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On the Correspondence Between Integer Sequences and Vacillating Tableaux

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Abstract: A fundamental identity in the representation theory of the partition algebra is $n^k = \sum_\lambda f^\lambda m_k^\lambda$ for $n \geq 2k$, where λ ranges over integer partitions of n, f^λ is the number of standard Young tableaux of shape λ, and m_k^λ is the number of vacillating tableaux of shape λ and length $2k$. Using a combination of RSK insertion and jeu de taquin, Halverson and Lewandowski constructed a bijection DI_n^k that maps each integer sequence in $[n]^k$ to a pair of tableaux of the same shape, where one is a standard Young tableau and the other is a vacillating tableau. In this paper, we study the fine properties of Halverson and Lewandowski's bijection and explore the correspondence between integer sequences and the vacillating tableaux via the map DI_n^k for general integers n and k. In particular, we characterize the integer sequences $\boldsymbol{i}$ whose corresponding shape, λ, in the image $DI_n^k(\boldsymbol{i})$, satisfies $\lambda_1 = n$ or $\lambda_1 = n - k$.

Keywords: Deletion-insertion process; Integer sequences; Vacillating tableaux
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1. Introduction

A fundamental identity in the representation theory of the partition algebra is

$$n^k = \sum_{\lambda \vdash n} f^\lambda m_k^\lambda, \tag{1}$$

for $n \geq 2k$, where the sum is over integer partitions λ of n, f^λ is the number of standard Young tableaux (SYT) of shape λ, and m_k^λ is the number of vacillating tableaux of shape λ and length $2k$. This identity reflects a combinatorial analogue of the Schur-Weyl duality between the symmetric group algebra and the partition algebra. Halverson and Lewandowski [10] constructed an elegant bijective proof of Identity (1). Using a deletion-insertion algorithm based on the Robinson-Schensted-Knuth (RSK) insertion algorithm and jeu de taquin (`jdt`), Halverson and Lewandowski associate to each integer sequence $\boldsymbol{i} \in [n]^k$, where $[n] := \{1, 2, \ldots, n\}$, a pair of tableaux of the same shape, one being a standard Young tableau and the other a vacillating tableau. The purpose of the present work is to study the properties of Halverson and Lewandowski's algorithm and to explore the correspondence between integer sequences and the shape of the associated tableaux.

We begin by providing the necessary definitions. A partition of a positive integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $|\lambda| := \lambda_1 + \cdots + \lambda_\ell = n$. We say that the size of λ is n and denote it as $\lambda \vdash n$. In addition, we let the empty partition $\emptyset$ be the only integer partition of 0.

A partition λ can be visually represented by the Young diagram whose j-th row contains λ_j boxes. We adopt the English notation in which the diagrams are left justified with the first row at the top. A *Young tableau* of shape λ is an array obtained by filling each box of the Young diagram of λ with an integer. A Young tableau is *semistandard* if the entries are weakly increasing in every row and strictly increasing in every column. A semistandard Young tableau (SSYT) is *partial* if the entries are all distinct, in which case we call it a *partial tableau.* The *content* of a Young tableau T, denoted as content(T), is the multiset of all the entries in T. If the content of an SSYT T with shape $\lambda \vdash n$ is exactly $[n]$, then we say T is a *standard Young tableau* (SYT). Throughout this paper, we use f^λ to denote the number of SYTs of shape λ. By convention, we set $f^\emptyset = 1$.

Below is the definition of a vacillating tableau as introduced in [10]. To emphasize the dependency on the parameter n, we call it an n-vacillating tableau.

Definition 1.1 ([10]). *Let $n > 0$ and $k \geq 0$ be integers. An* n-vacillating tableau of shape λ and length $2k$ *is a sequence of integer partitions*

$$\Gamma = ((n) = \lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \lambda^{(1\frac{1}{2})}, \dots, \lambda^{(k-\frac{1}{2})}, \lambda^{(k)} = \lambda)$$

such that for each integer $j = 0, 1, \dots, k-1$,

(a) $\lambda^{(j)} \supseteq \lambda^{(j+\frac{1}{2})}$ *and* $|\lambda^{(j)}/\lambda^{(j+\frac{1}{2})}| = 1$,

(b) $\lambda^{(j+\frac{1}{2})} \subseteq \lambda^{(j+1)}$ *and* $|\lambda^{(j+1)}/\lambda^{(j+\frac{1}{2})}| = 1$.

Note that the above conditions imply that $\lambda^{(j)} \vdash n$ and $\lambda^{(j+\frac{1}{2})} \vdash (n-1)$ for each integer j.

Let $\mathcal{VT}_{n,k}(\lambda)$ denote the set of n-vacillating tableaux of shape λ and length $2k$. Following [10], we use m_k^λ for the cardinality of $\mathcal{VT}_{n,k}(\lambda)$. From the definition and using induction, we have that $\lambda^{(j)}$ has at least $n-j$ boxes in the first row. Hence, m_k^λ is non-zero only when the first entry λ_1 of λ satisfies $\lambda_1 \geq n-k$.

There is an equivalent notion of vacillating tableau that does not depend on the parameter n, which was independently introduced by Chen et.al. in [5]. To distinguish from the n-vacillating tableau, we call this second notion the *simplified vacillating tableau.* Given an integer partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t) \vdash n$, let $\lambda^* = (\lambda_2, \dots, \lambda_t) \vdash (n-\lambda_1)$. For an n-vacillating tableau $\Gamma = (\lambda^{(j)} : j = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \dots, k)$ in $\mathcal{VT}_{n,k}(\lambda)$, the *simplified vacillating tableau* Γ^* is the sequence $(\mu^{(j)} : \mu^{(j)} = (\lambda^{(j)})^*$ for $j = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \dots, k)$. One can also define the simplified vacillating tableau in terms of integer partitions as in Definition 2.

Definition 1.2 ([5]). *A* simplified vacillating tableau Γ^* *of shape μ and length $2k$ is a sequence $(\mu^{(j)} : j = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \dots, k)$ of integer partitions such that $\mu^{(0)} = \emptyset$, $\mu^{(k)} = \mu$, and for each integer $j = 0, 1, \dots, k-1$,*

(a) $\mu^{(j)} \supseteq \mu^{(j+\frac{1}{2})}$ *and* $|\mu^{(j)}/\mu^{(j+\frac{1}{2})}| = 0$ *or* 1,

(b) $\mu^{(j+\frac{1}{2})} \subseteq \mu^{(j+1)}$ *and* $|\mu^{(j+1)}/\mu^{(j+\frac{1}{2})}| = 0$ *or* 1.

Denote by $\mathcal{SVT}_k(\mu)$ the set of simplified vacillating tableaux of shape μ and length $2k$. If $n \geq 2k$, it is clear that $\Gamma \leftrightarrow \Gamma^*$ is a bijection between $\mathcal{VT}_{n,k}(\lambda)$ and $\mathcal{SVT}_k(\lambda^*)$.

The set of simplified vacillating tableaux has a close tie with discrete structures. In [5] a bijection is constructed between set partitions on $[k]$ and simplified vacillating tableaux of the empty shape and length $2k$. This bijection reveals the symmetry between maximal crossings and maximal nestings of set partitions. In general, a simplified vacillating tableau of shape λ can be represented by a pair consisting of an arc diagram and a standard Young tableau of shape λ, which yields many combinatorial formulas and identities. See, for example, [3–5, 7]. The same idea was further extended to fillings of Young diagrams, stack polyominoes, and moon polyominoes, under the language of growth diagrams, to describe the properties of maximal increasing and decreasing chains in those fillings. For more on those connections, see [9, 12, 14, 15].

On the other hand, the combinatorial properties of n-vacillating tableaux for general n and k are not well understood. As mentioned previously, the notion of n-vacillating tableau was introduced in [10] to prove Identity (1). For any integer sequence $\boldsymbol{i} = (i_1, i_2, \dots, i_k) \in [n]^k$, Halverson and Lewandowski define an iterative delete-insert process, which we denote by DI_n^k, that associates to $\boldsymbol{i}$ a pair $DI_n^k(\boldsymbol{i}) = (P_n(\boldsymbol{i}), \Gamma_n(\boldsymbol{i}))$, where $P_n(\boldsymbol{i})$ is an SYT of some shape λ of size n, and $\Gamma_n(\boldsymbol{i})$ is an n-vacillating tableau of the same shape λ. The exact definition of $DI_n^k(\boldsymbol{i})$ is given in Section 2. The combinatorial properties of DI_n^k have not been thoroughly studied. For example, there is no known growth diagram description of DI_n^k yet.

The map DI_n^k is an analog of the classical RSK algorithm, which maps a permutation $\sigma \in \mathfrak{S}_n$ to a pair of SYT of the same shape μ of size n. In [8] Greene gave a global description of the shape μ in terms of the longest increasing/decreasing subsequences of σ. The objective of this paper is to investigate the relation between the integer sequence $\boldsymbol{i}$ and the shape λ of the tableaux in the image $DI_n^k(\boldsymbol{i})$.

Definition 1.3. *Let $\boldsymbol{i} \in [n]^k$ be an integer sequence of length k, where n and k are positive integers. Assume $DI_n^k(\boldsymbol{i}) = (P_n(\boldsymbol{i}), \Gamma_n(\boldsymbol{i}))$ for some integer partition λ. Then we say λ is the* VT-shape of $\boldsymbol{i}$. *Define the* VT-index of $\boldsymbol{i}$, *denoted by $\mathrm{vt}_n(\boldsymbol{i})$, to be the size of the shape λ^*. That is, $\mathrm{vt}_n(\boldsymbol{i}) = n - \lambda_1$ is the number of boxes in the Young diagram of λ that are not in the first row.*

See Example 2.1 (cf. Section 2) for the integer sequence $\boldsymbol{i} = (3, 2, 5)$ and its image under DI_6^3. For this integer sequence, the VT-shape is $(3, 2, 1)$, and hence $\mathrm{vt}_6(\boldsymbol{i}) = 3$.

For an integer sequence $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in [n]^k$, clearly $0 \leq \mathrm{vt}_n(\boldsymbol{i}) \leq k$. The exact value of the VT-index of $\boldsymbol{i}$ depends on n. We proved in [3] that when $n \geq 2k + \max(i_1, i_2, \ldots, i_k)$, the vacillating tableau $\Gamma_n^*(\boldsymbol{i})$ is stabilized and is called the *limiting vacillating tableau*, in which case $\mathrm{vt}_n(\boldsymbol{i}) = k$.

In this paper, we study the cases when $\mathrm{vt}_n(\boldsymbol{i}) = 0$ or k for general integers n and k. Section 2 provides the definition of the bijection DI_n^k as well as its main ingredients: the RSK row insertion and jeu de taquin. In Section 3, we characterize the sets of integer sequences for three special VT-shapes: $\lambda = (n)$, $\lambda = (n-k, 1^k)$ for $n \geq k+1$, and $\lambda = (n-k, k)$ for $n \geq 2k$. These sets are in one-to-one correspondences with set partitions of $[k]$ with up to n blocks, the decreasing integer sequences of length k of $[n-1]$, and subdiagonal lattice paths from $(0, 0)$ to $(n-k, k)$, respectively. Note that the VT-index is 0 for the first case and is k for the latter two cases.

In Section 4, we characterize the set of all integer sequences in $[n]^k$ with VT-index k when $n \geq k+1$. More precisely, for each $\lambda \vdash n$ with $\lambda_1 = n - k$, let $\mathcal{I}_k(\lambda)$ be the set of integer sequences in $[n]^k$ whose VT-shape is λ. We introduce a set $\mathcal{R}_k(\lambda)$ of permutations and show that an integer sequence $\boldsymbol{i} \in [n]^k$ has VT-shape λ if and only if $\boldsymbol{i}$ can be obtained from a permutation in $\mathcal{R}_k(\lambda)$ via some simple transformations. The transformations are described in Algorithm A of Section 4, while its inverse is described in Algorithm B. Combining these two algorithms, we develop a test which checks whether a sequence $\boldsymbol{i} \in [n]^k$ has VT-index k.

An important intermediate object in the algorithms of Section 4 is the *bumping sequence*. In Section 5, we characterize the bumping sequences and then reinterpret this characterization in terms of a reparking problem.

Finally, in Section 6, we describe another bijective proof of Identity (1) by Colmenarejo, Orellana, Saliola, Schilling, and Zabrocki [6], which we refer to as the COSSZ bijection. This bijection maps integer sequences in $[n]^k$ to pairs of tableaux of the same shape, with one being an SYT and the other being a standard multiset tableau. We explain when an integer sequence corresponds to a shape λ with $\lambda_1 = n - k$ under the COSSZ bijection.

2. The RSK algorithm and the deletion-insertion process

In this section, we recall the RSK algorithm and the bijection DI_n^k constructed by Halverson and Lewandowski [10]. The main ingredients of the map DI_n^k are the row insertion algorithm and a special case of Schützenberger's jeu de taquin algorithm, which removes a box containing an entry x from a partial tableau and produces a new partial tableau. We adopt the description from [10], while the full version and in-depth discussion of these algorithms can be found in [18, Chapter 3] or [19, Chapter 7].

The RSK row insertion. Let T be a partial tableau of partition shape λ with distinct entries. Let x be a positive integer that is not in T. The operation $x \xrightarrow{RSK} T$ is defined as follows.

(a) Let R be the first row of T.

(b) While x is less than some element in R:

   i) Let y be the smallest element of R greater than x;
   ii) Replace $y \in R$ with x;
   iii) Let $x := y$ and let R be the next row.

(c) Place x at the end of R (which is possibly empty).

The result is a partial tableau of shape μ such that $|\mu/\lambda| = 1$. For each occurrence of Step (b), we say that x *bumps* y to the next row.

The RSK algorithm is an algorithm in algebraic combinatorics that iterates the above insertion procedure. Below is Knuth's construction [11] that applies the algorithm to a two-line array of integers

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}, \tag{2}$$

where (u_j, v_j) are arranged in the non-decreasing lexicographic order from left to right, that is, $u_1 \leq u_2 \leq \cdots \leq u_n$ and $v_j \leq v_{j+1}$ if $u_j = u_{j+1}$.

The RSK algorithm. [11] Given the two-line array in For. 2, construct a pair of Young tableaux (P, Q) of the same shape by starting with $P = Q = \emptyset$. For $j = 1, 2, \ldots, n$:

(a) Insert v_j into tableau P using the RSK row insertion. This operation adds a new box to the shape of P. Assume the new box is at the end of the i-th row of P.

(b) Add a new box with entry u_j at the end of the i-th row of Q.

We call P the *insertion tableau* and Q the *recording tableau.*

Let A and B be two totally ordered alphabets. If $u_j \in A$ and $v_j \in B$ for all j, we say that the two-line array in For. (2) is a generalized permutation from A to B.

Theorem 2.1 (Knuth)**.** *There is a one-to-one correspondence between generalized permutations from A to B and pairs of SSYTs (P, Q) of the same shape, where content$(P) \subseteq B$ and content$(Q) \subseteq A$.*

In particular, there are two special cases of Knuth's RSK algorithm that correspond to permutations and words, as the ones developed by Robinson [17] and Schensted [20].

(i) When $(u_1\ u_2\ \cdots\ u_n) = (1\ 2\ \cdots\ n)$ and $(v_1\ \cdots\ v_n)$ ranges over all permutations of $[n]$, the correspondence gives a bijection between permutations of length n and pairs of SYTs of the same shape.

(ii) When $(u_1\ u_2\ \cdots\ u_n) = (1\ 2\ \cdots\ n)$ and $v_i \in \mathbb{Z}^+$, the correspondence gives a bijection between integer sequences of length n and pairs of Young tableaux of the same shape $\lambda \vdash n$, where P is an SSYT with content in $\mathbb{Z}^+$, and Q is an SYT.

Jeu de taquin. Let T be a partial tableau of shape λ with distinct entries. Let x be an entry in T. The following operation will delete x from T and yield a partial tableau.

(a) Let $c = T_{i,j}$ be the box of T containing x, which is the j-th box from the left in the i-th row of T.

(b) While c is not a corner, i.e., a box both at the end of a row and the end of a column, do

  i) Let c' be the box containing $\min\{T_{i+1,j}, T_{i,j+1}\}$; if only one of $T_{i+1,j}, T_{i,j+1}$ exists, then the minimum is taken to be that single entry;

  ii) Exchange the positions of c and c'.

(c) Delete c.

We denote this process by $x \xleftarrow{\texttt{jdt}} T$ and write `jdt` to denote jeu de taquin.

The bijection DI_n^k from the set of integer sequences in $[n]^k$ to the set of pairs consisting of an SYT and an n-vacillating tableau is built by iterating alternatively between the RSK row insertion and jeu de taquin.

The bijection DI_n^k. [10] Let $(i_1, i_2, \ldots, i_k) \in [n]^k$ be an integer sequence of length k. First, we define a sequence of partial tableaux recursively: the 0-th tableau is the unique SYT of shape (n) with the filling $1, 2, \ldots, n$, namely,

$$T^{(0)} = \boxed{1}\boxed{2}\boxed{\quad\cdots\quad}\boxed{n}\ .$$

Then for integers $j = 0, 1, \ldots, k-1$, the partial tableaux $T^{(j+\frac{1}{2})}$ and $T^{(j+1)}$ are defined by

$$T^{(j+\frac{1}{2})} = \left(i_{j+1} \xleftarrow{\texttt{jdt}} T^{(j)}\right), \tag{3}$$

$$T^{(j+1)} = \left(i_{j+1} \xrightarrow{RSK} T^{(j+\frac{1}{2})}\right). \tag{4}$$

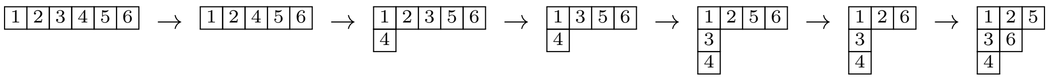
Note that $T^{(j+1)}$ is always an SYT. Let $\lambda^{(m)}$ be the shape of $T^{(m)}$ for all integral and half-integral indices m, and $\lambda = \lambda^{(k)}$ be the shape of the last partial tableau $T^{(k)}$. Finally, let

$$P_n(\boldsymbol{i}) = T^{(k)}, \qquad \Gamma_n(\boldsymbol{i}) = \left(\lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \lambda^{(1\frac{1}{2})}, \ldots, \lambda^{(k)}\right), \tag{5}$$

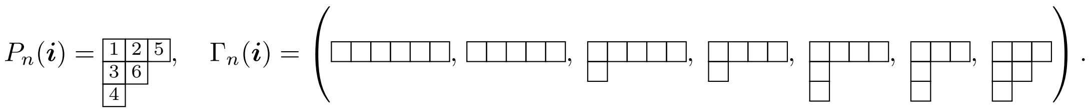
so $P_n(\boldsymbol{i})$ is an SYT of shape λ and $\Gamma_n(\boldsymbol{i})$ is an n-vacillating tableau of shape λ and length $2k$. The image of $\boldsymbol{i}$ under the map DI_n^k is given by $DI_n^k(\boldsymbol{i}) = (P_n(\boldsymbol{i}), \Gamma_n(\boldsymbol{i}))$.

Note that the map DI_n^k is a well-defined bijection for all positive integers n and k. Hence, Identity (1) is indeed true for all $n, k \in \mathbb{Z}_{>0}$.

Example 2.1. *Let $n = 6$, $k = 3$ and $\boldsymbol{i} = (3, 2, 5)$. We apply the map DI_6^3 to obtain the $(T^{(j)})$ sequence as follows.*



Thus, for $DI_6^3(\boldsymbol{i})$, $\lambda = (3, 2, 1)$, and



3. Integer sequences with special VT-shapes

Let n and k be positive integers. Given a partition $\lambda \vdash n$, we are interested in characterizing integer sequences $\boldsymbol{i} \in [n]^k$ with the VT-shape λ. Let $\mathcal{I}_k(\lambda)$ be the set of all such integer sequences $\boldsymbol{i}$. From the definition of DI_n^k, we have that DI_n^k induces a bijection between $\mathcal{I}_k(\lambda)$ and $\mathcal{SYT}(\lambda) \times \mathcal{VT}_{n,k}(\lambda)$, where $\mathcal{SYT}(\lambda)$ is the set of all SYT with shape λ. As a result,

$$|\mathcal{I}_k(\lambda)| = f^\lambda \cdot |\mathcal{VT}_{n,k}(\lambda)|. \tag{6}$$

Next, we characterize the set $\mathcal{I}_k(\lambda)$ for three special shapes: $\lambda = (n), (n-k, 1^k)$, and $(n-k, k)$.

Case 1.

$\lambda = (n)$.

It is proved in [2, 16] that the number $|\mathcal{VT}_{n,k}((n))|$ of n-vacillating tableaux with length $2k$ and shape (n) equals $\sum_{i\le n} S(k,i)$, where $S(k,i)$ is the Stirling number of the second kind and counts the number of set partitions of $[k]$ into exactly i blocks. When $n \ge k$, the sum $\sum_{i\le n} S(k,i)$ is the Bell number $B(k)$, and the equation $|\mathcal{VT}_{n,k}((n))| = B(k)$ is proved in [5, Theorem 2.4]. Since $f^{(n)} = 1$, by (6), we have

$$|\mathcal{I}_k((n))| = f^{(n)} \cdot |\mathcal{VT}_{n,k}((n))| = \sum_{i \le n} S(k,i).$$

Theorem 3.1 characterizes all the sequences $\boldsymbol{i} \in \mathcal{I}_k((n))$.

Theorem 3.1. *Let n and k be positive integers. An integer sequence $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in [n]^k$ is in $\mathcal{I}_k((n))$ if and only if $\boldsymbol{i}$ satisfies the following condition: For each r such that $1 \le r \le k$, if $i_r = m$ for some $m < n$, then there exists an integer s such that $r < s \le k$ and $i_s = m + 1$. In particular, $i_k = n$.*

Proof. We first show that if $\boldsymbol{i} \in \mathcal{I}_k((n))$, then it must have the described property. It is easy to check that the sequence $(n, \ldots, n)$ is in $\mathcal{I}_k((n))$ and satisfies the property. If $\boldsymbol{i} \neq (n, \ldots, n)$, let $m < n$ be an integer appearing in $\boldsymbol{i}$ and $r \le k$ be the latest position such that $i_r = m$. Then in the delete-insert process of DI_n^k when $i_r = m$ is removed by `jdt` and then inserted back using the RSK row insertion, we obtain the tableau $T^{(r)}$, in which m is in the first row and $m + 1$ is in a row strictly below m. (Here we use the English notation for Young diagrams, hence "below" means a row with a larger index.) Assume that the integer $m + 1$ does not appear in $i_{r+1}, \ldots, i_k$. Then we can prove by induction that in the remaining rounds of deletion-insertion $m + 1$ always stays below m. For $j = r + 1, \ldots, k$:

- When deleting i_j by `jdt` from $T^{(j)}$:
  - (i) Assume $i_j < m$. The operation of `jdt` does not move any entry to a row with a larger index. If in the process of `jdt`, $m + 1$ is moved up by exchanging with i_j from position $(t + 1, s)$ to (t, s), then we claim that m cannot be in row t. Otherwise, assume m is at (t, x) with $x < s$. Then the box $(t + 1, x)$ is in the tableau $T^{(j)}$ and occupied by an entry y such that $m < y < m + 1$, which is impossible. Therefore m must be in a row with an index less than t.

    If in the process of `jdt`, $m + 1$ is not moved up, then it stays below m.
  - (ii) Assume $i_j > m + 1$. Then i_j is only exchanged with entries $c > i_j$. Hence the process does not change the position of m or $m + 1$.
- When inserting i_j to $T^{(j+\frac{1}{2})}$ by row insertion:

(i) Assume $i_j < m$. If the insertion path of i_j contains m, assume m is bumped from row t to row $t+1$. Then $m+1$ is either already at a row below $t+1$, or bumped by m to row $t+2$.

If the insertion of i_j does not contain m, since the row insertion algorithm will not move any entry to a row with a smaller index, $m+1$ remains below m.

(ii) Assume $i_j > m+1$. Then i_j only bumps entries larger than $m+1$ and does not change the positions of m and $m+1$.

In any case, $m+1$ stays below m. Therefore we cannot get the shape (n) at the end, which is a contradiction.

Conversely, if $\boldsymbol{i}$ is a sequence satisfying the property in the statement, for any integer m appearing in $\boldsymbol{i}$ whose last appearance is at position r, delete m and insert m will leave m in the first row of $T^{(r)}$. This is because all the later integers in $\boldsymbol{i}$ must be larger than m. Otherwise, if there is a later integer smaller than m, the property implies that there must be another m appearing after position r, contradicting the choice of r. It follows that m will stay in the same box of the first row until the end of the process. The same is true for any integer t that is not appearing in $\boldsymbol{i}$: it will stay in the t-th box of the first row throughout the process. Therefore the ending tableau $T^{(k)}$ must have shape (n). □

Notice that a sequence satisfying the condition in Theorem 3.1 corresponds uniquely to a set partition of $[k]$, where $i_r = i_s$ if and only if r and s are in the same block. If we order the blocks in a set partition according to their largest elements in decreasing order, then the correspondence can be written as: $i_r = m$ if and only if r belongs to the $(n-m+1)$-th block. Hence, $\mathcal{I}_k((n))$ are in one-to-one correspondence with set partitions of $[k]$ with up to n blocks, which gives $|\mathcal{I}_n^k((n))| = \sum_{i\le n} S(k,i)$.

Case 2.

$\lambda = (n-k, 1^k)$ with $n \ge k+1$, i.e., λ is a hook shape with $k+1$ cells in the first column.

There is exactly one n-vacillating tableau of length $2k$ with shape $(n-k,1^k)$, namely, $\lambda^{(i)} = (n-i, 1^i)$ and $\lambda^{(i+\frac{1}{2})} = (n-i-1, 1^i)$ for $i = 0, 1, \ldots, k-1$. Hence, $|\mathcal{VT}_{n,k}((n-k,1^k))| = 1$, which by (6) implies

$$|\mathcal{I}_k((n-k,1^k))| = f^{(n-k,1^k)} = \binom{n-1}{k}.$$

Theorem 3.2. *For positive integers n and k such that $n \ge k+1$, an integer sequence $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k((n-k,1^k))$ if and only if $n > i_1 > i_2 > \cdots > i_k$.*

Proof. Let $\boldsymbol{i} = (i_1, \ldots, i_k)$ be such that $n > i_1 > \cdots > i_k$. Let $\Gamma(\boldsymbol{i}) = (T^{(0)}, T^{(\frac{1}{2})}, T^{(1)}, T^{(1\frac{1}{2})}, \ldots, T^{(k)})$ denote the sequence of partial tableaux obtained by applying DI_n^k to $\boldsymbol{i}$. Since $i_1 < n$, removing i_1 by `jdt` and inserting it back by RSK will make $T^{(1)}$ an SYT of shape $(n-1,1)$ with i_1+1 at position $(2,1)$. We assume, for induction, that $T^{(j)}$ is an SYT of shape $(n-j,1^j)$ with i_j+1 at position $(2,1)$. Since $i_{j+1} < i_{j+1}+1 < i_j+1$, by removing i_{j+1} from $T^{(j)}$ using `jdt`, we obtain a partial tableau of shape $(n-j-1,1^j)$ that still has i_j+1 at position $(2,1)$. Inserting i_{j+1} into $T^{(j+\frac{1}{2})}$ using RSK bumps $i_{j+1}+1$ from the first row to position $(2,1)$ and bumps each integer at position $(s,1)$ to $(s+1,1)$ for all $2 \le s \le j$. Hence, $T^{(j+1)}$ is of shape $(n-j-1,1^{j+1})$ with $i_{j+1}+1$ at position $(2,1)$ and this completes the induction. It follows that $T^{(k)}$ has the shape $(n-k,1^k)$.

Let $I = \{(i_1,\ldots,i_k) \mid n > i_1 > \cdots > i_k\}$. The above paragraph shows that $I \subseteq \mathcal{I}_k((n-k,1^k))$. But $|I| = \binom{n-1}{k} = |\mathcal{I}_k((n-k,1^k))|$. It follows that $I = \mathcal{I}_k((n-k,1^k))$, as desired. □

Case 3.

$\lambda = (n-k,k)$ with $n \ge 2k$.

There is exactly one n-vacillating tableau of length $2k$ with shape $(n-k,k)$, namely, $\lambda^{(i)} = (n-i,i)$ and $\lambda^{(i+\frac{1}{2})} = (n-1-i,i)$ for $i = 0,1,\ldots,k-1$. Therefore, $|\mathcal{VT}_{n,k}((n-k,k))| = 1$ and $|\mathcal{I}_k((n-k,k))| = f^{(n-k,k)}$.

Given an SYT P of shape $(n-k,k)$ with the second row being $\boldsymbol{b} = (b_1, b_2, \ldots, b_k)$, we obtain a sequence $\boldsymbol{i}$ by dividing $(b_1, b_2, \ldots, b_k)$ into disjoint maximal contiguous segments and replace each segment $a, a+1, \ldots, a+\ell-1$ with $a-1, a-1, \ldots, a-1$ (ℓ copies). We show in Theorem 3.3 that the sequence $\boldsymbol{i}$ obtained by the above operation is in $\mathcal{I}_k((n-k,k))$. In fact, P is the SYT component $P_n(\boldsymbol{i})$ when we apply DI_n^k to $\boldsymbol{i}$.

Theorem 3.3. *The above operation gives a bijection ϕ between $\mathcal{I}_k((n-k,k))$ and the set of SYTs of shape $(n-k,k)$. In fact, if $DI_n^k(\boldsymbol{i}) = (P(\boldsymbol{i}), T(\boldsymbol{i}))$, then $\phi(\boldsymbol{i}) = P(\boldsymbol{i})$.*

To illustrate this construction, let's first consider the following example.

Example 3.1. *Let $n = 15$ and $k = 7$. Consider the SYT*

$$P = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 & 9 & 11 & 15 \\ \hline 3 & 5 & 6 & 10 & 12 & 13 & 14 & \\ \cline{1-7}\end{array}$$

whose second row is $\boldsymbol{b} = (3, 5, 6, 10, 12, 13, 14)$. The segments are $3 \,|\, 5, 6 \,|\, 10 \,|\, 12, 13, 14$ which gives $\boldsymbol{i} = (2, 4, 4, 9, 11, 11, 11)$. One can check that in $DI^7_{15}(\boldsymbol{i})$, the SYT is exactly P.

Proof of Theorem 3.3. Notice that any SYT of shape $(n-k, k)$ can be represented by a lattice path from $(0, 0)$ to $(n-k, k)$ that stays on or below the line $y = x$. The lattice path is recorded by a sequence of east (E) and north (N) steps, say, $P_1P_2 \cdots P_n$, where $P_i = E$ if i is in the first row of the SYT and $P_i = N$ if i is in the second row. Let $\boldsymbol{v} = (v_1, v_2, \ldots, v_k)$ be the x-coordinates of the N-steps in the lattice path. Then $\boldsymbol{v}$ is a weakly increasing sequence larger than or equal to $(1, 2, \ldots, k)$ entry-wise. The vectors $\boldsymbol{b}$ and $\boldsymbol{v}$ determine each other by the simple relation that $b_i = i + v_i$. We write

$$\boldsymbol{v} = a_1^{r_1} a_2^{r_2} \cdots a_s^{r_s}, \tag{7}$$

where $a_1 < a_2 < \cdots < a_s$. Note that we must have $a_1 \geq r_1, a_2 \geq r_1 + r_2, \ldots, a_s \geq r_1 + \cdots + r_s$. Then the sequence $\boldsymbol{i}$ obtained by dividing $(b_1, b_2, \ldots, b_k)$ into disjoint maximal contiguous segments and replacing each segment $a, a+1, \ldots, a+\ell-1$ with $a-1, a-1, \ldots, a-1$ (ℓ copies) can be written as

$$\boldsymbol{i} = \boldsymbol{v} + \boldsymbol{\epsilon},$$

where

$$\boldsymbol{\epsilon} = 0^{r_1} r_1^{r_2} (r_1 + r_2)^{r_3} \cdots (r_1 + \cdots + r_{s-1})^{r_s}.$$

Claim: For the above $\boldsymbol{i}$, $\boldsymbol{v}$ and $\boldsymbol{b}$, applying DI^k_n to $\boldsymbol{i}$, we have $P_n(\boldsymbol{i}) = P$, where P is the SYT of shape $\lambda = (n-k, k)$ whose second row is $\boldsymbol{b}$.
Proof of the Claim. If $\boldsymbol{v}$ is given by (7), then

$$\boldsymbol{i} = a_1^{r_1} (a_2 + r_1)^{r_2} \cdots (a_s + r_1 + \cdots + r_{s-1})^{r_s}$$

and

$$\boldsymbol{b} = (a_1 + 1, \ldots, a_1 + r_1, a_2 + r_1 + 1, \ldots, a_2 + r_1 + r_2, \ldots, a_s + 1 + r_1 + \cdots + r_{s-1}, \ldots, a_s + r_1 + \cdots + r_s).$$

Let $\Gamma(\boldsymbol{i}) = (T^{(0)}, T^{(\frac{1}{2})}, T^{(1)}, T^{(1\frac{1}{2})}, \ldots, T^{(k)})$ be the sequence of partial tableaux obtained by applying DI^k_n to $\boldsymbol{i}$. We show by induction that for each integer i, $T^{(i)}$ is the SYT of shape $(n-i, i)$ whose second row consists of the first i entries of $\boldsymbol{b}$. Since $a_1 < n$, removing a_1 by `jdt` and inserting it back by RSK results in $T^{(1)}$ being an SYT of shape $(n-1, 1)$ with $a_1 + 1$ at position $(2, 1)$. Let $i - 1 \geq 1$. Assume that $T^{(i-1)}$ is the SYT of shape $(n-i+1, i-1)$ whose second row comprises the first $(i-1)$ entries of $\boldsymbol{b}$. We can write $i = r_1 + \cdots + r_{j-1} + m$ for some $j \geq 1$ and $1 \leq m \leq r_j$. The i-th entry of $\boldsymbol{i}$ is $a_j + r_1 + \cdots + r_{j-1}$.

Now consider $T^{(i)}$. By the inductive hypothesis, the largest entry in the second row of $T^{(i-1)}$ is $a_j + i - 1 = a_j + r_1 + \cdots + r_{j-1} + m - 1$. Since there are $i - 1$ entries in the second row and the largest entry in the second row is $a_j + i - 1$, there are at least $(a_j + i - 1) - (i - 1) = a_j \geq r_1 + r_2 + \cdots + r_j \geq i$ entries less than $a_j + i - 1$ in the first row of $T^{(i-1)}$.

As $a_j + r_1 + \cdots + r_{j-1}$ is the largest integer less than $a_j + i - 1$ in the first row, there are at least $i - 1$ entries less than $a_j + r_1 + \cdots + r_{j-1}$ in the first row of $T^{(i-1)}$. Consequently, $a_j + r_1 + \cdots + r_{j-1}$ is in the first row with no entry right below it. Therefore, removing $a_j + r_1 + \cdots + r_{j-1}$ by `jdt` and inserting it back by RSK will move the smallest entry greater than $a_j + r_1 + \cdots + r_{j-1}$ in the first row, i.e., $(a_j + i - 1) + 1 = a_j + r_1 + \cdots + r_{j-1} + m$, to the second row, resulting in the desired pattern in $\boldsymbol{b}$. □

We illustrate the proof of Theorem 3.3 next.

Example 3.2. *For the integer sequence in* (3.1), $\boldsymbol{i} = (2, 4, 4, 9, 11, 11, 11) = 2^1 4^2 9^1 11^3$ *and hence* $\boldsymbol{\epsilon} = 0^1 1^2 3^1 4^3$. *The x-coordinates of the North steps are $\boldsymbol{v} = (2, 3, 3, 6, 7, 7, 7)$. We illustrate the corresponding lattice path in Figure 1. We can write $\boldsymbol{v} = 2^1 3^2 6^1 7^3$. Then the sequence $\boldsymbol{i} = \boldsymbol{v} + \boldsymbol{\epsilon} = (2, 4, 4, 9, 11, 11, 11)$, as expected. One can check that the lattice path determined by $\boldsymbol{v}$ corresponding to SYT $P = P_{15}(\boldsymbol{i})$ in* (3.1).

The proof of Theorem 3.3 also establishes a correspondence between sequences in $\mathcal{I}_k((n-k, k))$ and lattice paths from $(0, 0)$ to $(n-k, k)$ below the line $y = x$. Let $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k((n-k, k))$. Assume $\boldsymbol{i} = c_1^{r_1} c_2^{r_2} \cdots c_s^{r_s}$, where $c_1 < c_2 < \cdots < c_s$. Let $\boldsymbol{\epsilon} = 0^{r_1} r_1^{r_2} (r_1 + r_2)^{r_3} \cdots (r_1 + \cdots + r_{s-1})^{r_s}$. Then, $i_j - \epsilon_j$ is the x-coordinate of the North step of a lattice path from $y = j - 1$ to $y = j$ for $1 \leq j \leq k$. It follows that $|\mathcal{I}_k((n-k, k))| = \frac{n-2k+1}{n+1}\binom{n+1}{k} = \frac{n-2k+1}{n-k+1}\binom{n}{k}$, [13, Cor.10.3.2]. In particular, when $n = 2m$ and $k = m$, this gives a bijection between $\mathcal{I}_k((m, m))$ and the set of Dyck paths from $(0, 0)$ to (m, m), which is counted by the m-th Catalan number.

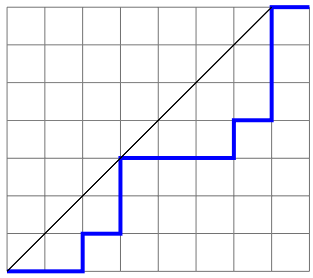


Figure 1: Lattice path corresponding to $\boldsymbol{v} = (2, 3, 3, 6, 7, 7, 7)$.

4. Integer sequences with maximal VT-index

In this section, we characterize all integer sequences in $[n]^k$ with VT-index k. More precisely, for each $\lambda \vdash n$ with $\lambda_1 = n - k$, (or equivalently, $|\lambda^*| = k$), we introduce a set $\mathcal{R}_k(\lambda)$ of permutations, which has the same cardinality as $\mathcal{I}_k(\lambda)$. We establish a bijection between these two sets and show that for $n \geq k + 1$, an integer sequence $\boldsymbol{i} \in [n]^k$ has the VT-shape λ if and only if $\boldsymbol{i}$ can be obtained from a permutation in $\mathcal{R}_k(\lambda)$ via some simple transformations, as defined in Algorithm A in this section.

Lemma 4.1. *Let $\lambda^* = (\lambda_2, \lambda_3, \ldots, \lambda_l)$ be an integer partition of a positive integer k. Let $n \geq k + \lambda_2$ and $\lambda = (n - k, \lambda_2, \lambda_3, \ldots, \lambda_l)$ be an integer partition of n. Then the number of n-vacillating tableaux of shape λ and length $2k$ is f^{λ^*}, the number of SYTs of shape λ^*. Consequently, $|\mathcal{I}_k(\lambda)| = f^{\lambda} \cdot f^{\lambda^*}$.*

Proof. Given $\Gamma_n = (\lambda^{(j)} : j = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, k)$ in $\mathcal{VT}_{n,k}(\lambda)$, let Γ_n^* be the corresponding simplified vacillating tableau $(\mu^{(j)} : \mu^{(j)} = (\lambda^{(j)})^*$ for $j = 0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, k)$. Since $|\mu^{(k)}| = |\lambda^*| = k$, we must have that $\mu^{(j)} = \mu^{(j+\frac{1}{2})}$ and $|\mu^{(j+1)}/\mu^{(j)}| = 1$ for all $j = 0, 1, \ldots, k-1$. Hence Γ_n is in one-to-one correspondence with the SYT Q^* of $\mu^{(k)} = \lambda^*$, where the integer $j \in [k]$ is at the box of $\mu^{(j+1)}/\mu^{(j)}$ in Q^*. This gives $|\mathcal{VT}_{n,k}(\lambda)| = f^{\lambda^*}$. □

Let $n \geq k + 1$. For a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ written in the one-line notation, let $\mathtt{IS}(w)$ be the length of the longest increasing subsequences of w. Define $\mathcal{R}_k^n \subseteq \mathfrak{S}_n$ by letting

$$\mathcal{R}_k^n = \{w_1 w_2 \cdots w_n \in \mathfrak{S}_n : \ w_1 < w_2 < \cdots < w_{n-k} \text{ and } \mathtt{IS}(w) = n - k\}.$$

Let $(P(w), Q(w))$ be the pair of SYTs obtained by applying the RSK algorithm to w, where $P(w)$ is the insertion tableau and $Q(w)$ is the recording tableau. Then $w \in \mathcal{R}_k^n$ if and only if λ, the shape of $P(w)$ and $Q(w)$, satisfies $\lambda_1 = n - k$ and the entries in the first row of $Q(w)$ are $1, 2, \ldots, n - k$. Let $\mathcal{R}_k(\lambda)$ be the permutations $w \in \mathcal{R}_k^n$ such that the shape of $P(w)$ is λ. Then we have $|\mathcal{R}_k(\lambda)| = f^{\lambda} \cdot f^{\lambda^*}$ and

$$\mathcal{R}_k^n = \biguplus_{\substack{\lambda \vdash n \\ \lambda_1 = n-k}} \mathcal{R}_k(\lambda).$$

Note that for $\lambda \vdash n$ with $\lambda_1 = n - k$, $\mathcal{I}_k(\lambda)$ and $\mathcal{R}_k(\lambda)$ have the same size. We define an explicit bijection ψ between these two sets, and then use the permutations in $\mathcal{R}_k(\lambda)$ to characterize the integer squences in $\mathcal{I}_k(\lambda)$.

In the rest of this section, we always assume that $\lambda \vdash n$ with $\lambda_1 = n - k$, (and hence $|\lambda^*| = k$).

A bijection ψ from $\mathcal{I}_k(\lambda)$ to $\mathcal{R}_k(\lambda)$.

Let $\boldsymbol{i} \in \mathcal{I}_k(\lambda)$.

(a) Let $DI_n^k(\boldsymbol{i}) = (P_n(\boldsymbol{i}), \Gamma_n(\boldsymbol{i}))$, where $P_n(\boldsymbol{i}) \in \mathcal{SYT}(\lambda)$ and $\Gamma_n(\boldsymbol{i}) \in \mathcal{VT}_{n,k}(\lambda)$.
Assume $\Gamma = \Gamma_n(\boldsymbol{i}) = (\lambda^{(i)} : i = 0, \frac{1}{2}, 1, \ldots, k)$. Let $\Gamma^* = \Gamma_n^*(\boldsymbol{i}) = (\mu^{(i)} = (\lambda^{(i)})^* : i = 0, \frac{1}{2}, 1, \ldots, k)$ be the corresponding simplified vacillating tableau. Then $\mu^{(0)} = \emptyset$, $\mu^{(k)} = \lambda^*$, and for each integer $i = 0, 1, \ldots, k-1$, $\mu^{(i)} \subset \mu^{(i+1)}$ and $|\mu^{(i)}| = i$.

(b) Let Q^* be an SYT of shape λ^* that corresponds to the sequence $\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(k)}$. That is, entry i of Q^* is in the unique box of $\mu^{(i)}/\mu^{(i-1)}$.

(c) Create a SYT Q of shape λ from Q^* by adding $n - k$ to all the k entries in Q^* (so all these entries form the set $\{n - k + 1, n - k + 2, \ldots, n\}$), followed by adding a top row with entries $1, 2, \ldots, n - k$.

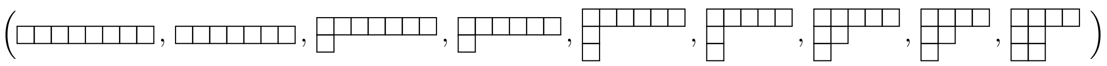
(d) Let w be the unique permutation such that $RSK(w) = (P_n(\boldsymbol{i}), Q)$. Then $w \in \mathcal{R}_k(\lambda)$.

(e) We define $\psi(\boldsymbol{i}) = w$.

Example 4.1. *Let $n = 8$, $k = 4$, and $\boldsymbol{i} = (3, 2, 6, 2)$. Then $DI_8^4(\boldsymbol{i}) = (P_8(\boldsymbol{i}), \Gamma_8(\boldsymbol{i}))$ of shape $\lambda = (4, 2, 2)$ such that*

$$P_8(\boldsymbol{i}) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 8 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}$$

and

$\Gamma_8(\boldsymbol{i}) =$  .

Therefore

$$Q^* = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \qquad \textit{and} \qquad Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 7 \\ \cline{1-2} 6 & 8 \\ \cline{1-2}\end{array}.$$

The pair $(P_8(\boldsymbol{i}), Q)$ corresponds to $w = 45783162$ under RSK. Hence $\psi((3, 2, 6, 2)) = 45783162$.

The inverse of ψ. The map ψ is a bijection since all the steps can be reversed. Starting with $w \in \mathcal{R}_k(\lambda)$, we can recover the integer sequence $\boldsymbol{i}$ via the following steps.

(a′) Apply the RSK algorithm to w to get the insertion tableau $P(w)$ and the recording tableau $Q(w)$. Let λ be the shape of $P(w)$.

(b′) Note that the first row of $Q(w)$ must be exactly $1, 2, \ldots, n-k$. Removing the first row and reducing every remaining entry of $Q(w)$ by $n - k$, we get an SYT Q^* of shape λ^*.

(c′) Let $\Gamma' = (\mu^{(i)} : i = 0, \frac{1}{2}, 1, \ldots, k)$ be the simplified vacillating tableau of length $2k$ and shape λ^* by letting $\mu^{(k)} = \lambda^*$, $\mu^{(0)} = \emptyset$, and $\mu^{(i)} = \mu^{(i+\frac{1}{2})}$ be the shape of Q^* restricted to the boxes with entries $\{1, 2, \ldots, i\}$, for $i = 1, 2, \ldots, k-1$.

(d′) Let Γ be the n-vacillating tableau whose simplified version is Γ'.

(e′) Then $\boldsymbol{i} = \psi^{-1}(w)$ is be the sequence such that $DI_n^k(\boldsymbol{i}) = (P(w), \Gamma)$. Since DI_n^k is a bijection, $\boldsymbol{i} \in [n]^k$ exists and is unique.

The bijection ψ gives a one-to-one correspondence between $\mathcal{I}_k(\lambda)$ and $\mathcal{R}_k(\lambda) \subseteq \mathfrak{S}_n$. The above definition uses the map DI_n^k and the RSK algorithm, hence it is not easy to obtain a characterization of the integer sequences in $\mathcal{I}_k(\lambda)$.

Next, we give a more direct description of the correspondence ψ. We show that the sequence $\boldsymbol{i} \in \mathcal{I}_k(\lambda)$ and $w = \psi(\boldsymbol{i}) \in \mathcal{R}_k(\lambda)$ can be obtained from each other by some simple operations without using jeu de taquin or row insertion. Algorithm A below provides the operations transforming w to $\boldsymbol{i}$, while Algorithm B is the reversed operations of Algorithm A. A key ingredient of these algorithms is an intermediate sequence $\boldsymbol{t} = (t_1, t_2, \ldots, t_k)$, which is called *a bumping sequence* because they are the integers that get bumped out from the first row in the last k rounds when we apply the RSK algorithm in the process of $w \xrightarrow{RSK} (P(w), Q(w))$. From the correspondence ψ, t_j is also the integer that gets bumped out from the first row when we apply the row-insertion $i_j \to T^{(j-\frac{1}{2})}$ in the process of $DI_n^k(\boldsymbol{i})$ for $\boldsymbol{i} \in \mathcal{I}_k(\lambda)$.

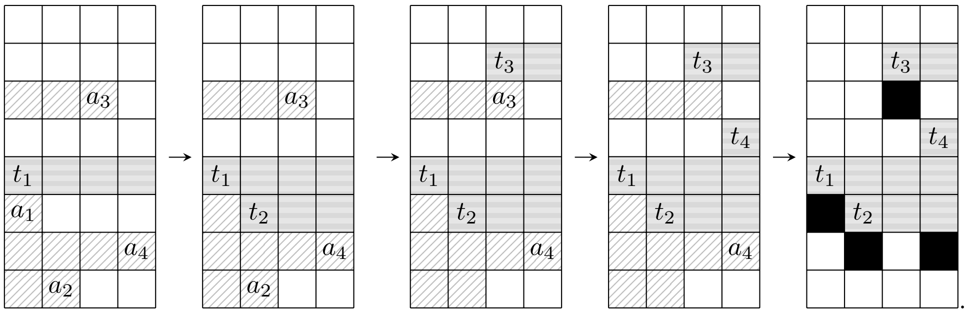
Algorithm A: from $\mathcal{R}_k(\lambda)$ to $\mathcal{I}_k(\lambda)$.

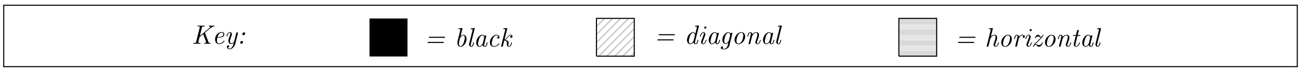
1. Start with $w = b_1 b_2 \cdots b_{n-k} a_1 a_2 \cdots a_k \in \mathcal{R}_k(\lambda)$.
2. For $j = 1$ to k, let
$$t_j := \min\{x \in \mathbb{Z} : \; x > a_j \text{ and } x \neq t_1, \ldots, t_{j-1}, a_{j+1}, \ldots, a_k\}.$$
3. For $j = 1$ to k, let
$$i_j := \max\{c \in \mathbb{Z} : \; c < t_j \text{ and } c \neq t_1, t_2, \ldots, t_{j-1}\}.$$
4. The output is the integer sequence $\boldsymbol{i} = (i_1, i_2, \ldots, i_k)$.

The above procedure can be visualized using an $n \times k$ rectangular grid in $\mathbb{N}^2$. Let (r, c) be the cell in row r from the bottom and column c from the left. Given $w = b_1 \cdots b_{n-k}\; a_1 \cdots a_k \in \mathcal{R}_k(\lambda)$, we use diagonal shading  to shade the cell (a_j, j) and all the cells in the same row on its left $\{(a_j, j') \mid 1 \leq j' < j \leq k\}$ for all $1 \leq j \leq k$.

Starting from $j = 1$ to $j = k$, let t_j be the row index (from bottom) of the lowest un-shaded cell above (a_j, j) in column j, and use horizontal shading to shade the cells on the right of (t_j, j), that is, cells (t_j, j') for all $j \le j' \le k$). After all the cells (t_j, j) are determined, remove the diagonal shading but keep the horizontal ones, and color in black the highest non-shaded cell (i_j, j) below cell (t_j, j). Then $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k(\lambda)$.

Example 4.2. *The following figures demonstrate the implementation of Algorithm A. Let* $n = 8$*,* $k = 4$ *and* $w = 45783162 \in \mathcal{R}_4((4,2,2))$*. We start with* $(a_1, a_2, a_3, a_4) = (3, 1, 6, 2)$*. The process yields* $\boldsymbol{t} = (4, 3, 7, 5)$ *and then* $\boldsymbol{i} = (3, 2, 6, 2) \in \mathcal{I}_4((4,2,2))$*, which corresponds to the black cells in the last grid.*





Theorem 4.1. *Let* $\lambda \vdash n$ *with* $\lambda_1 = n - k$*. Assume that* $\boldsymbol{i}$ *is an integer sequence in* $\mathcal{I}_k(\lambda)$ *and* $\psi(\boldsymbol{i}) = w$*. Then Algorithm A transforms* w *to* $\boldsymbol{i}$*.*

Proof. Let $w = b_1 \cdots b_{n-k}\, a_1 \cdots a_k$. Since $w \in \mathcal{R}_k(\lambda)$, we have that $b_1 < b_2 < \cdots < b_{n-k}$, and in the process of applying the RSK algorithm to w, each term a_i bumps a distinguished number $t_i \in \{b_1, \ldots, b_{n-k}, a_1, \ldots, a_{i-1}\}$ out of the first row. By the definition of row-insertion, t_i is the minimal integer that is larger than a_i and not equal to $t_1, \ldots, t_{i-1}$. Hence, t_i is given by Step (2) of Algorithm A, and the insertion tableau of $\boldsymbol{t} = (t_1, \ldots, t_k)$ under the RSK algorithm is $P(w)^*$, which is obtained from the insertion tableau $P(w)$ of w by removing the first row.

Let $P_0(\boldsymbol{t}) = \emptyset$ and $P_j(\boldsymbol{t})$ be the insertion tableau of $(t_1, \ldots, t_j)$ under RSK for $j = 1, \ldots, k$. Let $(T^{(0)}, T^{(\frac{1}{2})}, T^{(1)}, \ldots, T^{(k)})$ be the sequence of tableaux defined when applying DI_n^k to $\boldsymbol{i}$. By the definition of ψ, $P_j(\boldsymbol{t})$ can be obtained from $T^{(j)}$ by removing the first row, and the first row of $T^{(j)}$ contains exactly integers in $[n] \setminus \mathrm{content}(P_j(\boldsymbol{t}))$. It follows that the shape of $T^{(j)}$ (resp. $T^{(j-\frac{1}{2})}$) is obtained from that of $P_j(\boldsymbol{t})$ by adding a first row of $n - j$ (resp. $n - j - 1$) boxes.

Recall that in the inverse map of DI_n^k, for $j = k, k-1, \ldots, 1$, we get the integer i_j as the unique number such that $T^{(j)} = (i_j \xrightarrow{RSK} T^{(j-\frac{1}{2})})$. From the definition of row-insertion, we see that i_j is the largest number in the first row of $T^{(j)}$ that is smaller than t_j, as described by Step (3) of Algorithm A. □

Corollary 4.1. *Let* $n \ge k + 1$*. The set of integer sequences* $\boldsymbol{i} \in [n]^k$ *with VT-index* k *can be obtained by applying Algorithm A to permutations in* $\mathcal{R}_k^n$*.*

Example 4.3. *Let* $n = 4$ *and* $k = 2$*. There are five integer sequences* $(i_1, i_2) \in [4]^2$ *with VT-index* 2*, and* $\mathcal{R}_2^4 = \{1432, 2413, 2431, 3412, 3421\}$*. By Algorithm A we get that the set of integer sequences with VT-index* 2 *are the union of* $\mathcal{I}_2((2,2)) = \{(2,2), (1,3)\}$ *and* $\mathcal{I}_2((2,1,1)) = \{(3,2), (3,1), (2,1)\}$*.*

Algorithm A can be easily inverted, giving a direct way to obtain $w = \psi(\boldsymbol{i})$ for $\boldsymbol{i} \in \mathcal{I}_k(\lambda)$. We now describe the inverse operations in Algorithm B.

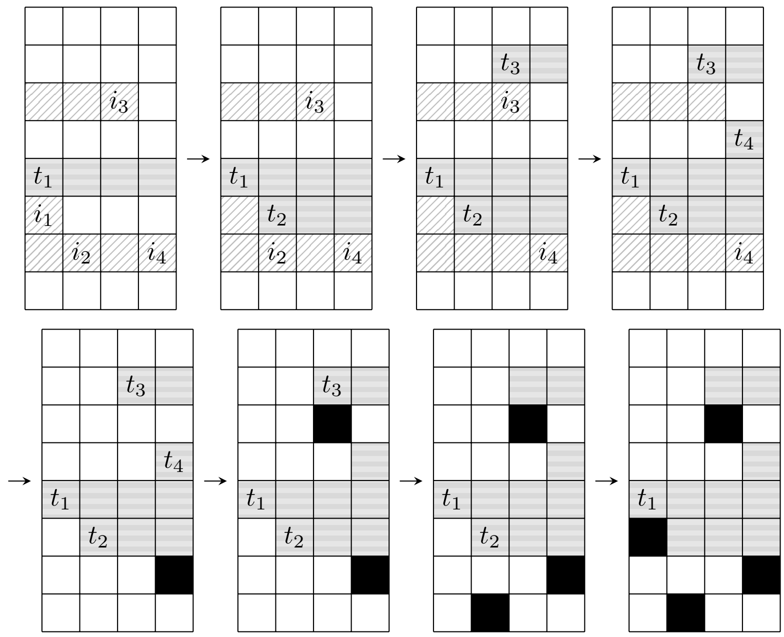
Algorithm B: from $\mathcal{I}_k(\lambda)$ to $\mathcal{R}_k(\lambda)$

1. Start with $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k(\lambda)$.
2. For $j = 1$ to k, let
$$t_j := \min\{x \in \mathbb{Z} : \ x > i_j \text{ and } x \neq t_1, \ldots, t_{j-1}, i_{j+1}, \ldots, i_k\}.$$
3. For $j = k$ to 1, let
$$a_j := \max\{c \in \mathbb{Z} : \ c < t_j \text{ and } c \neq t_1, \ldots, t_{j-1}, a_{j+1}, \ldots, a_k\}.$$
4. Let $w = b_1 b_2 \cdots b_{n-k}\ a_1 a_2 \cdots a_k$, where $b_1 < b_2 < \cdots < b_{n-k}$ are the elements in $[n] \setminus \{a_1, \ldots, a_k\}$.

Algorithm B can be visualized as operations in a rectangle grid. Given $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k(\lambda)$, for each $j = 1, \ldots, k$ use diagonal shading ▨ to shade all the cells (i_j, j') for $1 \leq j' \leq j$. Starting from $j = 1$ to $j = k$, find the lowest row t_j above (i_j, j) in column j such that the cell (t_j, j) is not shaded either diagonally ▨ or horizontally ▤, and shade all the cells (t_j, j') horizontally ▤ for all $j \leq j' \leq k$.

Next, remove the diagonal shading but keep the horizontal ones. Starting from $j = k$ to $j = 1$, find the highest row $a_j \leq t_j$ such that (a_j, j) is not shaded and row a_j does not have a black cell. Color cell (a_j, j) black. Let $b_1 < b_2 < \cdots < b_{n-k} = n$ be the elements in $[n] \setminus \{a_1, a_2, \ldots, a_k\}$ in an increasing order. Then $w = b_1 \cdots b_{n-k}\ a_1 \cdots a_k \in \mathcal{R}_k(\lambda)$ and $\boldsymbol{t} = (t_1, t_2, \ldots, t_k)$ is the corresponding bumping sequence.

Example 4.4. *Let $n = 8$ and $k = 4$. We start with $\boldsymbol{i} = (3, 2, 6, 2) \in \mathcal{I}_4((4, 2, 2))$. The grids in the first row shows how to obtain the bumping sequence $\boldsymbol{t} = (4, 3, 7, 5)$, and the second row shows how to obtain $a_4 = 2$, $a_3 = 6$, $a_2 = 1$ and $a_1 = 3$, which are represented by the black squares. From this we obtain $w = 45783162 \in \mathcal{R}_4((4, 2, 2))$.*



Given an integer sequence $\boldsymbol{i} \in [n]^k$, by applying Algorithm B, if either in Step (2), $t_j > n$ for some j, or in Step (3) $a_j \leq 0$ for some j, then the VT-index of $\boldsymbol{i}$ is not k.

Example 4.5. *Let $n = 4$ and $k = 2$.*

(i) For $\boldsymbol{i} = (4, 4)$, Step (2) of Algorithm B gives $\boldsymbol{t} = (5, 6) \notin [n]^k$. Hence the VT-index of $(4, 4)$ is not 2.

(ii) For $\boldsymbol{i} = (1, 1)$, Step (2) gives $\boldsymbol{t} = (2, 3)$ and then Step (3) gives $a_2 = 1, a_1 = 0$. Hence the VT-index of $(1, 1)$ is not 2.

However, there exists an integer sequence $\boldsymbol{i} \in [n]^k$ that can pass through Algorithm B and yield $\boldsymbol{t} \in [n]^k$ and $w \in \mathcal{R}_k^n$, yet $\boldsymbol{i}$ does not have VT-index k. For example, by applying Algorithm B to $(1, 2)$ we obtain $\boldsymbol{t} = (3, 4)$ and $w = 3412$, yet $\mathrm{vt}_4((1, 2)) = 1$. In fact, we have $\psi^{-1}(3412) = (2, 2)$.

Moreover, the correspondence between $w \in \mathcal{R}_k^n$ and the bumping sequence $\boldsymbol{t}$ is one-to-one. More precisely, for a permutation $w = b_1 \cdots b_{n-k}\ a_1 \cdots a_k$ with $b_1 < b_2 < \cdots < b_{n-k}$, $w \in \mathcal{R}_k^n$ if and only of Step (2) of Algorithm A gives a sequence $\boldsymbol{t} = (t_1, \ldots, t_k) \in [n]^k$. This follows from the RSK algorithm and Schensted's Theorem. Conversely, for a sequence $\boldsymbol{t} = (t_1, \ldots, t_k) \in [n]^k$, if Step (3) of Algorithm B gives a sequence $(a_1, \ldots, a_k) \in [n]^k$, then $w = b_1 \cdots b_{n-k}\ a_1 \cdots a_k \in \mathcal{R}_k^n$, where $b_1 < b_2 < \cdots < b_{n-k}$ are the elements in $[n]\backslash\{a_1, \ldots, a_k\}$. This follows from the fact that row-insertion is invertible.

The above argument implies that we can combine Algorithms A and B together to build a test to check whether a sequence $\boldsymbol{i} \in [n]^k$ has VT-index k.

Theorem 4.2. *The VT-index of a sequence $\boldsymbol{i} \in [n]^k$ is k if and only if the following two conditions hold.*

(a) The sequences $\boldsymbol{t}$ and $(a_1, a_2, \ldots, a_k)$ obtained by applying Algorithm B to $\boldsymbol{i}$ are in $[n]^k$.

(b) Applying Step (3) of Algorithm A to $\boldsymbol{t}$, we recover the sequence $\boldsymbol{i}$.

Note that condition (b) of Theorem 4.2 is not sufficient. For instance, in (ii) of Example 4.5, when $n = 4$, $k = 2$, and $\boldsymbol{i} = (1, 1)$, applying Step (2) of Algorithm B, we get $\boldsymbol{t} = (2, 3)$. Then applying Step (3) of Algorithm A to $(2, 3)$, we recover $(1, 1)$. Yet, $(1, 1)$ has VT-index 1. The missing step is Step (3) of Algorithm B.

In our algorithms the three sequences, $\boldsymbol{i}$, $\boldsymbol{t}$, and $(a_1, \ldots, a_k)$, are closely related. We know that the sequence $(a_1, \ldots, a_k)$ consists of the last k terms of permutations in $\mathcal{R}_k^n$. In the next section, we give an explicit characterization of the bumping sequence $\boldsymbol{t}$.

5. The bumping sequences and a reparking problem

From the construction of Algorithms A and B in Section 4, $\boldsymbol{t}$ is a bumping sequence of a permutation $w \in \mathcal{R}_k^n$ if and only if $\boldsymbol{t}$ yields $(a_1, a_2, \ldots, a_k) \in [n]^k$ in Step (3) of Algorithm B. It raises the question: Given a sequence $\boldsymbol{t} \in [n]^k$ with distinct terms, under what condition does Step (3) of Algorithm B yield a sequence $(a_1, a_2, \ldots, a_k) \in [n]^k$? We answer this question in Theorem 5.1.

For a sequence of distinct integers $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, let $g(a_i)$ be the maximal length among all of the increasing subsequences of $\boldsymbol{a}$ that end at a_i. The following proposition is well-known.

Proposition 5.1. *[1, Chapter 8.4] Let $P(\boldsymbol{a})$ be the insertion tableau of $\boldsymbol{a}$ under the RSK algorithm. Then the entry in the j-th box of the first row of $P(\boldsymbol{a})$ is a_i, where $i = \max\{k : g(a_k) = j\}$. In particular, when inserting a_i, it bumps the term a_t where t is the largest index less than i satisfying $g(a_t) = j$.*

Theorem 5.1. *Let $n > k$ and $(t_1, \ldots, t_k) \in [n]^k$ be a sequence of distinct integers. Let $t_{\max} = \max\{t_1, \ldots, t_k\}$ and $t_{\min} = \min\{t_1, \ldots, t_k\}$. Let $m_1 < \cdots < m_j$ denote the elements in $\{t_{\min}, t_{\min} + 1, \ldots, t_{\max}\}\backslash\{t_1, \ldots, t_k\}$. Let $l = \mathtt{IS}(t_1, \ldots t_k, m_1, \ldots, m_j)$ be the length of the longest increasing subsequence of $(t_1, \ldots, t_k, m_1, \ldots, m_j)$. Then $(t_1, \ldots, t_k)$ is the bumping sequence of a permutation $w \in \mathcal{R}_k^n \subseteq \mathfrak{S}_n$ if and only if $t_{\min} > l - j$.*

Proof. Suppose that $\boldsymbol{t} = (t_1, \ldots, t_k) \in [n]^k$ is the bumping sequence corresponding to a permutation $w \in \mathcal{R}_k^n$. By the definition of RSK, the sequence $\boldsymbol{t}$ can be generated by applying Step (2) of Algorithm A to w. We will construct two sequences, $\boldsymbol{u}$ and $\boldsymbol{t_m}$, and prove $t_{\min} > l - j$ by comparing the RSK insertion tableaux of these two sequences. Consider the sequence

$$\boldsymbol{u} = (t_1, \ldots, t_k, 1, 2, \ldots, t_{\min} - 1, m_1, \ldots, m_j, t_{\max} + 1, \ldots, n). \tag{8}$$

Since $\boldsymbol{t}$ is Knuth equivalent to the reading word of its RSK insertion tableau $P(\boldsymbol{t})$, and the first row of the tableau $P(w)$ is exactly $(1, 2, \ldots, t_{\min} - 1, m_1, \ldots, m_j, t_{\max} + 1, \ldots, n)$, we conclude that $\boldsymbol{u}$ shares the same RSK insertion tableau as the permutation w. Moreover, the length of the first row of this insertion tableau is at least as long as the length of the first row of the insertion tableau of the sequence $\boldsymbol{t_m} = (t_1, \ldots t_k, m_1, \ldots, m_j, t_{\max} + 1, \ldots, n)$, which is $l + n - t_{\max}$ by Schensted's Theorem. Therefore $n - k \geq l + n - t_{\max}$. Noting that $t_{\max} = t_{\min} + k + j - 1$, we get $t_{\min} \geq l - j + 1 > l - j$.

Next, we show that if $t_{\min} > l - j$, there exists a sequence $w \in \mathcal{R}_k^n$ such that its bumping sequence is $\boldsymbol{t}$. To accomplish this, we proceed with the following construction. Assume that $t_{\min} > l - j$. Consider the sequence $\boldsymbol{u}$ as defined in (8). Then the maximal size of increasing subsequences of $\boldsymbol{u}$ is $\max\{t_{\min} - 1 + j + n - t_{\max}, l + n - t_{\max}\} = t_{\min} - 1 + j + n - t_{\max}$. By Proposition 5.1, the entries in the first row of the insertion tableau $P(\boldsymbol{u})$ of $\boldsymbol{u}$ are $1, 2, \ldots, t_{\min} - 1, m_1, \ldots, m_j, t_{\max} + 1, \ldots, n$. Consequently, the insertion tableau for the bumping sequence $\boldsymbol{t}$ is positioned below the first row of $P(\boldsymbol{u})$. We can then apply the inverse RSK algorithm to construct a sequence w with the same insertion tableau as $P(\boldsymbol{u})$. The recording tableau for w has entries $1, 2, \ldots, n - k$ in the first row. Below the first row, the recording tableau is formed by adding $n - k$ to each entry of the recording tableau for the bumping sequence $(t_1, \ldots, t_k)$. Clearly, such a sequence w belongs to $\mathcal{R}_k^n$. $\square$

Comparing Step (2) of Algorithm A and Step (3) of Algorithm B, we observe that there is a symmetry between the sequence $\boldsymbol{a} = (a_1, \dots, a_k)$ and $\boldsymbol{t}$. Namely, $\boldsymbol{t}$ is the bumping sequence of $\boldsymbol{a}$ if and only if $\boldsymbol{a}^{rc}$ is the bumping sequence of $\boldsymbol{t}^{rc}$, where for a sequence $\boldsymbol{x} = (x_1, \dots, x_k) \in [n]^k$,

$$\boldsymbol{x}^{rc} := (n+1-x_k, n+1-x_{k-1}, \dots, n+1-x_1)$$

is the reverse complement of $\boldsymbol{x}$. Hence, Theorem 5.1 gives the following characterization of permutations in $\mathcal{R}_k^n$.

Corollary 5.1. *Let $n \geq k+1$ and $(a_1, a_2, \dots, a_k) \in [n]^k$ be a sequence of distinct integers. Let $a_{\max} = \max\{a_1, a_2, \dots, a_k\}$ and $a_{\min} = \min\{a_1, \dots, a_k\}$. Let $n_1 < \cdots < n_j$ denote the elements in the set $\{a_{\min}, a_{\min}+1, \dots, a_{\max}\} \setminus \{a_1, \dots, a_k\}$. Let $l' = \mathtt{IS}(n_1, \dots, n_j, a_1, \dots, a_k)$. Then $(a_1, \dots, a_k)$ is the last k entries of a permutation $w \in \mathcal{R}_k^n \subseteq \mathfrak{S}_n$ if and only if $n+1-a_{\max} > l' - j$.*

Proof. The sequence $\boldsymbol{a} = (a_1, \dots, a_k)$ is obtained from taking the last k entries of a permutation $w \in \mathcal{R}_k^n$ if and only if $\boldsymbol{a}^{rc}$ is a bumping sequence as described in Theorem 5.1. Hence, $\min\{n+1-a_i\} > l' - j$, where $l' = \mathtt{IS}(n+1-a_k, \dots, n+1-a_1, n+1-n_j, \dots, n+1-n_1)$, which is equal to $\mathtt{IS}(n_1, \dots, n_j, a_1, \dots, a_k)$. ☐

Theorem 5.1 and Corollary 5.1 can be described in terms of the following *reparking problem.*

Suppose that there are k cars parked on a street with n parking spots numbered 1 through n from the left to right, where car i is parked at position x_i. Assume $n_1 < \cdots < n_j$ are the empty spots between $x_{\min}$ and $x_{\max}$.

1. Starting with car 1, for each car j where $j = 1, 2, \dots, k$, we ask it to move one by one to the nearest available spot to its right. All cars can find new spots to park if and only if the number of empty spots at the end of the street is at least $l' - j$, where $l' = \mathtt{IS}(n_1, \dots, n_j, x_1, \dots, x_k)$.

2. Starting with car k, for each car j where $j = k, k-1, \dots, 1$, we ask it car to move one-by-one to the nearest available spot on its left. All cars can find new spots to park if and only if the number of empty spots at the beginning of the street is at least $l - j$, where $l = \mathtt{IS}(x_1, \dots, x_k, n_1, \dots, n_j)$.

Example 5.1. *Let $k = 5$ and assume that cars are parked at positions $\boldsymbol{x} = (3, 2, 5, 8, 9)$. Then $j = 3$ and $(n_1, n_2, n_3) = (4, 6, 7)$.*

1. *For $j = 1, 2, \dots, k$, move car j one-by-one to the closest available spot on its right. Then the cars are moved to spots $(4, 3, 6, 10, 11)$. The cars occupy two new spots on the right. Note that $l' = \mathtt{IS}(4, 6, 7, 3, 2, 5, 8, 9) = 5$ and $2 = l' - j$.*

2. *For $j = k, k-1, \dots, 1$, we move car j one-by-one to the closest available spot on its left. Then the cars are moved to spots $(2, 1, 4, 6, 7)$, where car 1 moves to spot 2, car 2 moves to spot 1, car 3 moves to spot 4, car 4 moves to spot 6, and car 5 moves to spot 7. The cars occupy one new spots on the left. Note that $l = \mathtt{IS}(3, 2, 5, 8, 9, 4, 6, 7) = 4$ and $l - j = 1$.*

6. Final Remark

Recently, there is another bijective proof of Identity (1) constructed by Colmenarejo, Orellana, Saliola, Schilling, and Zabrocki [6], which we call the COSSZ bijection. This bijection maps integer sequences in $[n]^k$ to pairs of tableaux of the same shape $\lambda \vdash n$ with one being a standard Young tableau of content $[n]$ and the other being a standard multiset tableaux of content $[k]$. A standard multiset tableau of shape λ is a filling of the Young diagram of λ with disjoint subsets of $[k]$ that is increasing along each row and each column, where disjoint subsets are ordered by their maximum elements, i.e., for two disjoint subsets A and B of $[k]$, $A \prec B$ if and only if $\max(A) \leq \max(B)$. In particular, $\emptyset \prec A$ if A is non-empty.

Under the COSSZ bijection, each integer sequence $\boldsymbol{i} \in [n]^k$ is mapped to pairs of tableaux of the same shape $\lambda \vdash n$. We can again ask: for which $\boldsymbol{i}$ does the shape λ satisfy $\lambda_1 = n - k$? We show that such integer sequences can again be characterized by the set $\mathcal{R}_k^n \subseteq \mathfrak{S}_n$, in a more direct way compared to the integer sequences under DI_n^k. Note that the COSSZ bijection follows the same idea as in [5], using only the RSK insertion (Knuth's version) but not jeu de taquin. The difference between the result here and those in Section 4 shows the intrinsic complexity and the subtle relation between the row-insertion and jeu de taquin.

Below is the description of the COSSZ bijection, which is defined for all integers $n \geq 1$ and $k \geq 0$. Let $\boldsymbol{i} = (i_1, i_2, \dots, i_k) \in [n]^k$ be an integer sequence.

(a) Let $M_r = \{j : i_j = r\}$ be the positions of integer r in the sequence $\boldsymbol{i}$. Then the non-empty M_r's form a set partition P of $[k]$. Assume that P has t blocks.

(b) Form a two-line array

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

where $a_1 \leq a_2 \leq \cdots \leq a_n$ is the non-decreasing rearrangement of $\{\emptyset^{n-t}\} \cup P$, ordered by their largest elements, and $w = b_1 b_2 \cdots b_n$ is a permutation of length n such that

(i) $b_1 < b_2 < \cdots < b_{n-t}$ are the integers $x \in [n]$ such that $M_x = \emptyset$. That is, $M_{b_1} = \cdots = M_{b_{n-t}} = \emptyset$.

(ii) $a_i = M_{b_i}$ if $M_{b_i} \neq \emptyset$.

(c) Apply the RSK algorithm to the two-line array A to obtain a pair of tableaux (S, T) of the same shape λ, where $S \in \mathcal{SYT}(\lambda)$ is the insertion tableau of $b_1 b_2 \cdots b_n$, and T is the recording tableau, whose entries are subsets M_r's, for $r = 1, 2, \ldots, n$.

The image of $\boldsymbol{i}$ is then (S, T).

Example 6.1. *Let $n = 6$, $k = 4$, and $\boldsymbol{i} = (3, 2, 6, 2)$. Then $M_2 = \{2, 4\}$, $M_3 = \{1\}$, $M_6 = \{3\}$ and $M_1 = M_4 = M_5 = \emptyset$. Hence the two-line array is*

$$A = \begin{pmatrix} \emptyset & \emptyset & \emptyset & \{1\} & \{3\} & \{2,4\} \\ 1 & 4 & 5 & 3 & 6 & 2 \end{pmatrix}.$$

Applying Knuth's RSK, we obtain that the image of $\boldsymbol{i}$ under the COSSZ bijection is

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \qquad T = \begin{array}{|c|c|c|c|}\hline \emptyset & \emptyset & \emptyset & 3 \\ \hline 1 \\ \cline{1-1} 24 \\ \cline{1-1}\end{array}.$$

Assume $n \geq k+1$. We consider when the image of $\boldsymbol{i}$ has a shape $\lambda \vdash n$ with $\lambda_1 = n - k$. In that case, each entry not in the first row of T must be a non-empty subset of $[k]$, hence the partition formed by non-empty M_r's has at least k blocks. But it is a set partition of $[k]$, hence each non-empty M_r must be a singleton block and the entries of the first row of T must all be the empty set. It follows that $a_1 = a_2 = \cdots = a_{n-k} = \emptyset$ and $a_{n-k+j} = \{j\}$ for $j \in [k]$. Therefore $b_1 < b_2 < \cdots < b_{n-k}$ and $b_{n-k+j} = i_j$, i.e., $\boldsymbol{i}$ is exactly $(b_{n-k+1}, b_{n-k+2}, \ldots, b_n)$, the last k integers of the second row of the two-line array.

Note that $w = b_1 b_2 \cdots b_n$ is a permutation of length n. The shape of P and T is λ implies that $\texttt{IS}(b_1 b_2 \cdots b_n) = \lambda_1 = n - k$. Hence, $w \in \mathcal{R}_k^n$. In summary, under the COSSZ bijection, an integer sequence $\boldsymbol{i} \in [n]^k$ is mapped to a pair of tableaux of shape $\lambda \vdash n$ with $\lambda_1 = n - k$ if and only if $\boldsymbol{i}$ consists of the last k entries of a permutation $w \in \mathcal{R}_k^n$.

Example 6.2. *Let $n = 8$, $k = 4$, and $\boldsymbol{i} = (3, 1, 6, 2)$. Then $M_1 = \{2\}$, $M_2 = \{4\}$, $M_3 = \{1\}$, $M_6 = \{3\}$ and $M_4 = M_5 = M_7 = M_8 = \emptyset$. Hence the two-line array is*

$$A = \begin{pmatrix} \emptyset & \emptyset & \emptyset & \emptyset & \{1\} & \{2\} & \{3\} & \{4\} \\ 4 & 5 & 7 & 8 & 3 & 1 & 6 & 2 \end{pmatrix}.$$

Applying Knuth's RSK, we obtain that the image of $\boldsymbol{i}$ under the COSSZ bijection is

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 8 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}, \qquad T = \begin{array}{|c|c|c|c|}\hline \emptyset & \emptyset & \emptyset & \emptyset \\ \hline 1 & 3 \\ \cline{1-2} 2 & 4 \\ \cline{1-2}\end{array}$$

and the shape of S and T is $\lambda = (4, 2, 2)$, which satisfies $\lambda_1 = n - k$. Observe that $\boldsymbol{i}$ consists of the last four letters of the permutation $w = 45783162$, which is in $\mathcal{R}_4^8$.

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