cdots < i_{k}\).\n
If \(|t_{i}| = 1\) for all \(i = 1, \ldots, m\), then \(t = [x_{i_{1}}, \ldots, x_{i_{n}}]\) is already a standard term with \(\ell = 0\). Here \(i_{1}, \ldots, i_{n}\) form a permutation of \(1, \ldots, n\), and we have \(n!\) possibilities in this case.

Suppose \(|t_{j}| > 1\) for some \(j\), where \(j\) is as large as possible. Then \(|t_{j+1}| = \cdots = |t_{m}| = 1\). We can transform \(t_{j}\) to the rightmost bracketing of its variables in any prescribed order by (i), (ii), and Lemma 2.1, then switch its leftmost variable \(x_{i}\) with \([t_{0}, t_{1}, \ldots, t_{j-1}]\) by (i), and use (i), (ii), and Lemma 2.1 again to transform \(t_{j}\) to the standard form \([x_{i_{1}}, \ldots, x_{i_{k}}], x_{i_{k+1}}, \ldots, x_{i_{n}}]\), where \(i_{1}, \ldots, i_{k}\) can be in any prescribed order, say the increasing one. There are \(n(n-1) \cdots (\ell + 1)\) possibilities for \(i_{k+1}, \ldots, i_{n}\), and we must have \(3 \leq \ell \leq n\) since \(|t_{j}| > 1\).

Summing the numbers of possibilities in the above two cases with \(k = n - \ell\) in the second case gives the desired upper bound for \(s_{n}^{m}(\ast)\). Restricting the above argument to bracketings of \(x_{1}x_{2} \cdots x_{n}\) in \(B_{n}\), we obtain standard terms of the form \([x_{1}, \ldots, x_{\ell}], x_{\ell+1}, \ldots, x_{n}\) with \(\ell \in \{0, 3, 4, \ldots, n\}\). Thus \(s_{n}^{m}(\ast) \leq n - 1\). It is easy to see that if the upper bound for \(s_{n}^{m}(\ast)\) is reached, so is the upper bound for \(s_{n}^{m}(\ast)\).

It is clear that SC10 is anti-isomorphic to SC377. The latter satisfies the identities (i) and (ii). It remains to show that \(s_{n}^{m}(\ast) \neq t^\ast\) whenever \(s_{n}^{m}(\ast) \neq t^\ast\) and \(s \neq t\) are distinct standard terms in \(\mathcal{F}_{n}\). We may assume that \(s = [x_{i_{1}}, \ldots, x_{i_{k}}], x_{i_{k+1}}, \ldots, x_{i_{n}}]\) and \(t = [x_{j_{1}}, \ldots, x_{j_{k}}], x_{j_{k+1}}, \ldots, x_{j_{n}}]\) for some \(\ell, m \in \{0, 3, 4, \ldots, n\}\), where \(i_{1} < \cdots < i_{k}, j_{1} < \cdots < j_{m}\), and \(\ell \leq m\).

First, assume that \(i_{k} \neq j_{k}\) for some \(k \in \{m + 1, \ldots, n\}\). Let \(k\) be as large as possible. We have \(h(s) = 0 \neq 1 = h(t)\) if \(n - k\) is odd or \(h(s) = 1 \neq 0 = h(t)\) if \(n - k\) is even, where \(h(x_{i_{n+1}}) = \cdots = h(x_{i_{n}}) := 2\) and \(h(x) := 0\) for all \(x \notin \{x_{i_{1}}, \ldots, x_{i_{n}}\}\).

Next, assume that \(i_{k} = j_{k}\) for all \(k = m + 1, \ldots, n\). This implies that \(\ell < m\) (otherwise \(s = t\)). We have \(h(s) = 0 \neq 1 = h(t)\) if \(n - m\) is odd or \(h(s) = 1 \neq 0 = h(t)\) if \(n - m\) is even, where \(h(x_{i_{m+1}}) = \cdots = h(x_{i_{m}}) := 2\) and \(h(x) := 0\) for all \(x \notin \{x_{i_{1}}, \ldots, x_{i_{m}}\}\). \(\square\)

The upper bounds in the next result are reached by the 3-element groupoid SC2302, which can be viewed as subtraction on a finite field of three elements, or more generally, reached by the subtraction on any commutative group \((G, +)\) of exponent greater than 2 (cf. [6, Example 7.1.4]). It is clear SC2302 is anti-isomorphic to SC2155.
**Proposition 4.3.** A groupoid \((G, \ast)\) satisfying the identities below must have \(s_n^a(*) \leq 2^{n-2}\) and \(s_n^{ac}(*) \leq 2^n - 2\) for \(n = 2, 3, \ldots\), where the second inequality holds as an equality whenever the second does and both equalities hold for the subtraction operation — on any commutative group \((G,\ast)\) of exponent greater than 2, in particular, for \(SC2032\) (hence the anti-isomorphic \(SC2155\)).

\[
(i) \ (xy)z \approx (xz)y, \quad (ii) \ x(yz) \approx z(xy), \quad (iii) \ w(x(yz)) \approx (w(xy))z.
\]

**Proof.** Let \(t\) be an arbitrary term in \(F_n\). We show by induction on \(|t|\) that \(t\) can be transformed to a “standard” term \([x_{i_1}, \ldots, x_{i_n}, [x_{i_{k+1}}, \ldots, x_{i_n}]]\) for some \(k \in \{1, \ldots, n-1\}\), where the sets \(\{i_1, i_{k+2}, \ldots, i_n\}\) and \(\{i_2, \ldots, i_{k+1}\}\) respectively contain the indices of the leftmost two variables of \(t\) and either set of indices can be permuted arbitrarily. We first write \(t = [t_0, t_1, \ldots, t_m]\) with \(|t_0| = 1\). We may assume that \(|t_1| \geq |t_2| \geq \cdots \geq |t_m|\), thanks to the identity (i) and Lemma 2.1. We distinguish some cases below.

**Case 1:** \(m > 1\) and \(|t_1| = 1\). Then \(t = [x_{i_1}, \ldots, x_{i_n}]\), which is in standard form with \(k = n - 1\). The leftmost two variables of \(t\) are indexed by \(i_1 \in \{i_1\}\) and \(i_2 \in \{i_2, \ldots, i_{k+1}\}\), and we can permute \(i_2, \ldots, i_{k+1}\) by (i).

**Case 2:** \(m > 1\) and \(|t_1| > 1\). We can first apply (iii) repeatedly to transform \(t \rightarrow t'\), where \(i_n\) is the index of the leftmost variable of \(t\) and the leftmost variable of \(t'\) is the second leftmost variable of \(t\). By the induction hypothesis, we may assume that \(t' = [x_{i_1}, \ldots, x_{i_n}, [x_{i_{k+1}}, \ldots, x_{i_n}]]\), where \(i_{k+1}, i_{k+2}, \ldots, i_n\) can be permuted in all possible ways and so can be \(i_{k+2}, \ldots, i_{k-1}, i_k\), and the leftmost variable of \(t'\) is indexed by one of \(i_{k+1}, i_{k+2}, \ldots, i_k\). We then apply (ii) to switch \(x_{i_n}\) with \([x_{i_1}, \ldots, x_{i_k}, [x_{i_{k+1}}, \ldots, x_{i_n}]]\). By (i), we can switch \(i_n\) and each of \(i_{k+2}, \ldots, i_{n-1}\). Thus we are done for this case.

**Case 3:** \(m = 1\). Similarly to the above case, we can apply the induction hypothesis to \(t_1\) and then use (i) and (ii) to finish the argument for this case.

It follows that \(s_n^a(*)\) is bounded above by the number of nonempty proper subsets of \(\{1, \ldots, n\}\), which is clearly \(2^{n-2}\). Restricting the above argument to \(t \in B_m\), we must have \(1 \in \{i_1, i_{k+2}, \ldots, i_n\}\) and \(2 \in \{i_2, \ldots, i_{k+1}\}\). Thus \(s_n^a(*) \leq 2^{n-2}\); see also earlier work [4]. It is easy to see that \(s_n^a(*) = 2^n - 2\) implies \(s_n^{ac}(*) = 2^{n-2}\).

The usual subtraction — on \(\mathbb{R}\) or \(\mathbb{C}\) satisfies the identities (i), (ii), and (iii). We have \(s_n^a(-) = 2^n - 2\) and \(s_n^{ac}(-) = 2^n - 2\) by previous work [6, Example 7.1.4]. The same argument there is also valid for subtraction on any commutative group \((G,\ast)\) of exponent greater than 2 and in particular, for \(SC2302\).

**Remark 4.1.** If \((G,\ast)\) is a commutative group of exponent at most 2, then the subtraction coincides with addition and \(s_n^{ac}(*) = 1\) for all \(n \in \mathbb{N}_+\).

We provide another variety of groupoids \((G,\ast)\) with the same associative spectrum upper bound \(2^{n-2}\) as Proposition 4.3 but a different ac-spectrum upper bound \(1, 2, 9, 28, 75, 186, \ldots\) [13, A058877]. We show that both upper bounds are reached by two 3-element groupoids \(SC271\) and \(SC356\), which are anti-isomorphic to \(SC1610\) (by \(0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0\)) and \(SC2032\) (by \(0 \rightarrow 2, 1 \rightarrow 0, 2 \rightarrow 1\)), respectively.

\[
\begin{array}{|c|c|c|} \hline * & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 \\ \hline \end{array}
\]

\[
\begin{array}{|c|c|c|} \hline * & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ \hline \end{array}
\]

\[
\begin{array}{|c|c|c|} \hline * & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 2 \\ \hline \end{array}
\]

**Theorem 4.1.** A groupoid \((G, \ast)\) satisfying the identities below must have \(s_n^a(*) \leq 2^{n-2}\) and \(s_n^{ac}(*) \leq n(2^{n-1} - 1)\) for \(n = 2, 3, \ldots\), where the first upper bound is reached whenever the second one is.

\[
(i) \ (xy)z \approx (xz)y, \quad (ii) \ w(x(yz)) \approx w((xy)z), \quad (iii) \ wx(yz) \approx (wy)(xz)
\]

Moreover, both upper bounds are reached for the 3-element groupoids \(SC271\) and \(SC356\) (hence the anti-isomorphic \(SC1610\) and \(SC2032\)).

**Proof.** We transform an arbitrary term \(t \in F_n\), whose leftmost decomposition is \(t = [x_{a_0}, t_1, \ldots, t_m]\), to a “standard term” using (i), (ii), (iii), and Lemma 2.1. We may assume that \(|t_1| \leq |t_2| \leq \cdots \leq |t_m|\), thanks to the identity (i). Let \(x_{a_0}\) be the leftmost variable of \(t_i\) for \(i = 1, \ldots, m\). If there exists a positive integer \(j < m\) such that \(|t_j| > 1\), we can use the identity (ii) to transform \(t_{j+1}\) to \(t_{j+1}^{t_{j+1}}\) and then use the identity (iii) to switch \(t_j\) and \(x_{a_{j+1}}\). Repeating this, we obtain \([x_{a_0}, x_{a_1}, \ldots, x_{a_{m-1}}, (x_{a_m} t_m')]\) from \(t\), where \(\{b_1, \ldots, b_m\} = \{a_1, \ldots, a_m\}\).
We can assume that \( b_1 < b_2 < \cdots < b_n \), thanks to the identities (i) and (iii). Applying the identity (ii) repeatedly to \( x_{b_m} t_m \) gives \( \langle x_{b_m}, x_{c_1}, x_{c_2}, \ldots, x_{c_{n-m-1}} \rangle \). We may assume that \( c_1 < c_2 < \cdots < c_{n-m-1} \) by the identities \( w(xyz) \approx w((xy)z) \approx w((xz)y) \approx w(x(zy)) \).

It follows that every \( t \in F_n \) induces the same \( n \)-ary operation as a standard term

\[
[x_{a_1}, x_{a_2}, \ldots, x_{b_{m-1}}] \langle x_{b_m}, x_{c_1}, x_{c_2}, \ldots, x_{c_{n-m-1}} \rangle,
\]

where \( b_1 < \cdots < b_n \) and \( c_1 < \cdots < c_{n-m-1} \). This implies that \( s_n^*(\ast) \leq n(2^{n-1} - 1) \) since there are \( n \) possibilities for \( a \) and \( 2^{n-1} - 1 \) possibilities for \( b_1, \ldots, b_m \).

Restricting the above argument to \( t \in B_n \), we must have \( a = 1 \) and \( b_1 = 2 \) since \( a_1 = 2 \in \{b_1, \ldots, b_m\} \). Thus \( s_n^*(\ast) \leq 2^{n-2} \), and it is easy to see that the equality must hold when \( s_n^*(\ast) = n(2^{n-1} - 1) \).

One can check that SC271 and SC356 both satisfy the identities (i), (ii), and (iii). It remains to show that \( h(s) \neq h(t) \) for some assignment \( h : X_n \to \{0, 1, 2\} \), where \( s \) and \( t \) are terms in \( F_n \). We can assume that \( s \neq t \), without loss of generality. We must have \( i \in \{b_1, \ldots, b_m\} \).

For SC271, we have \( h(s) = 0 \neq 1 = h(t) \), where \( h(x_i) := 2 \) and \( h(x_j) := 1 \) for all \( x_i 
eq x_j \).

For SC356, we have \( h(s) = 1 \neq 2 = h(t) \), where \( h(x_i) := 0 \) and \( h(x_j) := 2 \) for all \( x_i 
eq x_j \).

\[ \Box \]

5. Upper bounds related to set partitions

In this section, we present a few varieties of groupoids, whose ac-spectra are related to set partitions. Recall that the restricted Bell number \( B_{n,m} \) counts partitions of the set \( \{1, 2, \ldots, n\} \) into unordered nonempty blocks of size at most \( m \) [12]; it gives the well-known Bell number \( B_n \) when \( m \geq n \). In particular, we have \( B_{n,2} = 1 \) for \( n = 0, 1 \) and \( B_{n,2} = B_{n-1,2} + (n-1)B_{n-2,2} \) for \( n \geq 2 \); see the sequence A000085 in OEIS [13] for other interpretations and closed formulas for \( B_{n,2} \). We also need the following definition by Csákány and Waldhauser [3].

**Definition 5.1.** Define a term \( t \) to be a nest if either \( |t| = 1 \) (a trivial nest) or there exists a term \( t' \) together with a variable \( z \) such that \( t = xt' \) or \( t = t'x \), \( |t'| = |t|-1 \), and \( t' \) is a nest. Each variable in \( t \) must be contained in a unique maximal nest, which is simply called a nest of \( t \). Every nontrivial nest must have a unique subterm of the form \( xixj \), and the variables \( x_i \) and \( x_j \) are called the eggs of this nest.

Our first result is concerned with a variety of groupoids including the following two 3-element groupoids.

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\[ SC79 \]

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\[ SC1701 \]

**Proposition 5.1.** A groupoid \((G, \ast)\) satisfying the identities below must have \( s_n^*(\ast) \leq F_{n+1} - 1 \) and \( s_n^{ac}(\ast) \leq B_{n,2} - 1 \) for \( n = 2, 3, \ldots \), where the first inequality holds as an equality whenever the second does.

(i) \( xy \approx yx \),  \quad (ii) \( \langle xwxy \rangle yz \approx \langle (xw)yz \rangle \)

Moreover, both upper bounds are reached by \( SC79 \) and \( SC1701 \).

**Proof.** Suppose \( s, t \in F_n \) have the same eggs of nests. We show by induction on \( n \) that \( s^* = t^* \). Let \( x_i \) and \( x_j \) be the eggs of a nest of \( s \); they must be the eggs of a nest of \( t \). The case \( n = 2 \) is trivial; assume \( n \geq 3 \) below. Thanks to the identity (i), we may assume \( s = [x_i, x_j, s_1, \ldots, s_l] \) and \( t = [x_i, x_j, t_1, \ldots, t_m] \). We may also assume that \( |s_1| \geq \cdots \geq |s_l| \) and \( |t_1| \geq \cdots \geq |t_m| \) by (ii). Assume \( |s_1| \leq |t_1| \), without loss of generality.

**Case 1:** \( |t_1| \geq |s_1| > 1 \). Replacing \( x_i, x_j \) with a new variable \( x_0 \) in both \( s \) and \( t \) gives full linear terms \( s' \) and \( t' \) in \( n-1 \) variables that share the same eggs of nests. It follows from the induction hypothesis that \( (s')^* = (t')^* \), and this implies \( s^* = t^* \).

**Case 2:** \( |t_1| = 1 \). Then \( |s_2| = \cdots = |s_l| = 1 \) and \( s \) has only two eggs \( x_i \) and \( x_j \). We must have \( |t_1| = 1 \) (otherwise \( t_1 \) contains eggs different from \( x_i \) and \( x_j \)) and thus \( |t_2| = \cdots = |t_m| = 1 \). We can use (ii) to make
sure \( s_1 = t_1 = x_k \) for some \( k \neq \{i,j\} \). Replacing \( x_kx_j \) with a new variable \( x_0 \) in both \( s \) and \( t \) gives \( s' \) and \( t' \) with eggs \( x_0 \) and \( x_k \). By the induction hypothesis, we have \( (s')^* = (t')^* \). This implies that \( s^* = t^* \).

Therefore, \( s_n^{ac}(\ast) \) is bounded above by \( B_{n-2} - 1 \), which is the number of partitions of \( \{1, \ldots, n\} \) into blocks of size one or two with at least one block of size two (since there is at least one nest with two eggs).

Restricting the above argument to bracketings of \( x_1, \ldots, x_n \), we have \( s_n^*(\ast) \leq F_{n+1} - 1 \) since the partitions associated with bracketings of \( x_1, \ldots, x_n \) must have two consecutive integers in each block of size two; see also Csákány and Waldhauser \([3, \S 5.6]\). It is easy to see that \( s_n^{ac}(\ast) = B_{n-2} - 1 \) implies \( s_n^*(\ast) = F_{n+1} - 1 \).

It is routine to verify that groupoids \( SC_79 \) and \( SC_{1701} \) satisfy identities (i) and (ii). It remains to verify that if \( s, t \in F_n \) are terms whose eggs of nests are not the same, then \( s \) and \( t \) induce distinct operations on \( SC_79 \) and on \( SC_{1701} \). Suppose that \( x_i \) and \( x_j \) are eggs of a nest in \( s \) but not eggs of any nest in \( t \). For \( SC_79 \), Csákány and Waldhauser \([3]\) observed that \( h(s) = 1 \neq 0 = h(t) \), where \( h(x_i) = h(x_j) := 2 \) and \( h(x) := 1 \) for all \( x \neq \{x_i, x_j\} \). For \( SC_{1701} \), we have \( h(s) = 1 \neq 0 = h(t) \), where \( h(x_i) = h(x_j) := 2 \) and \( h(x) := 0 \) for all \( x \neq \{x_i, x_j\} \). Thus \( s_n^{ac}(\ast) = B_{n-2} - 1 \) and \( s_n^*(\ast) = F_{n+1} - 1 \) for \( SC_79 \) and \( SC_{1701} \).

A set partition is \emph{rooted} if it has a distinguished singleton block called the \emph{root}. The number of rooted partitions of \( \{1, 2, \ldots, n\} \) is \( n_{B_{n-1}} = 1 \), if \( n = 1 \), \( 2, 6, 20, 75, 312, \ldots \) \([13, A052889]\). We show below that this number is the upper bound for the \( ac \)-spectra of a variety of groupoids and can be attained by the 3-element groupoids \( SC_41 \) and \( SC_{96} \). The Cayley tables of these two groupoids together with the anti-isomorphic groupoids \( SC_{398} \) and \( SC_{1069} \) are given below.

\[
\begin{array}{ccc|ccc|ccc|ccc}
 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
2 & 1 & 1 & 2 & 2 & 2 & 0 & 2 & 0 & 2 \\
SC_{41} & SC_{96} & SC_{398} & SC_{1069}
\end{array}
\]

\[\text{Theorem 5.1.} \quad \text{A groupoid} \ (G, \ast) \ \text{satisfying the identities below must have} \ s_n^*(\ast) \leq 2^n - 2 \ \text{for} \ n = 2, 3, \ldots \ \text{and} \ s_n^{ac}(\ast) \leq n_{B_{n-1}} \ \text{for} \ n = 1, 2, \ldots, \ \text{where the second inequality holds as an equality whenever the second does.}
\]

(i) \( x(yz) \approx x(yz), \) (ii) \( (xy)z \approx (xz)y. \)

Moreover, both upper bounds are reached by \( SC_{41} \) and \( SC_{96} \) (hence the anti-isomorphic \( SC_{398} \) and \( SC_{1069} \)).

\[\text{Proof.} \ \text{Let} \ t \ \text{be an arbitrary term in} \ F_n \ \text{with leftmost decomposition} \ t = [x_a, t_1, \ldots, t_m]. \ \text{Define a rooted set partition of} \ \{1, 2, \ldots, n\} \ \text{associated with} \ t: \ \text{we have a block consisting of the indices of the variables in} \ t_j \ \text{for all} \ j = 1, 2, \ldots, m \ \text{together with a singleton block} \ \{a\} \ \text{that is the root of this partition. By (i), (ii), and Lemma 2.1,} \ t \ \text{induces on} \ (G, \ast) \ \text{the same term operation as} \ [x_a, t_{\sigma(1)}, \ldots, t_{\sigma(n)}] \ \text{for any permutation} \ \sigma \in S_m. \ \text{It follows that terms in} \ F_n \ \text{associated with the same rooted partition must induce the same} n \text{-ary operation on} \ (G, \ast). \ \text{Thus} \ s_n^{ac}(\ast) \leq n_{B_{n-1}}. \]

The rooted set partition associated with a bracketing of \( x_1 \cdots x_n \) must have \( \{1\} \) as its root and the other blocks are intervals. The number of such “interval partitions” can be found by counting the number of ways of inserting bars into the \( n - 2 \) spaces between \( 2, \ldots, n \). Thus \( s_n^{ac}(\ast) \leq 2^n - 2. \)

If \( s_n^*(\ast) = n_{B_{n-1}} \) for \( n \geq 1 \), then \( s^* \neq t^* \) whenever \( s, t \in F_n \) are associated with distinct rooted set partitions, and restricting this to bracketings of \( x_1 \cdots x_n \) gives \( s_n^*(\ast) = 2^n - 2. \)

It is routine to check that \( SC_{41} \) and \( SC_{96} \) both satisfy the identities (i) and (ii). It remains to show that \( s^* \neq t^* \) whenever \( s \) and \( t \) are terms in \( F_n \) associated with distinct rooted set partitions. Suppose \( s = [x_a, s_1, \ldots, s_k] \) and \( t = [x_b, t_1, \ldots, t_m] \), where \( s_1, \ldots, s_k \) and \( t_1, \ldots, t_m \) are ordered according to the smallest index of the variables they contain. If \( a \neq b \) then \( s^* \neq t^* \) since

- \( h(s) = 0 \neq 1 = h(t) \) if \( \{0, 1, 2, \ast\} = SC_{41}, h(x_a) = 0 \) and \( h(x_i) = 2 \) for all \( i \neq a \), and
- \( h(s) = 0 \neq 2 = h(t) \) if \( \{0, 1, 2, \ast\} = SC_{96}, h(x_a) = 0 \) and \( h(x_i) = 2 \) for all \( i \neq a \).

Assume \( a = b \) below. Let \( j \) be the smallest integer such that \( s_j \) and \( t_j \) do not contain the same set of variables. The least index \( c \) of the variables of \( s_j \) must agree with that of \( t_j \). There exists another variable \( x_d \) in exactly one of \( s_j \) and \( t_j \), say the former. Then \( x_d \) is in \( t_k \) for some \( k > j \). We have

- \( h(s) = 1 \neq 0 = h(t) \) if \( \{0, 1, 2, \ast\} = SC_{41}, h(x_c) = h(x_d) = 0, \) and \( h(x_i) = 2 \) for all \( i \neq \{c, d\}, \) and
- \( h(s) = 2 \neq 0 = h(t) \) if \( \{0, 1, 2, \ast\} = SC_{96}, h(x_d) = h(x_c) = 2, \) and \( h(x_i) = 1 \) for all \( i \neq \{a, c\}. \)

Thus \( s^* \neq t^* \). \( \square \)
Next, we provide an ordered version of Theorem 5.1 that has the same associative spectrum upper bound but a different ac-spectrum upper bound. Recall that the \textit{ordered Bell number} or \textit{Fubini number} \(B_n^\alpha\) counts ordered partitions of the set \([1,2,\ldots,n]\) [13, A000670]. The number of rooted ordered set partitions of \([1,\ldots,n]\) is \(nB_n^\alpha = 1, 2, 9, 52, 375, \ldots [13, A052882]\). We note that \(nB_n^\alpha\) is also the upper bound for the ac-spectrum of a variety of groupoids and can be reached by the 3-element groupoids SC262, SC1812, and SC2446, which are anti-isomorphic to SC1441 (by \(0 \rightarrow 2, 1 \rightarrow 0\), and \(2 \rightarrow 1\)), SC1793 and SC2430, respectively.

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**Theorem 5.2.** A groupoid \((G, *)\) satisfying the identities below must have \(s^\alpha_n(*) \leq 2^{n-2}\) for \(n = 2, 3, \ldots\) and \(s^{ac}_n(*) \leq nB_n^\alpha\) for \(n = 1, 2, \ldots\), where the first inequality holds as an equality whenever the second does.

\((i)\) \(x(yz) \approx x(zy)\), \((ii)\) \(w(xyz) \approx w((xy)z)\)

Moreover, both equalities hold for SC262, SC1812, and SC2446 (hence the anti-isomorphic SC1441, SC1793, and SC2430).

**Proof.** By \((i), (ii),\) and Lemma 2.1, we can transform an arbitrary term \(t \in \mathcal{F}_n\) with leftmost decomposition \(t = [t_0, t_1, \ldots, t_m]\), where \(|t_0| = 1\), to \([t_0, t_1^c, \ldots, t_m^c]\). Thus, in terms of \(\mathcal{F}_n\), we induce the same same operation if they are associated with distinct rooted ordered set partitions. It follows that \(s^{ac}_n(*) \leq nB_n^\alpha\). Restricting the above argument to \(\mathcal{F}_n\), gives \(s^{ac}_n(*) \leq 2^{n-2}\), where the equality holds if \(s^{ac}_n(*) = nB_n^\alpha\).

It is routine to check that SC262, SC1812, and SC2446 all satisfy the identities \((i)\) and \((ii)\). It remains to show that \(s^* \neq t^*\) whenever \(s, t \in \mathcal{F}_n\) are associated with distinct rooted ordered set partitions of \([1,2,\ldots,n]\).

We can write \(s = [x_a, s_1, \ldots, s_t]\) and \(t = [x_b, t_1, \ldots, t_m]\). If \(a \neq b\) then \(s^* \neq t^*\) by the following:

- For SC262, we have \(h(s) = 1 \neq 0 = h(t)\), where \(h(x_a) := 1\) and \(h(x) := 0\) for all \(x \neq x_a\).
- For SC1812 and SC2446, one of \(h(s)\) and \(h(t)\) is 1 and the other is 2, where \(h(x) := 1\) for all \(x \neq t\) and \(m\) have different parities or \(h(x_a) := 1\) and \(h(x) := 2\) for all \(x \neq x_a\) otherwise.

Assume \(a \leq b\). Let \(j\) be the smallest integer such that \(\text{var}(s_j) \neq \text{var}(t_j)\). For SC262, we distinguish two cases.

**Case 1:** \(\text{var}(t_i) \equiv \text{var}(s_j)\) for all \(i\). Define \(h(x) := 2\) for all \(x \in \{x_a\} \cup \{s_j\}\) and \(h(x) := 0\) for all \(x \neq \text{var}(s_j)\). Then \(h(x_a) = 2, h(s_j) = 2, h(s_i) = 0\) for all \(i \neq j\), and \(h(t_i) \in \{0, 1\}\) for all \(i\). One can check that \(h(s) = 1 \neq 0 = h(t)\) when \(j = 1\) and \(h(s) = 0 \neq 1 = h(t)\) when \(j > 1\).

**Case 2:** \(\text{var}(t_k) \equiv \text{var}(s_j)\) for some \(k\). If \(\text{var}(t_k) \equiv \text{var}(s_j)\), then we are back to Case 1 by switching \(s\) and \(t\) and using \(t_k\) instead of \(s_j\), since \(\text{var}(s_j) \equiv \text{var}(t_k)\) for all \(i\). Thus, we may assume that \(\text{var}(s_j) = \text{var}(t_k)\), which implies \(j < k\) since \(\text{var}(s_i) = \text{var}(t_i)\) for all \(i < j\). Define

\[
h(x) := \begin{cases} 
2, & \text{if } x \in \{x_a\} \cup \text{var}(s_1) \cup \cdots \cup \text{var}(s_j) = \text{var}(t_1) \cup \cdots \cup \text{var}(t_{j-1}) \cup \text{var}(t_k); \\
0, & \text{if } x \not\in \{x_a\} \cup \text{var}(s_1) \cup \cdots \cup \text{var}(s_j) = \text{var}(t_1) \cup \cdots \cup \text{var}(t_{j-1}) \cup \text{var}(t_k). 
\end{cases}
\]

We have \(h(s_1) = \cdots = h(s_j) = 2, h(s_i) = 0\) for all \(i = j + 1, \ldots, \ell\), and thus \(h(s) = 1\). On the other hand, we have \(h(t_1) = \cdots = h(t_{j-1}) = h(t_k) = 2, h(t_i) = 0\) for all \(i \in \{j, \ldots, m\}\), and thus \(h(t) = 0 \neq h(s)\).

For SC1812, we may assume that \(\ell\) and \(m\) have the same parity by the all-1 substitution as discussed earlier. We distinguish some cases below.

**Case 1:** \(\text{var}(t_i) \not\equiv \text{var}(s_j)\) for all \(i\). We further distinguish two subcases below.

- Suppose that \(j\) is odd. Define \(h(x) := 0\) for all \(x \in \text{var}(s_j)\) and \(h(x) := 1\) for all \(x \neq \text{var}(s_j)\). Then \(h(x_a) = 1, h(s_j) = 0, h(s_i) \in \{1, 2\}\) for all \(i \neq j\), and \(h(t_i) \in \{1, 2\}\) for all \(i\). One can check that \(h(s) = 1\) if \(\ell\) is odd or \(h(s) = 2\) if \(\ell\) is even. On the other hand, we have \(h(t) = 1\) if \(m\) is even or \(h(t) = 2\) otherwise. Since \(\ell\) and \(m\) have the same parity, it follows that \(h(s) \neq h(t)\).

- Suppose that \(j\) is even. Defined by \(h(x) := 0\) for all \(x \in \text{var}(s_j)\) and \(h(x) := 2\) for all \(x \neq \text{var}(s_j)\). Then \(h(x_a) = 1, h(s_j) = 0, h(s_i) \in \{1, 2\}\) for all \(i \neq j\), and \(h(t_i) \in \{1, 2\}\) for all \(i\). One can check that \(h(s) = 1\) if \(\ell\) is even or \(h(s) = 2\) if \(\ell\) is odd. On the other hand, we have \(h(t) = 1\) if \(m\) is odd or \(h(t) = 2\) if \(m\) is even. Since \(\ell\) and \(m\) have the same parity, we must have \(h(s) \neq h(t)\).
Case 2: \( var(t_k) \subseteq var(s_j) \) for some \( k \). If \( var(t_k) \nsubseteq var(s_j) \), then we are back to Case 1 by switching \( s \) and \( t \) and using \( t_k \) instead of \( s_j \), since \( var(s_j) \nsubseteq var(t_k) \) for all \( i \). Thus we may assume that \( var(s_j) = var(t_k) \), which implies \( j < k \). We further distinguish two subcases below.

- Suppose that \( j \) and \( k \) have different parities. Define \( h(x) := 0 \) for all \( x \in var(s_j) \) and \( h(x) := 2 \) for all \( x \notin var(s_j) \). Then \( h(x_a) = 2, h(s_j) = h(t_k) = 0, h(s_i) \in \{1, 2\} \) for all \( i \neq j, \) and \( h(t_i) \in \{1, 2\} \) for all \( i \neq k \). One can check that \( h(s) = 1 \) if \( j \) has the same parity as \( \ell \) or \( h(s) = 2 \) otherwise. Similarly, \( h(t) = 1 \) if \( k \) has the same parity as \( m \) or \( h(t) = 2 \) otherwise. Since \( \ell \) and \( m \) have the same parity, we must have \( h(s) \neq h(t) \).

- Suppose that \( j \) and \( k \) have the same parity. Define

\[
h(x) := \begin{cases} 
0, & \text{if } x \in \{x_a\} \cup var(s_1) \cup \cdots \cup var(s_j) = var(t_1) \cup \cdots \cup var(t_{j-1}) \cup var(t_k); \\
1, & \text{if } x \notin \{x_a\} \cup var(s_1) \cup \cdots \cup var(s_j) = var(t_1) \cup \cdots \cup var(t_{j-1}) \cup var(t_k). 
\end{cases}
\]

Then \( h(x_a) = h(s_j) = h(t_k) = 0, h(s_i) = h(t_i) = 0 \) for all \( i = 1, \ldots, j - 1, h(s_i) \in \{1, 2\} \) for all \( i = j + 1, \ldots, \ell, \) and \( h(t_i) \in \{1, 2\} \) for all \( i \in \{j, \ldots, m\} \backslash \{k\} \). One can check that \( h(s) = 1 \) if \( j \) and \( \ell \) have different parities or \( h(s) = 2 \) otherwise (note that \( j < \ell \)). Similarly, \( h(t) = 1 \) if \( k \) and \( m \) have the same parity or \( h(t) = 2 \) otherwise. Since \( \ell \) and \( m \) have the same parity, we must have \( h(s) \neq h(t) \).

For SC2446, we may again assume that \( \ell \) and \( m \) have the same parity by the all-1 substitution. There exists a variable \( x_c \) in exactly one of \( s_j \) and \( t_j \), say the former. Then \( x_c \) is in \( t_k \) for some \( k > j \). We distinguish two cases below.

Case 1: \( j \) and \( k \) have different parities. Define \( h(x_a) = h(x_c) = 0 \) and \( h(x) := 1 \) for all \( x \notin \{x_a, x_c\} \). We have \( h(s_j) = 0 \) and \( h(s_i) \in \{1, 2\} \) for all \( i \neq j \). Thus \( h(s) = 1 \) if \( j \) has the same parity as \( \ell \) or \( h(s) = 2 \) otherwise. Similarly, we have \( h(t_k) = 0 \) and \( h(t_i) \in \{1, 2\} \) for all \( i \neq k \). Thus \( h(t) = 1 \) if \( k \) has the same parity as \( m \) or \( h(t) = 2 \) otherwise. Then \( h(s) \neq h(t) \) since \( \ell \) and \( m \) have the same parity.

Case 2: \( j \) and \( k \) have the same parity. Pick any variable \( x_{j'} \) in \( t_{k-1} \), which must be in \( s_{j'} \) for some \( j' > j \). The argument in the above paragraph is valid for \( j' \) and \( k - 1 \) if they have different parities. Otherwise \( j' \) and \( k \) must have different parities, and it follows that \( j' > j \). Define \( h(x_{j'}) = h(x_{j'}) = 0 \) and \( h(x) := 1 \) for all \( x \notin \{x_c, x_{j'}\} \). We have \( h(s_i) = h(s_i) = 0 \) and \( h(s_i) \in \{1, 2\} \) for all \( i \notin \{j, j'\} \). Thus \( h(s) = 1 \) if \( j' \) has the same parity as \( \ell \), or \( h(s) = 2 \) otherwise. Similarly, we have \( h(t_{k-1}) = h(t_{k-1}) = 0 \) and \( h(s_i) \in \{1, 2\} \) for all \( i \notin \{k - 1, k\} \). Thus \( h(t) = 1 \) if \( k \) has the same parity as \( m \), or \( h(t) = 2 \) otherwise. Then \( h(s) \neq h(t) \) since \( \ell \) and \( m \) have the same parity but \( j' \) and \( k \) have different parities.

\[ \square \]

6. Congruence on depths

In this section we discuss the natural occurrence of leaf depths in the study of associative and ac-spectra of groupoids and how it can help us generalize some of our results.

Using both identities and the left/right depth, Hein and the first author [4] determined the associative spectrum of a generalization of addition and subtraction to be the modular Catalan number

\[
C_{k,n} := \sum_{0 \leq j \leq (n-1)/k} \frac{(-1)^j}{n j} \binom{n}{j} \left(\frac{2n-jk}{n+1}\right),
\]

and we determined its ac-spectrum in our previous work [6]. These results are rephrased below to include Proposition 4.3 as a special case (using right depth instead of identities).

**Theorem 6.1** ([4,6]). Let \( (G, *) \) be a groupoid such that for all \( s, t \in F_n \), we have \( s^* = t^* \) whenever \( \rho_i(s) = \rho_i(t) \) (mod \( k \)) for \( i = 1, \ldots, n \). Then \( s_{\mathcal{G}}^{ac}(*) \leq C_{k,n-1} \) and

\[
s_{\mathcal{G}}^{ac}(*) \leq k!S(n, k) + n \sum_{0 \leq i \leq k-2} i!S(n-1, i)
\]

for \( n = 1, 2, \ldots \), where the first equality holds as an equality if the second one does. Moreover, both upper bounds are reached if “whenever” can be replaced with “if and only if” in the above condition. In particular, both upper bounds are attained by \((\mathbb{C}, +)\), where \( a*b := a + e^{i\pi/a^2}b \) for all \( a, b \in \mathbb{C} \).

Now we use the left depth to generalize Proposition 3.4 and Proposition 3.5 as follows.

**Theorem 6.2.** Let \( (G, *) \) be a groupoid such that for all \( s, t \in F_n \), we have \( s^* = t^* \) whenever \( s \) and \( t \) have the same leftmost variable \( x_i \), whose left depths in \( s \) and \( t \) are congruent modulo \( k \). Then \( s_{\mathcal{G}}^{ac}(*) \leq k \) and \( s_{\mathcal{G}}^{ac}(*) \leq kn \) for \( n = k + 1, \ldots \), where the first inequality holds as an equality if the second does. Moreover, both upper bounds are reached if “whenever” can be replaced with “if and only if” in the above condition.
Proof. First, suppose that $s^* = t^*$ whenever $s$ and $t$ have the same leftmost variable $x_i$ and the left depths of $x_i$ in $s$ and $t$ are congruent modulo $k$. Then every term in $F_n$ induces the same $n$-ary operation on $(G, \ast)$ as a standard term $[x_i, x_{i_1}, \ldots, x_{i_m}, \langle x_{i_{m+1}}, \ldots, x_{i_{m+k-1}} \rangle]$, where $i_1 < \cdots < i_{n-1}$ and $m \in \{0, \ldots, k-1\}$. The above standard term is determined by $x_i$ and $m$, for which there are $n$ and $k$ possibilities, respectively (the latter requires $n \geq k + 1$). Thus $s_n^k(*) \leq kn$. Similarly, the standard term of each bracketing in $B_n$ must begin with $x_i$. Thus $s_n^k(*) \leq k$. It is easy to see that $s_n^k(*) = kn$ implies $s_n^k(*) = k$.

Now suppose that $s^* = t^*$ if and only if $s$ and $t$ have the same leftmost variable $x_i$ and the left depths of $x_i$ in $s$ and $t$ are congruent modulo $k$. The "only if" part implies that $s^* \neq t^*$ if $s$ and $t$ correspond to different standard terms. Thus $s_n^k(*) = k$ and $s_n^k(*) = kn$.

Remark 6.1. Hein and the first author [4] observed that the congruence relation modulo $k$ on the left depths of the bracketings in $B_n$ is characterized by the identity $s_0[s_1, \ldots, s_{k+1}] \approx [s_0, s_1, \ldots, s_{k+1}]$ and showed that $C_{k,n-1}$ is the number of terms in $B_n$ avoiding subterms of the form $s_0[s_1, \ldots, s_{k+1}]$. We also have $s_n^k(*) \leq C_{k,n-1}$ for a groupoid $(G, \ast)$ satisfying a different identity $s_0[s_1, \ldots, s_{k+1}] = s_0(s_1[s_2, \ldots, s_{k+1}])$ (1)

since we can still use this identity to transform every bracketing in $B_n$ to some bracketing in $B_n$ that avoids subterms of the form $s_0[s_1, \ldots, s_{k+1}]$. Although not needed for the proof of the upper bound $s_n^k(*) \leq C_{k,n-1}$, we can even show that distinct bracketings $t, t' \in B_n$ both avoiding $s_0[s_1, \ldots, s_{k+1}]$ cannot be obtained from each other by the identity (1), using the technique due to Hein and the first author [4]. In fact, we know that $t$ and $t'$ correspond to two binary trees with $n$ leaves labeled $1, \ldots, n$ from left to right, which in turn correspond to two rooted plane trees $T$ and $T'$ with $n$ vertices labeled $1, \ldots, n$ in the preorder by contracting each northeast "long edge" in the drawings of $t$ and $t'$. If $t$ can be obtained from $t'$ by the identity (1), then a non-root vertex in $T$ must have its degree (the number of children) less than $k$ and congruent to the degree of the vertex with the same label in $T'$ modulo $k - 1$, and the leaves (degree-zero vertices) in $T$ must correspond to the leaves in $T'$. Thus the degrees of the vertices of $T$ must agree with those of $T'$, and this forces $T = T'$.

For $k = 3$, we suspect that $s_n^k(*) = C_{k,n-1}$ holds for SC64, which is anti-isomorphic to SC399.

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SC64

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SC399

In fact, our computations show that the initial terms of the associative spectrum and ac-spectrum of SC64 are 1, 1, 2, 5, 13, 35, 96, 267 and 1, 2, 12, 84, 710, respectively; the former sequence coincides with $C_{3,n-1}$ while the latter differs from the upper bound of $s_n^k(*)$ for $k = 3$ in Theorem 6.1, whose initial terms are 1, 2, 9, 40, 155, 546, 1813, 5804, 18159. One can check that SC64 satisfies at least the four identities below.

$w(x(yz)) \approx w(y(xz))$, $w((xy)z) \approx w((yz)x)$, $(w(x)yz) \approx ((w)yz)x$, $w((w(yz))z) \approx v(((w)x)yz)$

But these identities seem unrelated to the left/right depth modulo $k = 3$.

The first author, Mickey, and Xu [7] used the depth to find the associative spectrum of the double minus operation $a \ast b := -a - b$, and we determined the ac-spectrum of this operation in previous work [6]. Both proofs are valid for any field with at least three elements, giving the following result.

Theorem 6.3 ([7]). Suppose that two terms $s, t \in F_n$ induce the same $n$-ary operation on a groupoid $(G, \ast)$ whenever $d_i(s) = d_i(t) \pmod{2}$ for $i = 1, \ldots, n$. Then $s_n^k(*) \leq 2^n/3$ and $s_n^k(*) \leq (2^n - (-1)^n)/3$ for $n = 1, 2, \ldots$, where the first equality holds as an equality if the second one does. Moreover, both upper bounds are reached if "whenever" can be replaced with "if and only if" in the above condition. In particular, both upper bounds are achieved by the double minus operation on any field with at least three elements.

The two upper bounds in the above theorem are both well studied [13, A000975, A001045] from many other perspectives; the latter is known as the Jacobsthal sequence. The double minus operation on a field of three elements is actually the 3-element groupoid SC2346.

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SC2346
To generalize the above theorem, one could use a primitive root of unity \( \omega := e^{2\pi i/k} \) to define an operation \( a \ast b := \omega a + \omega b \) on the field of complex numbers, which reduces to the double minus operation when \( k = 2 \); for \( k \geq 3 \), the \( n \)-th term of the associative spectrum was shown in [10] to coincide with the number of equivalence classes of the equivalence relation on \( n \)-leaf binary trees that relates two trees if the depths of corresponding leaves are congruent modulo \( k \). Closed formulas for the associative spectrum and the ac-spectrum of this operation are yet to be determined.

7. Questions and remarks

We have some more questions other than those in the last section. Our computations suggest that a majority of the 3330 non-isomorphic 3-element groupoids have their ac-spectrum reaching the upper bound \( n!C_{n-1} \) and thus have their associative spectrum reaching the upper bound \( C_{n-1} \). Some other 3-element groupoids have smaller spectra, including those given earlier in this paper as examples for various upper bounds to be sharp. We also have computational data on the spectra of several other 3-element groupoids but do not have any general result on them.

For instance, our computations show that the first several terms of the associative spectrum and ac-spectrum of each of the following groupoids are 1, 1, 2, 5, 12, 28, 65, 151, 351 and 1, 2, 12, 96, 880, respectively; the former agrees with the initial terms of a trisection of the Padovan sequence [13, A034943].

\[
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 1 & 0 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
2 & 1 & 0 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
SC258 & SC685 & SC1594 & SC1600
\end{array}
\]

It is clear that SC258 and SC685 are anti-isomorphic to SC1594, SC1600, respectively. One can check that SC258 and SC685 both satisfy at least the following identities.

\[(wx)(yz) \approx (wx)(zy), \quad ((wx)yz) \approx ((wx)z)y, \quad (vw)(x(yz)) \approx (vw)((xy)z), \quad v((wx)(yz)) \approx (v(wx))(yz)\]

Next, consider the following 3-element groupoids.

\[
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 0 & 2 \\
1 & 2 & 0 & 2 \\
2 & 2 & 0 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 0 & 2 \\
1 & 1 & 1 & 0 \\
2 & 2 & 0 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
SC1414 & SC1477 & SC1693 & SC1717
\end{array}
\]

There is an anti-isomorphism between SC1414 and SC1717 and between SC1477 and SC1693 by swapping 1 and 2. It is routine to check that SC1414 and SC1693 both satisfy the identities \((wx)(yz) \approx (yz)(wx)\) and \(((wx)yz) \approx ((wx)z)y\). Computations show that the first several terms of its associative spectrum and ac-spectrum are 1, 1, 2, 5, 13, 35, 97, 275, 794, 2327 and 1, 2, 12, 96, 980; the former matches with the initial terms of a generalized Catalan number [13, A025242], which counts Dyck paths of length 2n avoiding \(UUDD\).

Computations also show that the first several terms of the associative spectrum and ac-spectrum of the following two anti-isomorphic groupoids are 1, 1, 2, 5, 14, 42, 132, 429, 1430 and 1, 2, 12, 108, 1340; the former agrees with \(C_{n-1}\) while the latter is less than \(n!C_{n-1}\).

\[
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 \\
2 & 1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
* & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 \\
\end{array}
\quad
\begin{array}{cccc}
\text{SC229} & \text{SC1553}
\end{array}
\]

One can check that SC229 satisfies the identity \(((wx)yz) \approx ((wx)z)y\).

It would be nice if the associative spectra and ac-spectra of the above 3-element groupoids (or even better, groupoids satisfying the same identities as the above groupoids) could be determined.

Another question is about the arithmetic mean on \(\mathbb{R}\). Csákány and Waldhauser [3] showed that its associative spectrum is \(C_{n-1}\). In previous work [6], we showed that its ac-spectrum is the number of ways to write 1 as an ordered sum of \(n\) powers of 2 [13, A007178]. It would be interesting to find the identities that could be used to characterize all the groupoids whose associative spectra and ac-spectra are bounded by the above and if possible, find a 3-element groupoid to achieve the upper bounds.

Lastly, we provide a generalization of a result in our earlier work [6], which asserts that an associative groupoid \((G,\ast)\) must have \(s_n^G(\ast) \leq n!\) and this upper bound holds as an equality if \((G,\ast)\) is noncommutative and has an identity element.
Theorem 7.1. For any groupoid $(G, \ast)$, we have $s_n^{ac}(\ast) \leq n! \cdot s_n^a(\ast)$. Moreover, this inequality holds as an equality if $(G, \ast)$ is noncommutative and has an identity element.

Proof. For a bracketing $t \in B_n$ and a permutation $\sigma \in \mathfrak{S}_n$, let $t$ denote the full linear term obtained by replacing the variable $x_i$ with $x_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$. Consider two full linear terms in $F_n$; they can be written as $s_\sigma$ and $t_\tau$, where $s, t \in B_n$ and $\sigma, \tau \in \mathfrak{S}_n$. It is clear that if $\sigma = \tau$, then $(s_\sigma)^* = (t_\tau)^*$ if and only if $s^* = t^*$. The inequality $s_n^{ac}(\ast) \leq n! \cdot s_n^a(\ast)$ follows immediately from this fact.

Assume now that $(G, \ast)$ is noncommutative and has a neutral element 0. Then there are elements $a, b \in G$ such that $a \ast b \neq b \ast a$. Assume that $\sigma \neq \tau$. Then there exist $i, j \in \{1, \ldots, n\}$ such that $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $\tau^{-1}(i) > \tau^{-1}(j)$. Let $h : X_n \rightarrow G$ be the assignment $x_i \mapsto a$, $x_j \mapsto b$ and $x \mapsto 0$ for all $x \in X_n \setminus \{x_i, x_j\}$. It is easy to see that $h(s_\sigma) = a \ast b$ and $h(t_\tau) = b \ast a$; hence $(s_\sigma)^* \neq (t_\tau)^*$. We conclude that $(s_\sigma)^* = (t_\tau)^*$ if and only if $s^* = t^*$ and $\sigma = \tau$, and the equality $s_n^{ac}(\ast) = n! \cdot s_n^a(\ast)$ follows.

It would be nice to find a sufficient and necessary condition for the upper bound in Theorem 7.1 to hold as an equality.

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References


