

Pattern-Restricted One-Cycle Permutations With a Pattern-Restricted Cycle Form

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ABSTRACT: In this paper, we consider permutations composed of one cycle that avoid the monotone decreasing permutation $k(k-1)\dots 21$, whose cycle form also demonstrates some pattern avoidance. If the cycle is written in the form with 1 appearing at the beginning of the cycle and this form avoids a pattern of length 3, we find answers in terms of continued fraction generating functions. We also consider the case that every cyclic rotation of the cycle form of the permutation avoids a pattern of length 4 and enumerate two such cases.

Keywords: Cyclic permutations; Generating functions; One-cycle permutations; Pattern avoidance

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1. Introduction and background

The symmetric group \mathcal{S}_n consists of permutations (i.e., bijections) of the set $[n] = \{1, 2, \dots, n\}$. These permutations can be written either in their cycle form as a product of disjoint cycles or in one-line form, as $\pi = \pi_1\pi_2\dots\pi_n$ where $\pi_k = \pi(k)$ for any $k \in [n]$. We say a permutation $\pi \in \mathcal{S}_n$ is a *one-cycle* permutation (also called a *cyclic* permutation) if it is composed of a single n -cycle. In this paper, we denote by $C(\pi)$ the cycle form of a one-cycle permutation π , starting with the element 1; that is, $C(\pi) = (1, c_2, c_3, \dots, c_n)$ where $c_{i+1} = \pi(c_i)$. For example, $C(41532) = (1, 4, 3, 5, 2)$.

Pattern avoidance is typically defined in terms of a permutation's one-line form. We say that a permutation $\pi = \pi_1\pi_2\dots\pi_n \in \mathcal{S}_n$ *avoids* a pattern $\sigma = \sigma_1\sigma_2\dots\sigma_k \in \mathcal{S}_k$ if there is no subsequence of π , written in its one-line form, that appears in the same relative order as σ . For example, the permutation $\pi = 32185476$ avoids the pattern 231, but it does contain the pattern 123 since the subsequence $\pi_2\pi_5\pi_7 = 257$ is in the same relative order as 123. Enumeration of pattern-avoiding permutations began in earnest in 1985 with [13] and has been a topic of study since that time.

One can also define pattern avoidance in the cycle form of a one-cycle permutation. For a permutation $\pi \in \mathcal{S}_n$, we say that the cycle form $C(\pi)$ avoids the pattern $\sigma \in \mathcal{S}_k$ if there is no subsequence of the list $1, c_2, c_3, \dots, c_n$ that is in the same relative order as σ . For example, the permutation $\pi = 47236851$ has $C(\pi) = (1, 4, 3, 2, 7, 5, 6, 8)$ has the property that the cycle form $C(\pi)$ avoids 231, but that it contains 123 since $c_3c_6c_7 = 356$ is a 123 pattern.

Pattern avoidance of the cycle form of a one-cycle permutation has been considered previously in [1], in which the authors consider one-cycle permutations where the one-line form of π avoids one pattern in \mathcal{S}_3 and the cycle form $C(\pi)$ avoids another pattern in \mathcal{S}_3 . It has also been considered in [3] in which both the one-line form of a permutation and its standard cycle form (i.e. its image under the so-called fundamental bijection) avoids a pattern in \mathcal{S}_3 . Interestingly, the avoidance of patterns in the cycle form of a permutation has recently been used in [4] to characterize so-called shallow permutations, which are equivalent to unlinked permutations [15]. Avoidance of the cycle form has also been considered when characterizing almost-increasing permutations in [2,8] and when counting occurrences of adjacent q -cycles in permutations in [6]. Another related topic is the study

of circular pattern avoidance [5, 7, 11, 14], in which all cyclic rotations of a permutation avoid a given pattern. This topic becomes particularly relevant in Section 3 of this paper.

In this paper, we consider one-cycle permutations that avoid the decreasing pattern $\delta_k = k(k-1)\dots 21$ in its one-line form and another pattern in its cycle form $C(\pi)$ or in all cyclic rotations of $C(\pi)$. In Section 2, we let $\mathcal{A}_n(\sigma; \tau)$ denote the set of permutations $\pi \in \mathcal{S}_n$ such that the one-line form avoids σ and the cycle form $C(\pi)$ avoids τ ; we also denote $a_{n,k}(\tau) = |\mathcal{A}_n(\delta_k; \tau)|$. For example, $\pi = 531269478$ avoids 4321 with $C(\pi) = (1, 5, 6, 9, 8, 7, 4, 2, 3)$ avoiding 213, so $\pi \in \mathcal{A}_9(4321; 213)$. In this paper, we find $\mathcal{A}_n(\delta_k; \tau)$ for $\tau \in \mathcal{S}_3 \setminus \{321\}$.

In particular, in Theorem 2.1 in Section 2.1, we find that for $k \geq 4$, $f_k(z; 213) = \sum_{n \geq 0} a_{n,k}(213)z^n$ is given by the height k continued fraction

$$f_k(z; 213) = \frac{z}{1 - \frac{z}{1 - \frac{z}{1 - \dots - \frac{z}{1 - z}}}}$$

In Section 2.2, we show $f_k(z; 312) = f_k(z; 213)$. We also find in Theorem 2.3 of Section 2.3 that if $f_k(z; 231) = \sum_{n \geq 0} a_{n,k}(231)z^n$, we have a generating function that depends on the continued fraction for $f_k(z; 213)$:

$$f_k(z; 231) = \frac{f_{k-1}(z; 213) ([f_{k-2}(z; 213)]^2 - f_{k-2}(z; 213) + 1)}{1 - z}$$

In Section 3, we let $\mathcal{A}_n^\circ(\sigma; \tau)$ denote the set of permutations $\pi \in \mathcal{S}_n$ such that the one-line form avoids σ and all cyclic rotations of the cycle form $C(\pi)$ avoid τ ; we denote $a_{n,k}^\circ(\tau) = |\mathcal{A}_n^\circ(\delta_k; \tau)|$. For example, $\pi = 392145678$ avoids 4321 with all cyclic rotations of $C(\pi) = (1, 3, 2, 9, 8, 7, 6, 5, 4)$ avoiding 1342, so $\pi \in \mathcal{A}_9^\circ(4321; 1342)$. Up to symmetry and cyclic rotation, there are three cases in \mathcal{S}_4 to consider, and we find the enumeration for two of these cases. In Theorem 3.1, we find that $a_{n,3}^\circ(1324) = 2^{n-2}$ and $a_{n,k}^\circ(1324) = F_{2n-3}$ for $k \geq 4$, where F_n is the n th Fibonacci number. In Theorem 3.2 we find that if $f_k^\circ(z; 1342) = \sum_{n \geq 0} a_{n,k}^\circ(1342)z^n$, then

$$f_k^\circ(z; 1342) = \frac{1 - 3z + 2z^2 + z^3}{(1 - z)^2(1 - 2z)} - \frac{2z^{k+1}}{(1 - z)^{k-1}(1 - 2z)}$$

We conclude the paper in Section 4 with some conjectures, open questions, and directions for further research.

2. Avoidance in the cycle form $C(\pi)$

In this section, we enumerate one-cycle permutations such that π avoids the pattern $\delta_k = k(k-1)\dots 21$ and the cycle form $C(\pi)$ avoids the pattern τ for some $\tau \in \mathcal{S}_3$. We solve this for all but one $\tau \in \mathcal{S}_3$; the case when $\tau = 321$ is left as an open question.

We first note that the cases when $\tau = 123$ or $\tau = 132$ is trivial. Indeed, the only permutation $\pi \in \mathcal{S}_n$ such that $C(\pi)$ avoids 123 is $\pi = n12\dots(n-1)$ with $C(\pi) = (1, n, n-1, \dots, 3, 2)$ and so $a_{n,k}(123) = 1$ for all $n \geq 1$ and $k \geq 3$. Similarly, the only permutation $\pi \in \mathcal{S}_n$ such that $C(\pi)$ avoids 132 is $\pi = 23\dots n1$ with $C(\pi) = (1, 2, 3, \dots, n)$ and so $a_{n,k}(132) = 1$ for all $n \geq 1$ and $k \geq 3$.

We will next consider the case when $\tau = 213$, followed by the case when $\tau = 231$, both of which give interesting answers in terms of generating functions.

2.1 Enumerating $\mathcal{A}_n(\delta_k; 213)$

In this section, we consider one-cycle permutations that avoid $\delta_k := k(k-1)\dots 21$ in one-line form and avoid 213 in cycle form. Notice that if π is a permutation avoiding δ_k , then π necessarily avoids δ_i for all $i < k$. Thus, $a_{n,k}(213) \geq a_{n,k-1}(213)$ for $k \geq 2$. Furthermore, since one-cycle permutations on $[n]$ avoiding 213 in cycle form are in one-to-one correspondence with permutations on $[n-1]$ avoiding 213, we have $a_{n,k}(213)$ is the $(n-1)$ -st Catalan number for $k \geq n$.

As an example, consider $n = 5$, and notice there are 14 one-cycle permutations on 5 elements whose cycle form avoids 213. All 14 of these permutations avoid $\delta_5 = 54321$ in one-line form, and all but the permutation 54132 avoid $\delta_4 = 4321$. There are eight one-cycle permutations that avoid 213 in cycle form while avoiding $\delta_3 = 321$ in one-line form, namely, 31452, 31524, 41253, 51234, 23451, 23514, 24153, and 25134.

Thus we have

$$a_{5,k}(213) = \begin{cases} 0 & \text{if } k \leq 2 \\ 8 & \text{if } k = 3 \\ 13 & \text{if } k = 4 \\ 14 & \text{if } k \geq 5. \end{cases}$$

$k \setminus n$	1	2	3	4	5	6	7	8	9
2	1	0	0	0	0	0	0	0	0
3	1	1	2	4	8	16	32	64	128
4	1	1	2	5	13	34	89	233	610
5	1	1	2	5	14	41	122	365	1094
6	1	1	2	5	14	42	131	417	1341
7	1	1	2	5	14	42	132	428	1416
8	1	1	2	5	14	42	132	429	1429
9	1	1	2	5	14	42	132	429	1430

Figure 1: The number of permutations in $\mathcal{A}_n(\delta_k; 213)$; that is, the number of one-cycle permutations of length n avoiding $k(k-1)\cdots 1$ in one-line form and avoiding 213 in cycle form.

Figure 2.1 shows values of $a_{n,k}(213)$ for other small values of n and k .

In general, we will show that the permutations in $\mathcal{A}_n(\delta_k; 213)$ can be enumerated using a self-convolution recurrence as stated in the following theorem. The corresponding generating function can be found in Corollary 2.1.

Theorem 2.1. For $n \geq 2$ and $k \geq 4$,

$$a_{n,k}(213) = \sum_{i=1}^{n-1} a_{i,k}(213)a_{n-i,k-1}(213),$$

with base cases

- $a_{1,1}(213) = 0$ and $a_{1,k}(213) = 1$ for $k \geq 2$;
- $a_{n,1}(213) = a_{n,2}(213) = 0$ for $n \geq 2$; and
- $a_{n,3}(213) = 2^{n-2}$ for $n \geq 2$.

The remainder of this section will be devoted to proving this theorem. We begin by introducing new notation. Let $\mathcal{B}_n(\delta_k; 213)$ be the set of permutations in $\mathcal{A}_n(\delta_k; 213)$ with the additional property that $\pi_1 = n$, and let $b_{n,k}(213) = |\mathcal{B}_n(\delta_k; 213)|$. It turns out that each permutation in $\pi \in \mathcal{A}_n(\delta_k; 213)$ can be decomposed into two unique permutations based on the value of π_1 ; if $\pi_1 = j$, one of the permutations in the decomposition is in $\mathcal{B}_j(\delta_k; 213)$, while the other is in $\mathcal{A}_{n+1-j}(\delta_k; 213)$.

Lemma 2.1. For $n \geq 2$ and $k \geq 3$,

$$a_{n,k}(213) = \sum_{j=2}^n b_{j,k}(213)a_{n+1-j,k}(213).$$

Proof. Let $j \in [2, n]$. We show that the number of permutations in $\mathcal{A}_n(\delta_k; 213)$ where $\pi_1 = j$ is equal to $b_{j,k}(213)a_{n+1-j,k}(213)$. To this end, let $\pi \in \mathcal{A}_n(\delta_k; 213)$ and suppose that $\pi_1 = j$. Since $C(\pi)$ avoids 213, we can write

$$C(\pi) = (1, j, c_3, c_4, \dots, c_{n-j+2}, c_{n-j+3}, \dots, c_n)$$

where $\{c_3, \dots, c_{n-j+2}\} = [j+1, n]$ and $\{c_{n-j+3}, \dots, c_n\} = [2, j-1]$. In one-line form, we then have

$$\pi = j\pi_2\pi_3 \dots \pi_{j-1}\pi_j \dots \pi_n$$

where $\{\pi_2, \dots, \pi_{j-1}\} = [1, j-1] \setminus \{c_{n-j+3}\}$ and $\{\pi_j, \dots, \pi_n\} = [j+1, n] \cup \{c_{n-j+3}\}$. Notice that this implies that if π did contain δ_k as a pattern, then the δ_k pattern must be composed of only elements of $[1, j]$ or only elements of $[j+1, n] \cup \{c_{n-j+3}\}$.

We now create two one-cycle permutations from π that both avoid δ_k . Define π' to be the permutation formed by deleting the elements in $[2, j]$ from $C(\pi)$. More formally,

$$C(\pi') = (1, c_3 - j + 1, c_4 - j + 1, \dots, c_{n-j+2} - j + 1).$$

Equivalently, in one-line form, π' can be thought of as deleting the elements in $\{\pi_1, \pi_2, \dots, \pi_{j-1}\}$ from π . Thus π' consists of the last $n - j + 1$ elements of π in the same relative order. Therefore π' is cyclic with $C(\pi')$ avoiding 213 and π' avoids δ_k in one-line form, and we have $\pi' \in \mathcal{A}_{n+1-j}(\delta_k; 213)$.

Similarly, define π'' to be the permutation formed by deleting the elements in $[j + 1, n]$ from $C(\pi)$. In this case,

$$C(\pi'') = (1, j, c_{n-j+3}, \dots, c_n),$$

and

$$\pi'' = j\pi_2\pi_3 \cdots \pi_{j-1}c_{n-j+3}.$$

We again see that π'' is cyclic, its cycle form avoids 213, and its one-line form avoids δ_k . The permutation π'' has the additional condition that 1 maps to j and thus $\pi'' \in \mathcal{B}_j(\delta_k; 213)$.

Conversely, suppose π' is any permutation in $\mathcal{A}_{n+1-j}(\delta_k; 213)$ and π'' is any permutation in $\mathcal{B}_j(\delta_k; 213)$ with

$$C(\pi') = (1, c'_2, c'_3, \dots, c'_{n+1-j})$$

and

$$C(\pi'') = (1, j, c''_3, c''_4, \dots, c''_j).$$

Create the permutation π by

$$C(\pi) = (1, j, c'_2 + j - 1, c'_3 + j - 1, \dots, c'_{n+1-j} + j - 1, c''_3, c''_4, \dots, c''_j).$$

Notice $C(\pi)$ avoids 213 and its one-line form must avoid δ_k since the one-line form of both π' and π'' did. Thus the process is invertible and the number of permutations in $\mathcal{A}_n(\delta_k; 213)$ where $\pi_1 = j$ is equal to $b_{j,k}(213)a_{n+1-j,k}(213)$. Summing over all possible j gives the desired result. \square

The following example, along with Figure 2, illustrates the bijective correspondence between $\mathcal{A}_n(\delta_k; 213)$ and $\mathcal{B}_j(\delta_k; 213) \times \mathcal{A}_{n+1-j}(\delta_k; 213)$.

Example 2.1. Consider the permutation $\pi' = 4315672 = (1, 4, 5, 6, 7, 2, 3) \in \mathcal{A}_7(4321; 213)$ and the permutation $\pi'' = 53412 = (1, 5, 2, 3, 4) \in \mathcal{B}_5(4321; 213)$. Then we can obtain a permutation $\pi \in \mathcal{A}_{11}(4321; 213)$ as described in the preceding proof:

$$\pi = 5\ 3\ 4\ 1\ 8\ 7\ 2\ 9\ 10\ 11\ 6 = (1, 5, 8, 9, 10, 11, 6, 7, 2, 3, 4),$$

as shown in Figure 2.

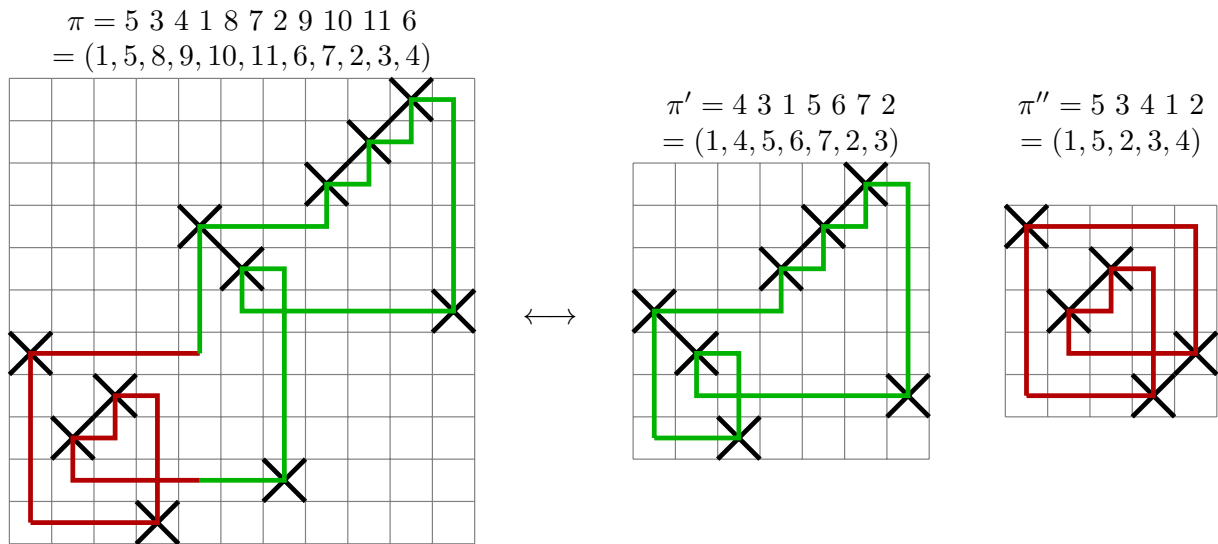


Figure 2: The permutation $\pi \in \mathcal{A}_{11}(4321; 213)$ with $\pi_1 = 5$ can be decomposed into a permutation $\pi' \in \mathcal{A}_7(4321; 213)$ and $\pi'' \in \mathcal{B}_5(4321; 213)$.

Because each element in $\mathcal{A}_n(\delta_k; 213)$ can be decomposed as two permutations in $\mathcal{B}_j(\delta_k; 213)$ and in $\mathcal{A}_{n+1-j}(\delta_k; 213)$, being able to enumerate $\mathcal{B}_j(\delta_k; 213)$ will help us enumerate $\mathcal{A}_n(\delta_k; 213)$. We will enumerate $\mathcal{B}_n(\delta_k; 213)$ by first counting all of those permutations in $\mathcal{B}_n(\delta_k; 213)$ with the additional condition $\pi_n = 2$. The next lemma shows that set of permutations in $\mathcal{B}_n(\delta_k; 213)$ with $\pi_n = 2$ is in bijective correspondence with $\mathcal{A}_{n-2}(\delta_{k-2}; 213)$.

Lemma 2.2. For $n \geq 3$, the number of permutations in $\mathcal{B}_n(\delta_k; 213)$ with $\pi_n = 2$ is equal to $a_{n-2,k-2}(213)$.

Proof. Suppose π is a one-cycle permutation on n elements where $C(\pi)$ avoids 213, and with the additional conditions that $\pi_1 = n$ and $\pi_n = 2$. Then

$$C(\pi) = (1, n, 2, c_4, \dots, c_n).$$

Now consider the permutation π' where $C(\pi')$ is formed by deleting n and 2 from $C(\pi)$. In other words,

$$C(\pi') = (1, c_4 - 1, \dots, c_n - 1).$$

In one-line form, we note that π' is formed from π by deleting n and 2 , and conversely, π is obtained from π' by inserting n at the front and 2 at the end of π' .

We further claim that if δ_{k-2} is a pattern in π' , then there must be an occurrence of δ_{k-2} that does not use the element 1 . Toward a contradiction, suppose we have an occurrence of δ_{k-2} in π' that uses 1 , namely $\pi'_{i_1} \dots \pi'_{i_{k-2}}$ with $\pi'_{i_{k-2}} = 1$. Then, we can show that 2 must occur after 1 in π' and thus the pattern induced by $\pi'_{i_1} \dots \pi'_{i_{k-3}} 2$ is also a δ_{k-2} pattern. To see this suppose 2 appears before 1 in π' . Then if $\pi_r = 1$ and $\pi_s = 2$ with $1 < s < r$, the pattern $s2r$ appears in $C(\pi')$, but this is a 213 pattern, contradicting that $C(\pi')$ avoids 213. Notice that if $s = 1$, then the cycle form starts $(1, 2, \dots)$, and thus we can delete the 2 and apply the same argument since this 2 cannot be part of a 213 pattern in $C(\pi)$ or a δ_k in the one-line form of π . This implies that π' avoids δ_{k-2} if and only if π avoids δ_k .

We have shown that $\pi \in \mathcal{B}_n(\delta_k; 213)$ with $\pi_n = 2$ if and only if $\pi' \in \mathcal{A}_{n-2}(\delta_{k-2}; 213)$ and therefore the result holds. \square

To count those permutations in $\mathcal{B}_n(\delta_k; 213)$ where $\pi_n = j \in [3, n-1]$, we can consider a decomposition based on j as seen in the next result.

Lemma 2.3. For $n \geq 3$ and $k \geq 2$,

$$b_{n,k}(213) = \sum_{j=2}^{n-1} b_{j,k}(213)a_{n-j,k-2}(213).$$

Proof. We show that the number of permutations in $\mathcal{B}_n(\delta_k; 213)$ where $\pi_n = j \in [3, n-1]$ is equal to $b_{j,k}(213)a_{n+1-j,k}(213)$. To this end, let $\pi \in \mathcal{B}_n(\delta_k; 213)$ and suppose that $\pi_n = j$.

Since $C(\pi)$ avoids 213, we can write

$$C(\pi) = (1, n, j, c_4, c_5, \dots, c_{n-j+2}, c_{n-j+3}, \dots, c_n)$$

where $\{c_4, \dots, c_{n-j+2}\} = [j+1, n-1]$ and $\{c_{n-j+3}, \dots, c_n\} = [2, j-1]$. This implies that π must be of the form

$$\pi = n\pi_2\pi_3 \dots \pi_{j-1}\pi_j \dots \pi_{n-1}j$$

where $\{\pi_2, \dots, \pi_{j-1}\} = [1, j-1] \setminus \{c_{n-j+3}\}$ and $\{\pi_j, \dots, \pi_{n-1}\} = [j+1, n-1] \cup \{c_{n-j+3}\}$. Now, if π contains δ_k as a pattern, it must either contain only elements of $[1, j] \cup \{n\}$ or only elements of $[j, n] \cup \{c_{n-j+2}\}$. Therefore, if π avoids δ_k , we can decompose π into two smaller one-cycle permutations, each of which avoids δ_k , namely, $\pi' \in \mathcal{B}_{n-j+2}(\delta_k; 213)$ and $\pi'' \in \mathcal{B}_j(\delta_k; 213)$, defined by

$$C(\pi') = (1, n-j+2, 2, c_4-j+2, c_5-j+2, \dots, c_{n-j+2}-j+2)$$

and

$$C(\pi'') = (1, j, c_{n-j+3}, \dots, c_n).$$

Notice that π' can be any permutation in $\mathcal{B}_{n-j+2}(\delta_k; 213)$ with $\pi'_1 = n-j+2$ (the largest element) and $\pi'_{n-j+2} = 2$, while π'' can be any permutation in $\mathcal{B}_j(\delta_k; 213)$ with $\pi''_1 = j$. By Lemma 2.2, the number of such π' is given by $a_{n-j,k-2}(213)$ and so the recurrence follows. \square

We again illustrate this decomposition with a specific example.

Example 2.2. Consider the permutation

$$\pi = 13 \ 1 \ 2 \ 3 \ 7 \ 4 \ 6 \ 9 \ 12 \ 5 \ 10 \ 11 \ 8 = (1, 13, 8, 9, 12, 11, 10, 5, 7, 6, 4, 3, 2).$$

Notice that $\pi \in \mathcal{B}_{13,4}$ and that $\pi_{13} = 8$. Following the proof of the preceding lemma, we define two new permutations, π' and π'' by

$$C(\pi') = (1, 7, 2, 3, 6, 5, 4) \quad \text{and} \quad C(\pi'') = (1, 8, 5, 7, 6, 4, 3, 2).$$

We note that $\pi' \in \mathcal{B}_7(\delta_4; 213)$ and $\pi'' \in \mathcal{B}_8(\delta_4; 213)$ as desired. A pictorial representation of this decomposition is shown in Figure 3.

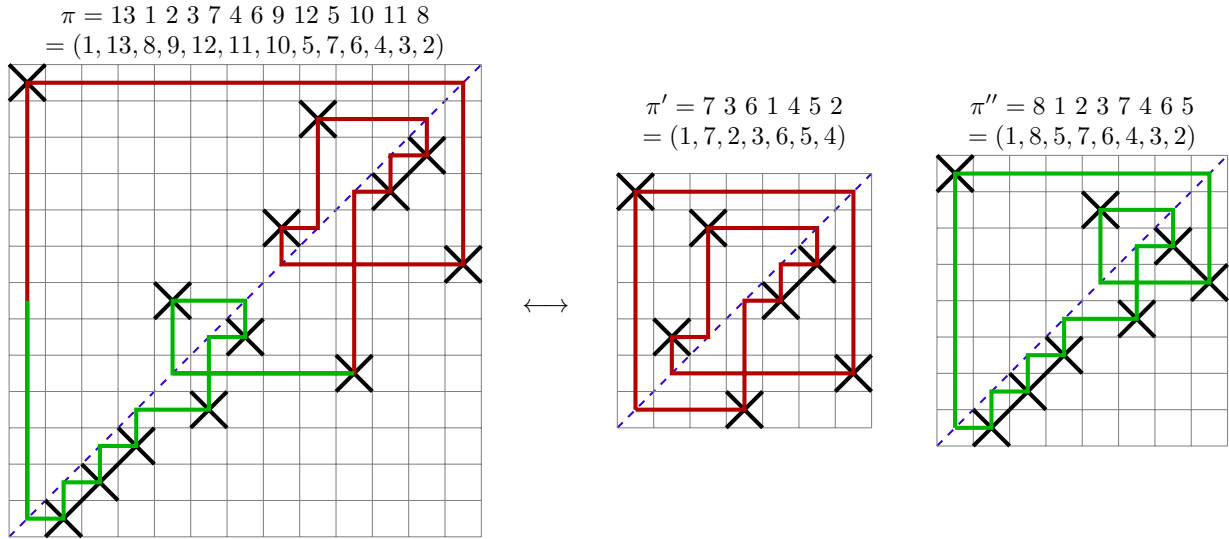


Figure 3: The permutation $\pi \in \mathcal{B}_{13}(4321; 213)$ with $\pi_{13} = 8$ can be decomposed into a permutation $\pi' \in \mathcal{B}_7(4321; 213)$ and $\pi'' \in \mathcal{B}_8(4321; 213)$.

Lemma 2.3 gives $b_{n,k}$ as a sum of products of $b_{j,k}$ and $a_{n-j,k-2}$ (for j from 2 to $n-1$) while Lemma 2.1 gives $a_{n,k}$ as a sum of products of $b_{j,k}$ and $a_{n+1-j,k}$ (for j from 2 to n). We can combine these results to get $b_{n+1,k} = a_{n,k-1}$ as seen in the next lemma.

Lemma 2.4. For $n \geq 3$ and $k \geq 3$,

$$b_{n+1,k}(213) = a_{n,k-1}(213).$$

Proof. We prove this by induction. It is straightforward to check the base cases. Let us assume the statement is true for all $N < n$ and $K < k$. Notice that by Lemmas 2.1 and 2.3, we can write

$$b_{n+1,k}(213) = \sum_{i=2}^n b_{i,k}(213)a_{n+1-i,k-2}(213)$$

and

$$a_{n,k-1}(213) = \sum_{j=2}^n b_{j,k-1}(213)a_{n+1-j,k-1}(213).$$

Using induction, we can say that $b_{j,k-1}(213) = a_{j-1,k-2}(213)$ and $a_{n+1-j,k-1}(213) = b_{n+2-j,k}(213)$. Reindexing, we can see that the new formula for $a_{n,k-1}(213)$ coincides with that for $b_{n+1,k}(213)$. \square

We are now ready to prove our main result as stated in Theorem 2.1.

Proof of Theorem 2.1. The base cases $a_{1,1}(213) = 0$ and $a_{1,k}(213) = 1$ for $k \geq 2$ are clear since there is only one permutation of length 1, which must avoid any pattern of length $k \geq 2$. For $n \geq 2$, it is also clear that $a_{n,1}(213) = a_{n,2}(213) = 0$ since it is impossible for any permutation in \mathcal{S}_n to avoid the pattern of length 1 and impossible for a one-cycle permutation to avoid 21. The base case $a_{n,3}(212) = 2^{n-2}$ for $n \geq 2$ can be found in [1, Theorem 3.24]. Finally, by combining Lemmas 2.1 and 2.4 and reindexing, the recurrence in the statement of theorem follows. \square

For any fixed k , the generating function for $a_{n,k}(213)$ can be described recursively as seen in the following corollary.

Corollary 2.1. Let $f_k(z; 213)$ be the generating function for the $a_{n,k}(213)$. Then for $k \geq 4$,

$$f_k(z; 213) = \frac{z}{1 - f_{k-1}(z; 213)}$$

with $f_1(z; 213) = 0$, $f_2(z; 213) = z$, and $f_3(z; 213) = \frac{z(1-z)}{1-2z}$.

Notice that since $f_3(z; 213)$ can also be written as

$$f_3(z; 213) = \frac{z}{1 - \frac{z}{1-z}},$$

$k \setminus n$	1	2	3	4	5	6	7	8	9
2	1	0	0	0	0	0	0	0	0
3	1	1	2	3	4	5	6	7	8
4	1	1	2	5	11	23	47	95	191
5	1	1	2	5	14	40	113	314	860
6	1	1	2	5	14	42	130	406	1267
7	1	1	2	5	14	42	132	427	1403
8	1	1	2	5	14	42	132	429	1428
9	1	1	2	5	14	42	132	429	1430

Figure 4: The number of permutations in $\mathcal{A}_n(\delta_k; 231)$; that is, the number of one-cycle permutations of length n avoiding $k(k-1)\cdots 1$ in one-line form and avoiding 231 in cycle form.

the generating function described in Corollary 2.1 is a continued fraction generating function. For example, for $k = 4$,

$$f_4(z; 213) = \frac{z}{1 - f_3(z; 213)} = \frac{z}{1 - \frac{z}{1 - \frac{z}{1 - z}}} = z + z^2 + 2z^3 + 5z^4 + 13z^5 + 34z^6 + 89z^7 + \dots,$$

where the coefficient of z^n gives the number of one-cycle permutations π on $[n]$ that avoid 4321 whose cycle form $C(\pi)$ avoids 213. Similarly, for $k = 5$,

$$f_5(z; 213) = \frac{z}{1 - f_4(z; 213)} = \frac{z}{1 - \frac{z}{1 - \frac{z}{1 - \frac{z}{1 - z}}}} = z + z^2 + 2z^3 + 5z^4 + 14z^5 + 41z^6 + 122z^7 + \dots,$$

where the coefficient of z^n gives the number of one-cycle permutations π on $[n]$ that avoid 54321 whose cycle form $C(\pi)$ avoids 213.

Remark 2.1. For $k \geq 4$, these sequences given by the recurrence and generating function in Theorem 2.1 and Corollary 2.1 can be found in OEIS [12, A080934]. This implies that the generating function $f_k(213)$ is also equal to

$$f_k(213) = z \cdot \frac{g_{k+1}(z)}{g_k(z)}$$

where $g_k(x)$ is the k -th Fibonacci polynomial defined by $g_0(z) = 0$, $g_1(z) = 1$, and $g_{i+1}(z) = g_i(z) + z \cdot g_{i-1}(z)$. It also follows that $a_n(\delta_k; 213)$ is equal to the number of permutations of length $n - 1$ that classically avoid both 231 and $k(k-1)\cdots 21$, as well as the number of rooted ordered trees on n nodes of height at most $k - 1$ and the number of Dyck paths of semi-length $n - 1$ and height at most $k - 1$. A direct bijection would be interesting but is not immediately obvious from the proof above because of the induction used in Lemma 2.7.

2.2 Enumerating $\mathcal{A}_n(\delta_k; 312)$

We now briefly consider $\mathcal{A}_n(\delta_k; 312)$. Since 312 is the reverse of 213, $C(\pi)$ avoids 213 if and only if $C(\pi)^r = (c_n, c_{n-1}, \dots, c_2, 1)$ avoids 312. Furthermore, we must have $C(\pi^{-1}) = (1, c_n, c_{n-1}, \dots, c_2)$, which also necessarily avoids 312 since the only change from $C(\pi)^r = (c_n, c_{n-1}, \dots, c_2, 1)$ is that the 1 was moved to the front. Finally, since the inverse of the decreasing monotone pattern is itself, we have that $a_{n,k}(213) = a_{n,k}(312)$. We summarize these results in the following theorem.

Theorem 2.2. For $n \geq 1$ and $k \geq 1$, $a_{n,k}(312) = a_{n,k}(213)$. Furthermore, if $f_k(z; 312)$ is the generating function for $a_{n,k}(312)$, then $f_k(312; z) = f_k(213; z)$.

2.3 Enumerating $\mathcal{A}_n(\delta_k; 231)$

In this section, we consider one-cycle permutations that avoid δ_k in one-line form and avoid 231 in cycle form. We again define $a_{n,k}(231) = |\mathcal{A}_n(\delta_k; 231)|$ with a summary of small values of $a_{n,k}(231)$ found in Figure 2.3. Interestingly, this case will use the results in Section 2.1. There are multiple useful correspondences between permutations in $\mathcal{A}_n(\delta_k; 231)$ and one-cycle permutations whose cycle avoids 213. The following lemma formalizes some of these facts.

Lemma 2.5. Let $n \geq 3$ and $k \geq 5$. Then

1. The number of permutations $\pi \in \mathcal{A}_n(\delta_k; 231)$ with $\pi_1 = n - 1$ and $\pi_n = 1$ is equal to $a_{n-2,k-2}(213)$.

2. The number of permutations $\pi \in \mathcal{A}_n(\delta_k; 231)$ with $\pi_1 = n$ and $\pi_n = 2$ is equal to $a_{n-2, k-2}(213)$.

3. The number of permutations $\pi \in \mathcal{A}_n(\delta_k; 231)$ with $\pi_1 = n$, $\pi_{n-1} = 1$, and $\pi_n = n - 2$ is $a_{n-3, k-2}(213)$.

Proof. We will find a one-to-one correspondence between permutations $\pi \in \mathcal{A}_n(\delta_k; 231)$ with $\pi_1 = n - 1$ and $\pi_n = 1$ and permutations $\pi' \in \mathcal{A}_n(\delta_k; 213)$ with the additional constraints that $\pi'_1 = n$ and $\pi'_n = 2$. When this correspondence is found, Lemma 2.2 can be used to show that the number of desired permutations is then $a_{n-2, k-2}(213)$.

To that end, let $\pi \in \mathcal{A}_n(\delta_k; 231)$ with $\pi_1 = n - 1$ and $\pi_n = 1$. In cycle form, we have

$$C(\pi) = (1, n - 1, c_3, c_4, \dots, c_{n-1}, n).$$

Consider the permutation $\pi' = \pi^{rc}$. Since $\pi_1 = n - 1$ and $\pi_n = 1$, we have $\pi'_1 = n$ and $\pi'_n = 2$. Furthermore, the cycle form of π' is complemented, and thus is given by

$$C(\pi') = (1, n, 2, n + 1 - c_3, n + 1 - c_4, \dots, n + 1 - c_{n-1}).$$

Thus $C(\pi')$ avoids 213, and π' avoids δ_k . Because this process is invertible, the result holds.

Now suppose $\pi \in \mathcal{A}_n(\delta_k; 231)$ with $\pi_1 = n$ and $\pi_n = 2$. Then

$$C(\pi) = (1, n, 2, c_4, c_5, \dots, c_n)$$

and

$$\pi = n\pi_2\pi_3 \cdots \pi_{n-1}2.$$

Let us define π'' by complementing $C(\pi)$ and deleting n , so that

$$C(\pi'') = (1, n - 1, n + 1 - c_4, n + 1 - c_5, \dots, n + 1 - c_n).$$

We claim that $\pi'' \in \mathcal{B}_{n-1}(\delta_k; 213)$, and in fact all such elements in $\mathcal{B}_{n-1}(\delta_{k-1}; 213)$ can be obtained this way. First, it is clear that $C(\pi'')$ avoids 231 since the elements after $n - 1$ must avoid 231. Next, let us notice that the permutation π^{rc} , obtained by complementing the cycle form of π , is given by $\pi^{rc} = (n - 1)(n + 1 - \pi_{n-1}) \cdots (n + 1 - \pi_2)1$ avoids δ_k . The relationship between π^{rc} and π'' can be seen as follows. If $\pi_\ell^{rc} = n$, then $\pi''_i = \pi_i^{rc}$ for all $i \in [n - 1] \setminus \{\ell\}$ and $\pi''_\ell = 1$.

To see that π'' does not contain the pattern δ_{k-1} , we use contradiction. If it did have a δ_{k-1} , then by the proof of Lemma 2.2, there is a δ_{k-1} pattern that does not use 1. This pattern would then also appear in π^{rc} , and together with the 1 at the end of π^{rc} , we would have a δ_k pattern in π^{rc} , which is a contradiction. Thus π'' avoids δ_{k-1} if and only if π^{rc} avoids δ_k . This, together with the fact that $a_{n-2, k-2}(213) = b_{n-1, k-1}(213)$ by Lemma 2.4 implies the second bullet point in the theorem.

Finally, suppose $\pi \in \mathcal{A}_n(\delta_k; 231)$ with $\pi_1 = n$, $\pi_{n-1} = 1$, and $\pi_n = n - 2$. Then

$$C(\pi) = (1, n, n - 2, c_4, c_5, \dots, c_{n-1}, n - 1)$$

and

$$\pi = n\pi_2 \cdots \pi_{n-2}1(n - 2).$$

In this case, consider the permutation π''' formed by deleting 1, n , and $n - 2$ from $C(\pi)$ and then taking the complement. In other words,

$$C(\pi''') = (1, n - 1 - c_4, n - 1 - c_5, \dots, n - 1 - c_{n-1}).$$

We note that $C(\pi''')$ is 213-avoiding since $C(\pi)$ is 231-avoiding and $231^c = 213$. In one-line form, π''' is formed by deleting 1, n , and $n - 2$ from the one-line form of π and then taking the reverse complement. Thus,

$$\pi''' = (n - 2 - \pi_{n-2})(n - 2 - \pi_{n-3}) \cdots (n - 2 - \pi_2).$$

Since $\delta_k^{rc} = \delta_k$, we see that π''' clearly avoids δ_k . However, any maximal decreasing sequence in π must use n and either 1 or $n - 2$, so π''' avoids δ_{k-2} and $\pi''' \in \mathcal{A}_{n-3}(\delta_k; 213)$. Because this process is invertible, we have the desired result. \square

We now illustrate the correspondences described in Lemma 2.5 with some examples.

Example 2.3. First, consider the permutation

$$\pi = 11\ 4\ 2\ 9\ 3\ 5\ 6\ 7\ 10\ 12\ 8\ 1 \quad \text{with} \quad C(\pi) = (1, 11, 8, 7, 6, 5, 3, 2, 4, 9, 10, 12)$$

in $\mathcal{A}_{12}(54321; 231)$ corresponding to the first bullet of Lemma 2.5. We can obtain

$$\pi' = \pi^{rc} = 11\ 5\ 1\ 3\ 6\ 7\ 8\ 10\ 4\ 11\ 9\ 2 \quad \text{with} \quad C(\pi') = (1, 12, 2, 5, 6, 7, 8, 10, 11, 9, 4, 3)$$

which is in $\mathcal{A}_{12}(54321; 213)$ and has the property that $\pi'_1 = 12$ and $\pi'_{12} = 2$.

Now, consider the permutation

$$\pi = 12\ 8\ 9\ 3\ 4\ 5\ 6\ 7\ 10\ 11\ 1\ 2 \quad \text{with} \quad C(\pi) = (1, 12, 2, 8, 7, 6, 5, 4, 3, 9, 10, 11)$$

in $\mathcal{A}_{12}(54321; 231)$ corresponding to the second bullet of Lemma 2.5. We can obtain π'' by complementing $C(\pi)$ and deleting $n = 12$, obtaining

$$\pi'' = C(\pi) = 11\ 1\ 2\ 3\ 6\ 7\ 8\ 9\ 10\ 4\ 5 = (1, 11, 5, 6, 7, 8, 9, 10, 4, 3, 2)$$

which is an element of $\mathcal{B}_{11}(4321; 213)$.

Finally, consider the permutation

$$\pi = 83527416 \quad \text{with} \quad C(\pi) = (1, 8, 6, 4, 2, 3, 5, 7) \in \mathcal{A}_8(\delta_6; 231)$$

corresponding to the third bullet of Lemma 2.5. Following the proof of Lemma 2.5, we create the corresponding permutation $\pi''' \in \mathcal{A}_5(\delta_4; 213)$ by deleting 1, 8, and 6 from π and then taking the reverse-complement. In this case, deleting the appropriate values yields 35274 or 24153. The reverse-complement is then $\pi''' = 31524 = (1, 3, 5, 4, 2)$ which is in $\mathcal{A}_5(\delta_4; 213)$.

As in Section 2.1, we continue by defining $\mathcal{B}_n(\delta_k; 231)$ to be the set of permutations in $\mathcal{A}_n(\delta_k; 231)$ with the additional property that $\pi_1 = n$, and we let $b_{n,k}(231) = |\mathcal{B}_n(\delta_k; 231)|$. We can show that any permutation in $\mathcal{A}_{n,k}(231)$ that is not in $\mathcal{B}_n(\delta_k; 231)$ can be decomposed into two unique permutations based on the value of π_1 . If $\pi_1 = j$, one of the permutations in the decomposition is in $\mathcal{A}_{j+1}(\delta_k; 231)$ with some additional constraints while the other is in $\mathcal{A}_{n+1-j}(\delta_k; 231)$. The subsequent lemma enumerates permutations in $\mathcal{A}_n(\delta_k; 231)$ based on this decomposition. Notice that the recurrence in the next lemma also depends on the values of $a_{n,k}(213)$ found in the previous section.

Lemma 2.6. For $n \geq 3$ and $k \geq 5$,

$$a_{n,k}(231) = b_{n,k}(231) + \sum_{j=2}^{n-1} a_{j-1,k-2}(213)a_{n+1-j,k}(231).$$

Proof. Notice that since $b_{n,k}(231)$ is the number of permutations $\pi \in \mathcal{A}_n(\delta_k; 213)$ with the property that $\pi_1 = n$, we need only consider the permutations in $\mathcal{A}_{n,k}(231)$ that do not begin with n . Let $j \in [2, n-1]$. We show that the number of permutations in $\mathcal{A}_n(\delta_k; 231)$ where $\pi_1 = j < n$ is equal to $a_{j-1,k-2}(213)a_{n+1-j,k}(231)$. To this end, let $\pi \in \mathcal{A}_n(\delta_k; 231)$ and suppose that $\pi_1 = j$. Since $C(\pi)$ avoids 231, we can write

$$C(\pi) = (1, j, c_3, c_4, \dots, c_j, c_{j+1}, \dots, c_n)$$

where $\{c_3, \dots, c_j\} = [2, j-1]$ and $\{c_{j+1}, \dots, c_n\} = [j+1, n]$, which is nonempty. In one-line form, we then have

$$\pi = j\pi_2\pi_3 \dots \pi_j\pi_{j+1} \dots \pi_n$$

where $\{\pi_2, \dots, \pi_j\} = [2, j-1] \cup \{c_{j+1}\}$ and $\{\pi_{j+1}, \dots, \pi_n\} = [j+1, n] \setminus \{c_{j+1}\} \cup \{1\}$. Notice that this implies that if π did contain δ_k as a pattern, then the δ_k pattern must be composed of only elements of $[1, j] \cup \{c_{j+1}\}$ or only elements of $[j+1, n] \cup \{1\}$.

We now create two one-cycle permutations from π that both avoid δ_k . Define π' to be the permutation formed by deleting the elements in $[2, j]$ from $C(\pi)$. More formally,

$$C(\pi') = (1, c_{j+1} - j + 1, \dots, c_n - j + 1).$$

Equivalently, in one-line form, π' can be thought of as deleting the elements in $[2, j] = \{\pi_1, \pi_2, \dots, \pi_j\} \setminus \{\pi_{c_j}\}$ from π . Thus π' is cyclic with $C(\pi')$ avoiding 231 and π' avoids δ_k in one-line form, and thus $\pi' \in \mathcal{A}_{n+1-j}(\delta_k; 231)$.

Similarly, define π'' to be the permutation formed by deleting the elements in $[j+1, n] \setminus \{c_{j+1}\}$ from $C(\pi)$. In this case,

$$C(\pi'') = (1, j, c_3, \dots, c_j, j+1)$$

and

$$\pi'' = j\pi_2\pi_3 \dots \pi_{j-1}\pi_j 1$$

where we take $\pi_{c_j} = j+1$. We again see that π'' is cyclic, its cycle form avoids 231, and its one-line form avoids δ_k . The permutation $\pi'' \in \mathcal{A}_{j+1}(\delta_k; 231)$ has the additional conditions that $\pi_1 = j$ and $\pi_{j+1} = 1$. By Lemma 2.5, the number of such permutations is equal to $a_{j-1,k-2}(213)$. \square

Example 2.4. Consider the permutation

$$5\ 8\ 2\ 3\ 4\ 9\ 6\ 7\ 11\ 1\ 10 = (1, 5, 4, 3, 2, 8, 7, 6, 9, 11, 10)$$

in $\mathcal{A}_{11}(4321; 231)$. Then, we can obtain the permutations $\pi' = 4523671 = (1, 4, 3, 2, 5, 6, 7)$ and $\pi'' = 562341 = (1, 5, 4, 3, 2, 6)$ as described in the preceding proof and as shown in Figure 5.

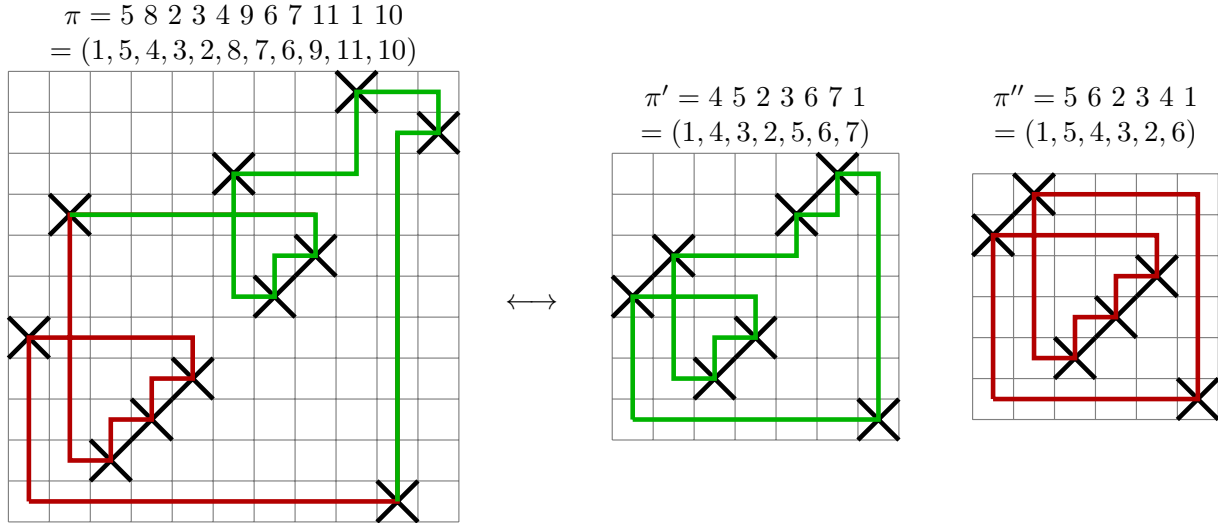


Figure 5: The permutation $\pi \in \mathcal{A}_{11}(4321; 213)$ with $\pi_1 = 5$ can be decomposed into a permutation $\pi' \in \mathcal{A}_7(4321; 231)$ and $\pi'' \in \mathcal{A}_6(4321; 231)$ with some additional constraints.

We now consider permutations in $\mathcal{B}_n(\delta_k; 231)$ and enumerate these based on one-cycle permutations that avoid 213 in cycle form and δ_{k-2} in one-line form.

Lemma 2.7. For $n \geq 3$ and $k \geq 5$,

$$b_{n,k}(231) = b_{n-1,k}(231) + \sum_{j=2}^{n-2} a_{j-1,k-2}(213)a_{n-j,k-2}(213).$$

Proof. We will count permutations $\pi \in \mathcal{B}_n(\delta_k; 231)$ based on the value of π_n . We first show that the number with $\pi_n = n - 1$ is $b_{n-1,k}(231)$. Let $\pi \in \mathcal{B}_n(\delta_k; 231)$ with $\pi_n = n - 1$. Then

$$C(\pi) = (1, n, n - 1, c_4, \dots, c_n).$$

Let π' be the permutation formed by deleting $n - 1$ from π . Then π' clearly avoids δ_k if and only if π does. Also, the cycle form of π' is

$$C(\pi') = (1, n - 1, c_4, \dots, c_n)$$

and thus, $\pi' \in \mathcal{B}_{n-1}(\delta_k; 231)$. Because this process is invertible, there are $b_{n-1,k}$ such permutations as desired.

Now suppose $\pi \in \mathcal{B}_n(\delta_k; 231)$ with $\pi_n = j$ for some $j \in [2, n - 2]$. Then

$$C(\pi) = (1, n, j, c_4, c_5, \dots, c_{j+1}, c_{j+2}, \dots, c_n)$$

where $\{c_4, \dots, c_{j+1}\} = [2, j - 1]$ and $\{c_{j+2}, \dots, c_n\} = [j + 1, n - 1]$. In one-line form, we then have

$$\pi = n\pi_2\pi_3 \cdots \pi_j\pi_{j+1} \cdots \pi_{n-1}j$$

where $\{\pi_2, \dots, \pi_j\} = [2, j - 1] \cup \{c_{j+2}\}$ and $\{\pi_{j+1}, \dots, \pi_{n-1}\} = [j + 1, n - 1] \setminus \{c_{j+2}\} \cup \{1\}$. We now create two one-cycle permutations from π that both avoid δ_k .

Let π' be the permutation formed by deleting the elements in $\{c_{j+3}, c_{j+4}, \dots, c_n\}$ from $C(\pi)$. Thus,

$$C(\pi') = (1, j + 2, j, c_4, \dots, c_{j+1}, j + 1),$$

and in one-line form

$$\pi' = (j + 2)\pi_2\pi_3 \cdots \pi_j1j.$$

Notice that π' avoids δ_k and that $C(\pi')$ avoids 231. By Lemma 2.5, there are $a_{j-1,k-2}(213)$ such permutations.

We form the second permutation in the decomposition, π'' , by deleting $\{c_4, \dots, c_{j+1}\}$ from $C(\pi)$. Thus,

$$C(\pi'') = (1, n - j + 2, 2, c_{j+2} - j + 2, c_{j+3} - j + 2, \dots, c_n - j + 2)$$

and

$$\pi'' = (n - j + 2)(c_{j+2} - j + 2)(\pi_{j+1} - j + 2)(\pi_{j+2} - j + 2) \cdots (\pi_{n-1} - j + 2)2.$$

Notice π'' must avoid δ_k because it is a subpattern of π which avoids δ_k . Since this permutation starts with its largest element and ends with 2, by Lemma 2.5, we see that the number of such permutations is $a_{n-j, k-2}(213)$.

Since, as we stated above, any permutation π of this form must look like

$$\pi = n\pi_2\pi_3 \cdots \pi_j\pi_{j+1} \cdots \pi_{n-1}j$$

where $\{\pi_2, \dots, \pi_j\} = [2, j - 1] \cup \{c_{j+2}\}$ and $\{\pi_{j+1}, \dots, \pi_{n-1}\} = [j + 1, n - 1] \setminus \{c_{j+2}\} \cup \{1\}$, any δ_k pattern must only consist of elements from either $[1, j - 1] \cup \{c_{j+1}, n\}$ or from $\{1\} \cup [j, n]$, this implies that doing the inverse process, building a permutation π from π' and π'' will result in a permutation that still avoids δ_k . \square

Given these results in Lemma 2.6, we can define a generating function for $b_{n,k}(231)$ based on the generating function for $a_{n,k}(213)$.

Corollary 2.2. *Let $f_k(z; 213)$ be the generating function for $a_{n,k}(213)$ and let $g_k(z; 231)$ be the generating function for $b_{n,k}(231)$. Then for $k \geq 5$,*

$$g_k(z; 231) = \frac{z}{1 - z} [1 - zf_{k-2}(z; 213) + (f_{k-2}(z; 213))^2]$$

Using this Corollary along with Lemma 2.6, we can now find the generating function for $a_{n,k}(231)$.

Theorem 2.3. *Let $f_k(z; 231)$ be the generating function for $a_{n,k}(231)$. Then for $k \geq 5$, $f_k(z; 231)$ can be written in terms of the generating functions for $a_{n,k}(213)$:*

$$f_k(z; 231) = \frac{f_{k-1}(z; 213) ([f_{k-2}(z; 213)]^2 - f_{k-2}(z; 213) + 1)}{1 - z}.$$

We also have

$$f_4(z; 231) = \frac{z(z^3 + z^2 - 2z + 1)}{(1 - z)(1 - 2z)},$$

$$f_3(z; 231) = \frac{z}{(1 - z)^2}$$

and $f_2(z; 231) = f_1(z; 231) = z$.

Proof. For $k \geq 5$, Corollary 2.2 and Lemma 2.6 combine to give us:

$$f_k(z; 231) = \frac{z([f_{k-2}(z; 213)]^2 - f_{k-2}(z; 213) + 1)}{(1 - z)(1 - f_{k-2}(z; 213))}.$$

Using Corollary 2.1, we can simplify this result as seen in the theorem. The case for $k = 3$ follows from [1, Theorem 4.16] and the case when $k \leq 2$ is clear.

All that remains is the case when $k = 4$. Let us first notice that $b_{n,4}(231) = n - 2$. Indeed, it is easy to see there are $b_{n-1,4}$ permutations that have $\pi_n = n - 1$, and one permutation that has $\pi_n = 2$. We need only show that there are no permutations with $\pi_n = j$ for $3 \leq j \leq n - 2$. If there were such a permutation, we would have $\pi = (1, n, j, c_3, \dots, c_{j+1}, c_{j+2}, \dots, c_n)$ with $c_i > j$ for $i \geq j + 2$ and $c_i < j$ for $3 \leq i \leq j + 1$. Thus $\pi_1\pi_{c_{j+1}}\pi_j\pi_{c_n} = nc_{j+2}c_31$ is a 4321 pattern. Next, let us see that any permutation in $\mathcal{A}_n(4321; 231)$ with $\pi_1 = j < n$ must be of the form

$$C(\pi) = (1, j, j - 1, \dots, 3, 2, c_{j+1}, \dots, c_n)$$

where the permutation obtained by removing the elements in $[2, j]$ is in $\mathcal{A}_{n-j+1}(4321; 231)$. If this were not the case and i was the first place $c_i < c_{i+1}$ for some $i < j + 1$, then $jc_{i+1}c_i1$ would be a 4321 pattern in π . Thus we know that

$$a_{n,4}(231) = b_{n,4}(231) + \sum_{j=2}^{n-1} a_{n-j+1}(231),$$

which together with the fact that $b_{n,4}(231) = n - 2$ implies the given generating function for $k = 4$. \square

For example, consider $k = 5$. We have

$$f_5(z; 231) = \frac{f_4(z; 213) ([f_3(z; 213)]^2 - f_3(z; 213) + 1)}{1 - z}$$

$$= \frac{z(1 - 2z)}{(z^2 - 3z + 1)(1 - z)} - \frac{z^2}{1 - 2z}.$$

3. Avoidance of a pattern in all cyclic rotations of $C(\pi)$

In this section, we will consider when the one-line form of a one-cycle permutation π avoids $\delta_k = k(k-1)\dots 21$ and *all* cycle forms (that is, all cyclic rotations of $C(\pi)$) avoid a given pattern τ . As noted in [5, 7, 14], the case where τ is of length 3 is trivial, and so here, we consider $\tau \in \mathcal{S}_4$. In particular, we consider the case $\tau \in \{1324, 1342\}$. Up to symmetry, the only remaining case to consider would be $\tau = 1234$, which we leave as an open question.

In this section, we use $\mathcal{A}_n^\circ(\delta_k; \tau)$ to denote those one-cycle permutations that avoid δ_k where all cyclic rotations of $C(\pi)$ avoid τ , and we let $a_{n,k}^\circ(\tau) = |\mathcal{A}_n^\circ(\delta_k; \tau)|$.

3.1 Enumerating $\mathcal{A}_n^\circ(\delta^k; 1324)$ and $\mathcal{A}_n^\circ(\delta^k; 1423)$

In [5, 14], the authors find the total number of one-cycle permutations π so that all rotations of $C(\pi)$ avoid 1342 is equal to F_{2n-3} , the $(2n-3)$ rd Fibonacci number.

Theorem 3.1. *For $n \geq 3$, $a_{n,3}^\circ(1324) = 2^{n-2}$ and for $k \geq 4$, $a_{n,k}^\circ(1324) = F_{2n-3}$.*

Proof. Let us first notice that if $\pi \in \mathcal{A}_n^\circ(\delta_k; 1324)$ then we must have $C(\pi) = (1, c_2, \dots, c_{r-1}, 2, c_{r+1}, \dots, c_n)$ with $\{c_2, \dots, c_{r-1}\} = [n-r+3, n]$ and $\{c_{r+1}, \dots, c_n\} = [3, n-r+2]$. Furthermore, if $\pi_1 \neq 2$, we must have that the elements after 2 appear in increasing order since otherwise we would have a 1324 pattern in the cycle form that begins with 2, with 2 acting as the 1 in the pattern and c_2 acting as the 4 in the pattern.

Now, if $k = 3$ we claim that we must have that $\pi_1 = 2$ or $\pi_2 = 1$. If not, then we have $C(\pi) = (1, c_2, \dots, c_{r-1}, 2, 3, \dots, n-r+2)$ for some r . Then, in π , we have the 321 pattern $\pi_1\pi_2\pi_{n-r+2} = \pi_1 31$ with $\pi_1 > 2$. It is also straightforward to see that for any permutation $\pi \in \mathcal{A}_n^\circ(321; 1324)$ with $C(\pi) = (1, 2, c_3, \dots, c_n)$, the permutation π' with $C(\pi') = (1, c_3 - 1, \dots, c_n - 1)$ is in $\mathcal{A}_{n-1}^\circ(321; 1324)$ and in fact any permutation in $\mathcal{A}_{n-1}^\circ(321; 1324)$ can be obtained this way. Similarly if π satisfies $C(\pi) = (1, c_2, \dots, c_{n-1}, 2)$, then the permutation π' with $C(\pi') = (1, c_2 - 1, \dots, c_{n-1} - 1)$ is in $\mathcal{A}_{n-1}^\circ(321, 1324)$. Together with the fact that $a_2^\circ(321, 1324) = 1$, the result that $a_{n,3}^\circ(1324) = 2^{n-2}$ follows.

For the second part of the theorem, it is enough to show that if all cyclic rotations of $C(\pi)$ avoid the pattern 1324, then π avoids 4321. As noted above, all permutations in $\mathcal{A}_n^\circ(\delta_k; 1324)$ must either have the property that $\pi_1 = 2$ or $C(\pi) = (1, c_2, \dots, c_{n+1-k}, 2, 3, \dots, k)$ for some $k > 2$ where the permutation obtained by deleting $[2, k]$ from $C(\pi)$ (equivalently, deleting $34\dots k1$ from positions $\{2, \dots, k\}$ of the one-line form) is also in $\mathcal{A}_{n-k+1}^\circ(\delta_k; 1324)$. Additionally, if $\pi_1 = 2$, deleting 2 from the cycle form (equivalently, deleting 2 from the front of one-line form), leaves you with a permutation in $\mathcal{A}_{n-1}^\circ(\delta_k; 1324)$. Since all the permutations in $\mathcal{A}_n^\circ(\delta_k; 1324)$ are built recursively by adding 2 to the front of the one-line form or $34\dots k1$ after the first position of the one-line form, it is impossible to introduce a new 4321 pattern to the one-line form, and so all permutations whose cycle forms avoid 1324 must avoid 4321 in its one-line form. \square

Finally, by considering the reverse of 1324, which is equivalent to 1423 under cyclic rotation, we have the following corollary.

Corollary 3.1. *For $n \geq 3$, $a_{n,3}^\circ(1423) = 2^{n-2}$. For $k \geq 4$, $a_{n,k}^\circ(1423) = F_{2n-3}$.*

3.2 Enumerating $\mathcal{A}_n^\circ(\delta^k; 1342)$

In this final section, we consider the case where $\tau = 1342$. In [5, 14], the authors find the the total number of one-cycle permutations π so that all rotations of $C(\pi)$ avoid 1342 is equal to $2^{n-1} - n + 1$.

Theorem 3.2. *If $f_k^\circ(z; 1342) = \sum_{n \geq 0} a_{n,k}^\circ(1342)z^n$, then $f_1^\circ(1342) = f_2^\circ(1342) = 1 + z$, and for $k \geq 3$,*

$$f_k^\circ(z; 1342) = \frac{1 - 3z + 2z^2 + z^3}{(1 - z)^2(1 - 2z)} - \frac{2z^{k+1}}{(1 - z)^{k-1}(1 - 2z)}.$$

Let us first address the case when $k = 3$, and then we will find a recurrence for the cases when $k \geq 4$.

Lemma 3.1. *For $n \geq 3$, $a_{n,k}^\circ(1342) = n - 1$.*

Proof. First, note that if $C(\pi) = (1, c_2, \dots, c_{k-1}, n, c_{k+1}, \dots, c_n)$, then since $C(\pi)$ avoids 1342, we must have that $\{c_2, \dots, c_{k-1}\} = [2, k-1]$ and $\{c_{k+1}, \dots, c_n\} = [k, n-1]$. That means $\pi = \pi_1\pi_2\dots\pi_n$ satisfies that $\{\pi_1, \dots, \pi_{k-1}\} = [2, k-1] \cup \{n\}$ and $\{\pi_k, \dots, \pi_n\} = \{1\} \cup [k, n-1]$. Since π avoids 321, we must have that $\pi_{k-1} = n$ and $\pi_k = 1$ or else either $n\pi_{k-1}1$ or $n\pi_k1$ would be a 321 pattern. However, this implies that the elements appearing before 1 and those appearing after n must be increasing, so $\pi = 23\dots(k-1)n1k(k+1)\dots(n-1)$. Since all cyclic rotations of the cycle form $C(\pi) = (1, 2, \dots, k-1, n, n-1, \dots, k+1, k)$ avoid 1342, there are $n-1$ permutations in $\mathcal{A}_n^\circ(\delta_k; 1342)$. \square

In order to avoid δ_k for $k \geq 4$, we will establish a recurrence. As before, $\mathcal{B}_n^\circ(\delta_k; 1342)$ are those permutations in $\mathcal{A}_n^\circ(\delta_k; 1342)$ with $\pi_1 = n$ and $b_{n,k}^\circ(1342) = |\mathcal{B}_n^\circ(\delta_k; 1342)|$.

Lemma 3.2. *For $n \geq 3$ and $k \geq 4$, the number of permutations in $\mathcal{A}_n^\circ(\delta_k; 1342)$ with $\pi_1 = n - 1$ and $\pi_n = 1$ is equal to $b_{n-1,k-1}^\circ(1342)$.*

Proof. Given $\mathcal{A}_n^\circ(\delta_k; 1342)$ with $\pi_1 = n - 1$, $\pi_n = 1$, and $\pi_j = n$, let $\pi' \in \mathcal{S}_n$ is defined by letting $\pi'_j = 1$ and $\pi'_i = \pi_i$ for $i \in [n - 1] \setminus \{j\}$. This corresponds to deleting n from the cycle form of π to obtain π' . We claim that the permutation $\pi' \in \mathcal{B}_{n-1}^\circ(\delta_{k-1}; 1342)$ and that all such permutations in $\mathcal{B}_{n-1}^\circ(\delta_{k-1}; 1342)$ can be obtained this way.

First notice that if there is a δ_{k-1} pattern in π' , that pattern together with the 1 at the end of π would be a δ_k pattern in π . Conversely, a δ_k pattern in π , then there is one that does not use n as the k in the $k \dots 21$ pattern (since we can use $\pi_1 = n - 1$ instead). By taking the first $k - 1$ elements of that pattern (which must not contain $\pi_n = 1$) we would have a δ_{k-1} pattern in π' .

Finally, since adding or removing the n at the end (which is cyclically adjacent to $1, n - 1$), we cannot introduce a new 1342 pattern to any cyclic rotation of the cycle forms of the permutations and so the result follows. \square

Lemma 3.3. *For $n \geq 5$ and $k \geq 4$, we have*

$$a_{n,k}^\circ(1342) = a_{n-1,k}^\circ(1342) + b_{n,k}^\circ(1342) + \sum_{r=3}^{n-1} b_{r,k-1}^\circ(1342).$$

Proof. Let $\pi \in \mathcal{A}_n^\circ(\delta_k; 1342)$. Note that if $\pi_1 = 2$, one can delete the 2 from both the cycle form and one-line form of the permutation and get a new permutation in $\pi \in \mathcal{A}_{n-1}^\circ(\delta_k; 1342)$ and that this process is invertible since that 2 cannot be part of a δ_k pattern in π and cannot add a 1342 pattern in any cyclic rotation of $C(\pi)$. Also, by definition there are $b_{n,k}^\circ(1342)$ permutations with $\pi_1 = n$.

Assume that $3 \leq \pi_1 \leq n - 1$, and let $r = \pi_1$. Then we claim that the permutation must be of the form $(1, r, c_3, \dots, c_r, n, n - 1, \dots, r + 1)$. Indeed, since π avoids 1342, all elements after n must be larger than elements before n . Furthermore, if there were some element $c_m < c_{m+1}$ with $m > r$, then $c_3 c_m c_{m+1} r$ would be a 1342 pattern in a cyclic rotation of $C(\pi)$. Therefore, in one-line form, π is of the form $\pi = r \pi_2 \dots \pi_r 1(r + 1)(r + 2) \dots (n - 1)$. Any δ_k pattern in π does not include any element of the increasing segment $(r + 1)(r + 2) \dots (n - 1)$ since n is the only element larger than those appearing before them. For this reason, the permutation obtained by deleting these elements (from both the one-line and cycle form of the permutation) is of the form $(1, r, c_3, \dots, c_r, r + 1)$ and still avoids δ_k , and that all such cycles can be formed this way. Thus, by Lemma 3.2, there are $b_{r,k-1}^\circ$ such permutations with $\pi_1 = r$. \square

Lemma 3.4. *For $n \geq 1$, we have $b_{n,3}^\circ(1342) = 1$. For $n \geq 4$ and $k \geq 4$, we have*

$$b_{n,k}^\circ(1342) = b_{n-1,k}^\circ(1342) + b_{n-1,k-1}^\circ(1342).$$

As a result we have the generating function $B_k(z; 1342) = \sum_{n \geq 1} b_{n,k}^\circ(1342)z^n$

$$B_k(z; 1342) = \frac{z(1 - z - z^2)}{1 - 2z} - \frac{z^{k+1}}{(1 - 2z)(1 - z)^{k-2}}.$$

Proof. We first claim that if $\pi \in \mathcal{B}_n^\circ(\delta_k; 1342)$, then we must have $\pi_n \in \{2, n - 1\}$. If not, then we would have $C(\pi) = (1, n, j, c_4, \dots, c_n)$ for some $3 \leq j \leq n - 2$. Notice that if 2 appears after $n - 1$ in $C(\pi)$, then $1j(n - 1)2$ is a 1342 pattern, and if 2 appears before $n - 1$, then $2(n - 1)nj$ is a 1342 pattern in a cyclic rotation of $C(\pi)$. Since this is a contradiction, we must have $\pi_n \in \{2, n - 1\}$. If $\pi_n = 2$, then by complementing $C(\pi)$ (corresponding to the reverse complement of π), we have a permutation $\pi^{rc} \in \mathcal{A}_n^\circ(\delta_k; 1342)$ with the property that $\pi_1 = n - 1$ and $\pi_n = 1$. By Lemma 3.2, there are $b_{n-1,k-1}^\circ(1342)$ such permutations. If $\pi_n = n - 1$, we can delete $n - 1$ from the cycle form (and equivalently from the one-line form) to obtain a permutation in $\mathcal{B}_n^\circ(\delta_k; 1342)$. Since this process is invertible, there are $b_{n-1,k}^\circ(1342)$ permutations with $\pi_n = n - 1$.

Finally, since the generating function in the statement of the theorem satisfies

$$B_k(z; 1342) = zB_k(z; 1342) + zB_{k-1}(z; 1342) + z - z^2 - z^3,$$

the theorem follows. \square

Proof of Theorem 3.2. The generating function $f_k^\circ(z; 1342)$ in the statement of the theorem follows from the recurrence in Lemma 3.3 and the generating function in Lemma 3.4. \square

4. Open Questions

In Section 2, we enumerate one-cycle permutations that avoid the decreasing pattern $\delta_k = k(k-1)\dots 21$ and whose cycle form $C(\pi)$ avoids a pattern τ of length 3, with the exception of $\tau = 321$, which remains an open question. Similarly, in Section 3, we enumerate one-cycle permutations that avoid the decreasing pattern $\delta_k = k(k-1)\dots 21$ such that every cyclic rotation of its cycle form avoids a pattern τ of length 4, with the exception of $\tau = 1432$, which also remains open.

There are many other interesting sequences that seem to appear when considering pattern-avoiding one-cycle permutations where the cycle form (or all cyclic rotations of the cycle form) also avoids a pattern or set of patterns. For example, we conjecture that $|\mathcal{A}_n^\circ(4123; 1324)|$ is the $(n-2)^{\text{nd}}$ Tetranacci number, that $|\mathcal{A}_n^\circ(2431; 1324)|$ is the $(n-1)^{\text{st}}$ Pell number, and $|\mathcal{A}_n^\circ(4132; 1324)|$ is the $(3n)^{\text{th}}$ Padovan number. Several other pairs of patterns (or sets of patterns) seem to give interesting sequences that appear in the Online Encyclopedia of Integer Sequences, which makes these open questions seem quite accessible to future researchers.

Finally, one could also consider pattern avoidance in other “cyclic” combinatorial objects, such as parking functions [10] or tensor words [9].

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