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An Invitation to the Riordan Group

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Abstract: The Riordan group is a group of infinite lower triangular matrices that are defined by two generating functions, g and f. The k^{th} column of each matrix has a generating function related to gf^k. There are many applications of Riordan arrays, among other things they can be used to count combinatorial objects and to prove combinatorial identities.

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1. Introduction

The original idea for the Riordan group evolved from a set of combinatorial examples and from Rota's umbral calculus. We could prove various identities very neatly using what is now called the fundamental theorem of Riordan arrays (FTRA) [42]. This also gives us the group structure. Since many of our identities came from John Riordan's famous book, "Combinatorial Identities" [39] and he had recently passed away, we named the group after him. The group structure allowed us a systematic way to invert identities which had been a major theme in his book. Several years later in 1994 an important paper by Renzo Sprugnoli [46] appeared which widely extended the range of examples using the Riordan group framework.

Following this, there were many talented other contributors from Italy. The next group to contribute was led by Gi-Sang Cheon and a strong group of mathematicians from South Korea. In the ensuing years, hundreds of papers have been published with contributors from all over the world.

A valuable bibliography through 2016 is due to Sprugnoli [47]. Typing in "Riordan array" on your browser will lead to several hundred articles. An introductory text is Paul Barry's "Riordan Array Primer" [6] and a research-level book is "The Riordan Group and Applications" [41]. There is also an annual international conference and some YouTube videos [53, 54].

This survey article on Riordan arrays is divided into six sections, not counting the introduction and the conclusion. Section 2 is on Preliminary Results and includes the Fundamental Theorem of Riordan Arrays, the Riordan group, and using $A-$ and $Z-$sequences to construct Riordan arrays. In Section 3, there are some interesting applications of Riordan arrays, such as counting paths and trees. Section 4 discusses subgroups and the relatively new result (2021) by A. Luzón, M. Morón and L. Felipe Prieto-Martínez describing the commutator subgroup of the Riordan group, [29]. There are many types of Riordan arrays. In Sections 2-4, we consider ordinary Riordan arrays. In Section 5, we write about the well-known exponential Riordan arrays and, in Section 6, we write about generalized Riordan arrays and double Riordan arrays. Section 7 is on involutions and pseudo-involutions. Most of Subsection 7.1 is new.

2. Preliminary Results

There is a folklore saying in the mathematics community to the effect that,

"Combinatorial identities are pretty boring except for the one that you are working on."

Here is an identity that we will use to introduce many of the ideas of the Riordan group. One item of interest here is that the right-hand side is always 2, for $n \geq 2$.

$$\sum_{k\geq 0} \frac{n}{n-k}\binom{n-k}{k}(-1)^k\, 2^{2n-2k} = 2$$

As a first step, we try out a few small values of n. For instance if $n = 6$ we get

$$\frac{6}{6}\binom{6}{0}64 - \frac{6}{5}\binom{5}{1}16 + \frac{6}{4}\binom{4}{2}4 - \frac{6}{3}\binom{3}{3} = 2.$$

For the moment ignore the negative signs and the powers of 2 and we get the coefficients 1, 6, 9, 2.

Similarly $n = 5$ gives us $1, 5, 5$ and $n = 4$ yields $1, 4, 2$. Collecting the lower terms give us the table

$$\begin{bmatrix} 1 & & & \\ 1 & & & \\ 1 & 2 & & \\ 1 & 3 & & \\ 1 & 4 & 2 & \\ 1 & 5 & 5 & \\ 1 & 6 & 9 & 2 \end{bmatrix}.$$

For the second step write the identity as a matrix times a column vector. We want a triangular matrix and would like 1"s on the main diagonal so we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & -5 & 0 & 1 & 0 \\ -2 & 0 & 9 & 0 & -6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ 64 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

We can only display a few rows but we have here an infinite lower triangular matrix and infinite column vectors.

For the third step, we find the generating functions for the columns of the matrix and the two-column vectors.

The left most column (the zeroth column) has the generating function $\frac{1-z^2}{1+z^2} := g$. The next column has the generating function $z\frac{1-z^2}{(1+z^2)^2}$ and the next is $z^2\frac{1-z^2}{(1+z^2)^3}$. If we let $f = \frac{z}{1+z^2}$ then the columns have the generating functions $g, gf, gf^2, ...$ The column vector for the powers of 2 is $\frac{1}{1-2z}$ and the column of almost all 2's has the generating function $\frac{1+z}{1-z.}$.

For the fourth step, we look at the matrix times column vector multiplication in terms of generating functions and we have

$$\begin{aligned} & 1\cdot\frac{1-z^2}{1+z^2} + \frac{1-z^2}{1+z^2}\cdot\frac{z}{1+z^2}\cdot 2 + \frac{1-z^2}{1+z^2}\cdot\left(\frac{z}{1+z^2}\right)^2\cdot 4 + \frac{1-z^2}{1+z^2}\cdot\left(\frac{z}{1+z^2}\right)^3\cdot 8 + \cdots \\ = \; & \frac{1-z^2}{1+z^2}\left\{1 + \frac{z}{1+z^2}\cdot 2 + \left(\frac{z}{1+z^2}\right)^2\cdot 4 + \left(\frac{z}{1+z^2}\right)^3\cdot 8 + \cdots\right\} \\ = \; & \frac{1-z^2}{1+z^2}\cdot\frac{1}{1-2\left(\frac{z}{1+z^2}\right)} = \frac{1-z^2}{1+z^2}\cdot\frac{1+z^2}{1-2z+z^2} = \frac{1+z}{1-z} \text{ as desired.} \end{aligned}$$

We can generalize what we just did by replacing the column vector $[1, 2, 4, 8, \cdots]^T \longleftrightarrow \frac{1}{1-2z}$ with $[a_0, a_1, a_2, \cdots] \longleftrightarrow A(z)$ and $[1, 2, 2, 2, \cdots]^T \longleftrightarrow \frac{1+z}{1-z}$ with $[b_0, b_1, b_2, \cdots] \longleftrightarrow B(z)$.

Now our computation becomes

$$\begin{aligned} [g, gf, gf^2, \cdots]\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} &= \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} \\ a_0 g + a_1 gf + a_2 gf^2 + \cdots &= B(z) \end{aligned}$$

$$\begin{aligned} g\left(a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \cdots\right) &= \\ gA\left(f\right) &= B\left(z\right) \\ \text{or } g\left(z\right) A\left(f\left(z\right)\right) &= B\left(z\right). \end{aligned}$$

This modest observation is very important and is called the **Fundamental Theorem of Riordan Arrays** which is abbreviated as the FTRA. More formally we have the following, which was first stated by Shapiro, Getu, Woan, and Woodson in [42]. But first, we need some notation.

Definition 2.1. *Let $g(z) = \sum_{k=0}^{\infty} g_k z^k$ and $f(z) = \sum_{k=1}^{\infty} f_k z^k$, where $g_0 \neq 0$ and $f_1 \neq 0$. Let $d_{n,k}$ be the coefficient of z^n in $g(z)\big(f(z)\big)^k$. Then $D = (d_{n,k})_{n,k\geq 0}$ is a* Riordan array*. We denote the Riordan array D by $D = \big(g(z), f(z)\big) = \big(g, f\big)$.*

Theorem 2.1. ***(The Fundamental Theorem of Riordan Arrays):*** *Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ and $B(z) = \sum_{k=0}^{\infty} b_k z^k$ and let A and B be the column vectors $A = (a_0, a_1, a_2, \cdots)^T$ and $B = (b_0, b_1, b_2, \cdots)^T$. Then $(g, f)A = B$, if and only if $B(z) = g(z)A(f(z))$.*

As a simple application, we can use FTRA to find row sums of Riordan arrays. For instance, going back to our example.

First, the row sums of the matrix without the minus signs are the Lucas numbers. To find row sums let $A\left(z\right) = \frac{1}{1-z} \longleftrightarrow [1, 1, 1, \cdots]^T$. Without the minus signs we will have $g = \frac{1+z^2}{1-z^2}$ and $f = \frac{z}{1-z^2}$. For the Lucas numbers, we have the generating function $B\left(z\right) = \frac{1+z^2}{1-z-z^2} = 1 + z + 3z^2 + 4z^3 + 7z^4 + 11z^5 + 18z^6 + \cdots$.

By the FTRA we have

$$\begin{aligned} \left(\frac{1+z^2}{1-z^2}, \frac{z}{1-z^2}\right)\frac{1}{1-z} &= \frac{1+z^2}{1-z^2} \cdot \frac{1}{1 - \left(\frac{z}{1-z^2}\right)} \\ &= \frac{1+z^2}{1-z^2} \cdot \frac{1}{\frac{1-z-z^2}{1-z^2}} \\ &= \frac{1+z^2}{1-z-z^2} \\ &= B\left(z\right). \end{aligned}$$

There is another identity we can treat. Returning the minus signs and taking row sums we have,

$$\begin{aligned} \left(\frac{1-z^2}{1+z^2}, \frac{z}{1+z^2}\right)\frac{1}{1-z} &= \frac{1-z^2}{1+z^2}\frac{1}{1-\frac{z}{1+z^2}} \\ &= \frac{1-z^2}{1+z^2}\frac{1}{\frac{1-z+z^2}{1+z^2}} \\ &= \frac{1-z^2}{1-z+z^2} \\ &= 1 + z\Big(\frac{1-z-2z^2-z^3+z^4+z^5}{1-z^6}\Big). \end{aligned}$$

Thus the sums after the first are a repeating six-cycle $1, -1, -2, -1, 1, 1$. This identity was a problem on the 1932 Tripos exam and appeared in Hardy's Pure Mathematics on page 445, [21].

What happens if we multiply one Riordan array by a second (G, F)? The k^{th} column of (G, F) has the generating function GF^k and we can apply the FTRA to express $(g, f)\, GF^k = gG\left(f\right)F^k\left(f\right)$.

Since this applies for each column of (G, F) we have a multiplication rule for Riordan arrays.

$$(g, f) * (G, F) = \left(gG\left(f\right), F\left(f\right)\right).$$

Now suppose that $f\left(z\right) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots$ and that $f_1 \neq 0$. Then f is a unit element in the domain of formal power series and has a compositional inverse which we denote as $\overline{f}\left(z\right)$ or as $\overline{f}$.

With the requirements that $g_0 \neq 0$ and $f_1 \neq 0$ we now have a group structure and we call this group the **Riordan group**, which will be denoted by $\mathcal{R}$. The identity is the usual matrix identity and is $(1, z)$. For the inverse we have

$$\left(g\left(z\right), f\left(z\right)\right)^{-1} = \left(\frac{1}{g\left(\overline{f}\left(z\right)\right)}, \overline{f}\left(z\right)\right) = \left(\frac{1}{g\left(\overline{f}\right)}, \overline{f}\right).$$

Theorem 2.2. *Let (g, f) and (G, F) be two Riordan arrays. Then the operation *, given by $(g, f) * (G, F) = (g(z)G(f(z)), F(f(z)))$ is the usual matrix multiplication which is an associative binary operation, $(1, z)$ is the identity element and the inverse of (g, f) is $\Big(\frac{1}{g(\overline{f})}, \overline{f}\Big)$.*

Using FTRA we can easily prove many combinatorial identities and the group structure gives a systematic way to invert identities.

The Riordan Group has many interesting and important subgroups. The set of all elements (g, f), such that g is an even function (with nonzero constant term) and f is an odd function, is called the Checkerboard Subgroup. The terminology comes from the fact that (g, f) has the appearance of a checkerboard. We also say that a generating function or an array is aerated if it has alternating zeros. Other subgroups are given in Section 3.

Given any Riordan array, every element of the array that is not in the leftmost column can be written as a linear combination of elements in the row that is directly above starting from the preceding column, see [40]. In addition, every element in the zeroth column other than the zeroth element can be expressed as a linear combination of the elements from the row that is directly above, see [40] and [33]. Hence, we can construct Riordan arrays by using the rows of the matrix. The following theorem tells us how to construct a Riordan array using its rows.

Theorem 2.3. *Let $D = (d_{n,k})$ be an infinite lower triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = \{a_i\}_{i=0}^{\infty}$ and $Z = \{z_i\}_{i=0}^{\infty}$ with $a_0 \neq 0$ and $z_0 \neq 0$ such that*

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}; k, n = 0, 1, 2, ... \tag{1}$$

$$d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}; n = 0, 1, 2, ... \tag{2}$$

The entry $d_{0,0}$ is given and must be nonzero, but we often set it equal to 1. The sequences $\{a_i\}_{i=0}^{\infty}$ and $\{z_i\}_{i=0}^{\infty}$, respectively, are called the $A-$sequence and $Z-$sequence of the Riordan array D.

The following theorem shows us how to find the $A-$ and $Z-$sequences of a Riordan array if we know the generating functions that determine the array.

Theorem 2.4. *Let $D = \big(g(z), f(z)\big)$ be a Riordan array. Let A be the generating function of the $A-$sequence and Z the generating function of the $Z-$sequence. Then*

$$A(z) = \frac{z}{\overline{f}(z)} \text{ and } Z(z) = \frac{1}{\overline{f}(z)} \cdot \left(1 - \frac{1}{g\big(\overline{f}(z)\big)}\right),$$

where $\overline{f}$ is the compositional inverse of f.

See [6] and [33] for more information about $A-$ and $Z-$sequences of Riordan arrays.

Another method of finding the $A-$ and $Z-$sequences of Riordan arrays is using the production (or Stieltjes) matrix.

Definition 2.2. *Let (g, f) be a Riordan array. The* production matrix or Stieltjes matrix *P is given by:*

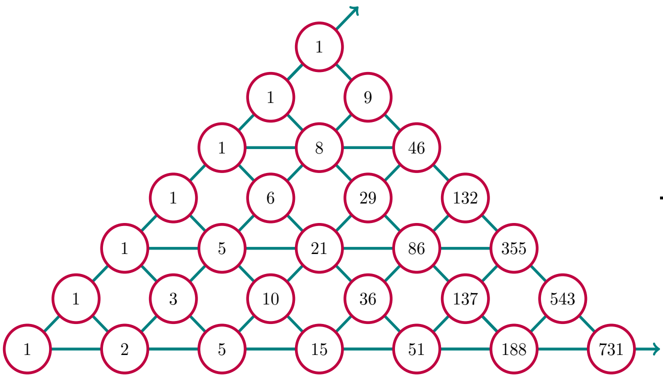
$$P = (g, f)^{-1} \cdot \overline{(g, f)},$$

where $\overline{(g, f)}$ is the truncated Riordan array with the first row omitted.

Theorem 2.5. *Let $R = (g, f)$ be a Riordan array. Then the production matrix P for R is of the form $\begin{pmatrix} Z & A & tA & t^2A & ... \end{pmatrix}$, where Z is the generating function for the $Z-$sequence and A is the generating function for the $A-$sequence of R.*

The background needed to use the Riordan group is very modest, a little abstract algebra, an acquaintance with generating functions, and a knowledge of some basic combinatorial objects such as the Catalan and Fibonacci numbers and Pascal's Triangle. The next example shows how to derive new results using some elementary tools.

Example 2.1. *Recall that given any positive integer n a Schröder path is a lattice path from the origin $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$, $(1, -1)$, and $(2, 0)$ that do not go below the x-axis. We start with the following figure of Schröder paths with no level steps at odd heights.*



$$\longrightarrow \begin{bmatrix} 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 2 & 0 & 1 & & & & & & \\ 0 & 3 & 0 & 1 & & & & & \\ 5 & 0 & 5 & 0 & 1 & & & & \cdots \\ 0 & 10 & 0 & 6 & 0 & 1 & & & \\ 15 & 0 & 21 & 0 & 8 & 0 & 1 & & \\ 0 & 36 & 0 & 29 & 0 & 9 & 0 & 1 & \\ & & & \cdots & & & & & \end{bmatrix}$$

This yields a double Riordan array rather than a Riordan array. This is just suggestive at this point, details are in Section 5.2. However, taking just rows at even heights or at odd heights yields two Riordan arrays which we call the distillates. Let E be the matrix whose columns are the rows at even height and O the matrix whose columns are the rows at odd height. Then the first few rows of each are

$$E = \begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 5 & 1 & & & \cdots \\ 15 & 21 & 8 & 1 & & \\ 51 & 86 & 46 & 11 & 1 & \\ & & \dots & & & \end{bmatrix}, \quad O = \begin{bmatrix} 1 & & & & & \\ 3 & 1 & & & & \\ 10 & 6 & 1 & & & \cdots \\ 36 & 29 & 9 & 1 & & \\ 137 & 132 & 57 & 121 & 1 & \\ & & \dots & & & \end{bmatrix}.$$

An obvious question is what are the g and f functions such that $E = (g, f)$? This problem can be solved using relations between the functions, see [17]. There are also two tools that can be used to assist us in answering the question, OEIS [43] and the Riordan calculator [38]. We leave it to the reader to answer this question.

3. Riordan Arrays, Trees, and Path Counts

In this section, we look at some more elementary applications of the Riordan group, look at some subgroups, and discuss the A and Z sequences. The Catalan numbers count ordered trees and we sketch these trees for $n = 0, 1, 2, 3$, and 4 edges.

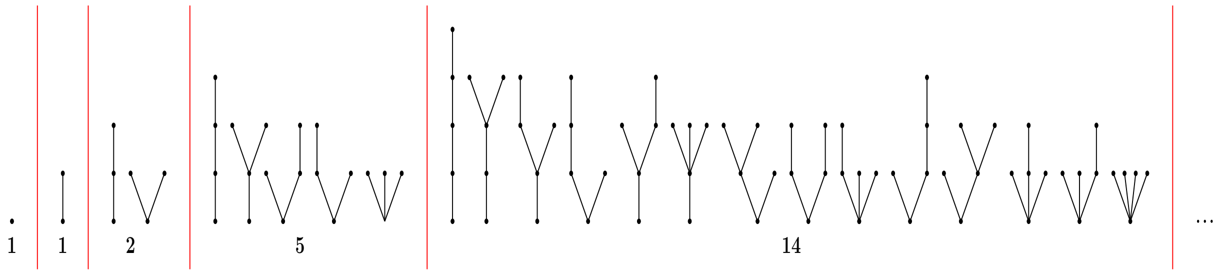


Figure 1: Counting Ordered Trees

Let's count vertices by height and we get the following matrix where n is the number of edges and k is the height. As usual, we show just the first few rows of an infinite matrix. We include more rows than usual because the k^{th} row has interesting divisibility properties if $2k - 1$ is a prime.

$$\mathbb{V} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 9 & 5 & 1 & 0 & 0 & 0 & 0 \\ 14 & 28 & 20 & 7 & 1 & 0 & 0 & 0 \\ 42 & 90 & 75 & 35 & 9 & 1 & 0 & 0 \\ 132 & 297 & 275 & 154 & 54 & 11 & 1 & 0 \\ 429 & 1001 & 1001 & 637 & 273 & 77 & 13 & 1 \end{bmatrix} = (C, zC^2)$$

Similarly, we can classify leaves by height and we get a similar matrix.

$$\mathbb{L} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 14 & 14 & 6 & 1 & 0 & 0 & 0 \\ 0 & 42 & 48 & 27 & 8 & 1 & 0 & 0 \\ 0 & 132 & 165 & 110 & 44 & 10 & 1 & 0 \\ 0 & 429 & 572 & 429 & 208 & 65 & 12 & 1 \end{bmatrix} = \left(1, zC^2\right)$$

These two are connected by the equation $\mathbb{TL} = \mathbb{V}$ where

$$\mathbb{T} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & & & \\ 1 & 1 & 0 & 0 & 0 & & & \\ 2 & 1 & 1 & 0 & 0 & & & \\ 5 & 2 & 1 & 1 & 0 & & & \\ 14 & 5 & 2 & 1 & 1 & & & \\ 42 & 14 & 5 & 2 & 1 & 1 & & \\ 132 & 42 & 14 & 5 & 2 & 1 & 1 & \\ 429 & 132 & 42 & 14 & 5 & 2 & 1 & 1 \end{bmatrix} = (C, z)\,.$$

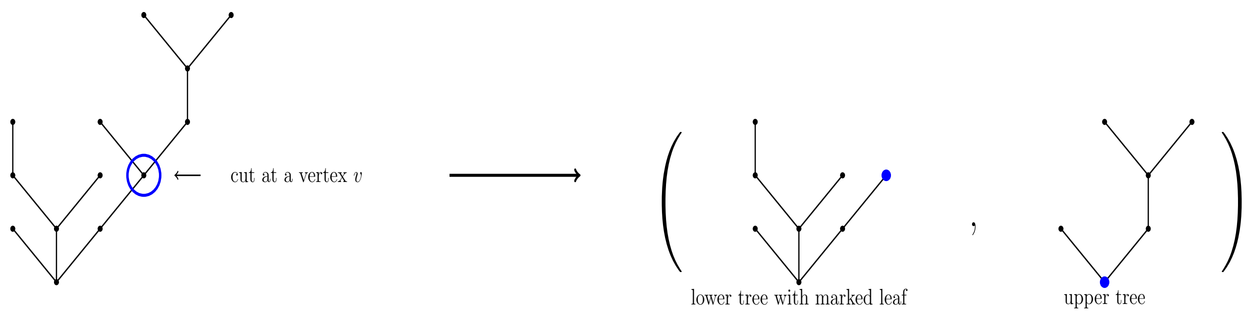
With this example in hand, there are two directions to explore. One is subgroups of the Riordan group, which we will explore in **Section 3**.

Here are the five basic equations for counting UUR trees. These are the trees where the possible up-degrees are the same for each vertex.

$$\begin{aligned} TL &= V && (3)\\ V &= (zT)' && (4)\\ L_1 &= zA'(zT) && (5)\\ L_k &= (L_1)^k && (6)\\ L &= \frac{1}{1-L_1} && (7) \end{aligned}$$

The generating functions involved are as follows. The generating function for a class of trees is $T = \sum_{n\geq 0} t_n z^n$ where t_n is the number of such trees with n edges. We essentially are assigning a z to each edge. The generating function V counts these same trees but with a marked vertex. Similarly, L is the generating function for these trees but with a marked leaf. Then L_1 is the generating function for these trees with a marked leaf at height 1, where $A(z)$ is the generating function of the A-sequence. Also, L_k counts these trees with a marked leaf at height k.

The following is a sketch of the proof for Equation (3): Consider a tree with a marked vertex v. Then cut the tree at v to produce two smaller trees. The lower subtree is a tree with a marked leaf where v was. The upper tree was sitting atop v. This translates to $V = TL$.



Here is a sketch of the proof for Equation (4): Since a tree with n edges has $n+1$ vertices we want $\sum_{n\geq 0}(n+1)\,t_n z^n = \left(\sum_{n\geq 0} t_n z^{n+1}\right)' = (zT)'$.

We explain Equation (6) as follows. Obviously $L = 1 + L_1 + L_2 + L_3 + \cdots$. But $L_2 = L_1^2$ and indeed $L_k = L_1^k$.

The following is a sketch of the proof for Equation (7): The A-sequence gives the number of possibilities for the updegree of a vertex. The number or weight for updegree k is a_k. Often it is just $a_k = 0$ meaning that no vertex has updegree k. Dually $a_k = 1$ means that a vertex of updegree k is permitted. For instance a Motzkin tree is one where each vertex can have updegree 0,1, or 2. Thus it has the A-sequence $1, 1, 1, 0, 0, 0, \cdots$ and A function $A(z) = A = 1 + z + z^2$. Complete binary trees have the A sequence $1, 0, 1, 0, 0, 0, \cdots$ and $A = 1 + z^2$. However incomplete binary trees can have a right or a left single edge so the A function for these is $1 + 2z + z^2$.

For ordered trees, we have $T = C$ the Catalan number generating function. Then

$$\begin{aligned} V &= \sum (n+1) C_n z^n = \sum (n+1) \frac{1}{n+1} \binom{2n}{n} z^n \\ &= \sum \binom{2n}{n} z^n = B = \frac{1}{\sqrt{1-4z}}. \end{aligned}$$

By Equation (3), we have $L = B/C = \frac{B+1}{2}$.

We have now proved two interesting results. The first is that for nontrivial ordered trees half of the vertices are leafs. The companion equation which follows from $B = 1 + 2zCB$ is $\frac{B-1}{2} = zBC = L - 1$.

Next, we want to compute the total height of all the leafs of all the trees with n edges. A quick count of the trees with 2 or 3 edges yields a total height 4 for the two trees with 2 edges and 16 for the five trees with 3 edges. The key observation is that if we pick a leaf at height k then there are $k - 1$ vertices between it and the root. Pick one of these $k - 1$ vertices and cut the path at that point. The appropriate generating function is $L(L-1)$. But for ordered trees, we then have

$$L(L-1) = \frac{B}{C} \cdot zBC = zB^2 = \frac{z}{1-4z} = \sum_{n \geq 1} 4^{n-1} z^n.$$

The total leaf height is 4^{n-1} for trees with n edges. This translates directly to all Dyck paths of $2n$ steps. The total height of all peaks is also 4^{n-1}.

We can obtain similar results for the many other types of UUR trees. We will work through one example here and comment on a few others.

The second example could be called Traffic Light trees. We consider ordered trees where the edges above a vertex can be either red or green. If both are present we require that all the green edges are to the left of all the red edges. Thus a vertex of updegree k has $k + 1$ possibilities for the edges above it ranging from all green to all red. Thus the generating function is $A(z) = \sum_{n \geq 0} (n+1) z^n = 1/(1-z)^2$. Differentiating gives

$$A'(z) = 2(1-z)^{-3}$$

From this it follows that $\overline{f} = z(1-z)^2$. Hence, $z = f\left(1 - f^2\right)$. Looking at the first few values of L_1 which are $0, 2, 6, 24, 110, \cdots$ suggest that $L_1 = 2g - 2$ where

$$g = 1 + zg^3 = \frac{1}{1 - zg^2} = \sum_{n \geq 0} \frac{1}{3n+1} \binom{3n+1}{n} z^n,$$

the generating function for ternary numbers.

It turns out that the T function is $g^2 = \sum_{n \geq 0} \frac{2}{3n+2} \binom{3n+2}{n} z^n$. Thus $V = (zT)' = \sum_{n \geq 0} \binom{3n+1}{n} z^n$.

We can express L as either V/T or as $\frac{1}{1-L_1}$ where

$$L_1 = zA'(zT) = z \cdot \frac{2}{(1-zT)^3} = \frac{2z}{(1-zg^2)^3} = 2zg^3 = 2(g-1) = 2z + 6z^2 + 24z^3 + 110z^4 + 546z^5 + \cdots$$

For the Traffic Light trees we have

$$\mathbb{L} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 6 & 4 & 0 & 0 \\ 0 & 24 & 24 & 8 & 0 \\ 0 & 110 & 132 & 72 & 16 \end{bmatrix} = (1, 2(g-1))$$

and

$$\mathbb{V} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 7 & 10 & 4 & 0 & 0 \\ 30 & 50 & 32 & 8 & 0 \\ 143 & 260 & 208 & 88 & 16 \end{bmatrix} = \left(g^2, 2(g-1)\right).$$

We now go back to the ordered trees and ask what the average number of leafs at height 1. Since $L_1 = zC^2$ we have

$$\frac{[z^n]\, zC^2}{[z^n]\, C} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\frac{1}{n+1}\binom{2n}{n}} = 1 \text{ for } n \geq 1 \text{ since } C = 1 + zC^2.$$

Thus, for all trees with n edges, there is on average one leaf of height 1 per tree. In terms of Dyck paths, the number of hills, which are subsequences UD starting and ending on the x-axis is on average 1.

By way of contrast, we have Motzkin trees where every vertex can have updegree 0, 1, or 2.

The generating function counting Motzkin trees is denoted $m(z)$ or m. The defining equation for this generating function is $m = 1 + zm + z^2m^2$. The three terms correspond to the root having degree 0, 1, or 2. Thus $m = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2} = 1 + z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + 51z^6 + \cdots = \sum_{n\geq 0} m_n z^n$. Note that $2z^2m = 1 - z - \sqrt{(1+z)(1-3z)}$ so the radius of convergence is $1/3$ where $1 - 3z = 0$ occurs. The ratio test then says that $\lim_{n\to\infty} \frac{m_{n+1}}{m_n} = 3$. The A-sequence is 1, 1, 1, 0, 0, 0, ... and thus $A(z) = 1 + z + z^2$, which implies $A'(z) = 1 + 2z$. With the L_1 equation we get

$$\begin{aligned} L_1 &= zA'(zm) = z(1 + 2zm) \\ \text{Then } [z^n](z(1+2zm)) &= 2m_{n-2} \text{ for } n \geq 2 \\ \text{Hence } \frac{2m_{n-2}}{m_n} &= 2 \cdot \frac{m_{n-2}}{m_{n-1}} \cdot \frac{m_{n-1}}{m_n} \to 2 \cdot \left(\frac{1}{3}\right)^2 = \frac{2}{9}. \end{aligned}$$

So, the average number of leaves at height 1 approaches just $2/9$.

To check this we see that

$$\begin{aligned} \frac{2m_{10}}{m_{12}} &= \frac{2 \cdot 2188}{15511} \simeq 0.282\,12 \\ \frac{2m_{20}}{m_{22}} &= 2 \cdot \frac{50\,852\,019}{400\,763\,223} \simeq 0.253\,78 \end{aligned}$$

showing the convergence, albeit slowly to $2/9 \simeq 0.22222$.

Note that for the Traffic Light trees

$$\frac{[z^n]\, L_1}{[z^n]\, T} = \frac{[z^n]\, 2(g-1)}{[z^n]\, g^2} = \frac{2 \cdot \frac{1}{3n+1}\binom{3n+1}{n}}{\frac{2}{3n+2}\binom{3n+2}{2}} = \frac{2n+2}{3n+1} \to \frac{2}{3}.$$

4. Subgroups

Many subgroups of $\mathcal{R}$ have been studied, both for their combinatorial and algebraic properties. For example, the Bell subgroup, given by $\{(g, f) \in \mathcal{R} : f = zg\} = \{(g, zg)\}$, the Associated subgroup, given by $\{(g, f) \in \mathcal{R} : g = 1\} = \{(1, f)\}$, and the Appell subgroup, given by $\{(g, f) \in \mathcal{R} : f = z\} = \{(g, z)\}$. Note that for all functions f and g, $(g, f) = (g, z) * (1, f)$. Thus $\mathcal{R}$ is the semidirect product of the Associated and Appell subgroups.

Here is a small list of subgroups.

Typical Element	Name
(g, zg)	Bell subgroup
$(1, f)$	Associated subgroup
(g, z)	Appell subgroup
(f', f)	Derivative subgroup
$\left(\frac{zf'}{f}, f\right)$	Hitting time subgroup
$\left(g^{even}, f^{odd}\right)$	Checkerboard subgroup
$\left(g^a, zg^b\right)$	(a, b)-Bell subgroup

The Bell subgroup is an (a, b)-Bell subgroup with $a = 1$ and $b = 1$. Another common well-known Riordan group subgroup that is used often in this article is the $2-$Bell subgroup which is an (a, b)-Bell subgroup with $a = 1$ and $b = 2$. The Pascal matrix is obviously in the Bell subgroup and less obviously in the Hitting time subgroup. The matrix $\mathbb{L}$ is in the Associated subgroup. The matrix $\mathbb{V}$ is in the 2-Bell subgroup as is the matrix $\left(\frac{1}{1-z}, \frac{z}{(1-z)^2}\right)$ from section one.

Of these seven subgroups, the only one that is a normal subgroup is the Appell subgroup. The verifications of the subgroups mentioned here are subgroups is usually straightforward. Here is an example.

Theorem 4.1. *The derivative subgroup is indeed a subgroup.*

Proof. Since $\frac{d}{dz}(z) = 1$ The identity $(1, z)$ is in the subgroup.

To show closure we multiply (f', f) and (F', F) and their product is $(f', f)(F', F) = (f'F'(f), F(f))$ which is in the subgroup since

$$\frac{d}{dz}(F(f(z))) = F'(f(z)) \cdot f'(z).$$

Similarly,

$$\frac{d}{dz}\left(\overline{f}(z)\right) = 1/f'\left(\overline{f}(z)\right) \text{ so } (f', f)^{-1} = \left(\frac{1}{f'\left(\overline{f}(z)\right)}, \overline{f}(z)\right) = \left(\overline{f}(z)', \overline{f}(z)\right).$$

□

The other subgroup verifications are similar or even less involved and are left to the reader.

A natural question to ask is, what is the commutator subgroup of $\mathcal{R}$? In 2021, Ana Luzón, Manuel Morón, and L. Felipe Prieto-Martínez proved the following, see [29].

Theorem 4.2. *The commutator subgroup of $\mathcal{R}$, denoted by $[\mathcal{R}, \mathcal{R}]$, is given by $[\mathcal{R}, \mathcal{R}] = \{(g, f) \in \mathcal{R} : g_0 = 1$ and $f_1 = 1\}$. Moreover, every element in $[\mathcal{R}, \mathcal{R}]$ is a commutator.*

Hence, the elements in $[\mathcal{R}, \mathcal{R}]$ are those Riordan arrays with $1s$ down the main diagonal.

5. Exponential Riordan Arrays

The other major class of examples and applications of Riordan arrays involve exponential generating functions. We now associate a sequence $(a_0, a_1, a_2, ...)$ with the generating function $A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$.

If $A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ and $B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!}$ are multiplied, then $C(z) = A(z)B(z) = \sum_{n\geq 0} c_n \frac{z^n}{n!}$, where $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$.

Let $g(z) = g = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$, with $g_0 \neq 0$ and $f(z) = f = \sum_{n=1}^{\infty} f_n \frac{z^n}{n!}$, with $f_1 \neq 0$. Then the exponential Riordan array they generate is denoted by $[g, f]$, where the k^{th} column has the generating function $\frac{g(z)(f(z))^k}{k!}$.

Here are five well-known exponential Riordan arrays.

1. $[e^z, z] = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \\ & ... & & \end{bmatrix}$

   The Pascal matrix is a rare example of an array that is both an ordinary Riordan array and an exponential Riordan array.

2. $[1, e^z - 1] = \Big(S(n,k)\Big)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 3 & 1 & & \\ 0 & 1 & 7 & 6 & 1 & \\ 0 & 1 & 15 & 25 & 10 & 1 \\ & & & ... & & \end{bmatrix}$

   The Stirling numbers of the second kind $S(n,k)$ count the number of ways to partition a set with n elements into k nonempty blocks. The row sums of the array are the Bell numbers and $S(n,k)k!$ counts the number of onto functions from an $n - set$ to a $k - set$.

3. $\left[1, ln\left(\frac{1}{1-z}\right)\right] = \Big(s(n,k)\Big)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 2 & 3 & 1 & & \\ 0 & 6 & 11 & 6 & 1 & \\ 0 & 24 & 50 & 35 & 10 & 1 \\ & & & ... & & \end{bmatrix}$

   These are the signless Stirling numbers of the first kind $s(n,k)$. These count the number of permutations in S_n whose disjoint decomposition has k cycles.

4. $[D(z), z] = \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 9 & 8 & 6 & 0 & 1 \\ 44 & 45 & 20 & 10 & 0 & 1 \\ & & \dots \end{bmatrix}$

   $D(z) = \sum_{n=0}^{\infty} d_n \frac{z^n}{n!} = \frac{e^{-z}}{1-z}$, is the derangement series. Recall that a derangement is a permutation with no fixed points and $(d_n)_{n\geq 0} = (1, 0, 1, 2, 9, 44, ...)$. The k^{th} column, for $k = 1, 2, 3, ...$ counts permutations with k fixed points.

5. The telephone exchange.

   Let T_n be the number of ways that n people can be off the phone or on with one other person. If k counts the number of people not engaged in a phone conversation, then we get the following exponential Riordan array

   $$[e^{\frac{z^2}{2}}, z] = \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 3 & 0 & 6 & 0 & 1 \\ 0 & 15 & 0 & 10 & 0 & 1 \\ 15 & 0 & 45 & 0 & 15 & 0 & 1 \\ & & & \dots \end{bmatrix}$$

   The row sums $(1, 1, 2, 4, 10, 26, 76, ...) = (T_n)_{n\geq 0}$. T_n is also the number of solutions of $\pi^2 = \iota$ in the symmetric group on n letters.

There is an FTRA for exponential Riordan arrays where we get a group structure. The identity element is $[1, z]$ and

$$[g(z), f(z)]^{-1} = \left[\frac{1}{g\left(\overline{f}(z)\right)}, \overline{f}(z)\right] = \left[\frac{1}{g\left(\overline{f}\right)}, \overline{f}\right].$$

The subgroup definitions are unchanged as well.

Given an exponential generating function $[g, f]$, the generating function for the row sums is $[g, f]e^z = g(z)e^{f(z)}$.

Example 5.1. *The row sums of the Stirling numbers of the first kind start* $1, 1, 2, 6, 24$, *and the assumption is that we are looking at* $(n!)_{n\geq 0}$. *To prove this we have*

$$\left[1, ln\left(\frac{1}{1-z}\right)\right] e^z = 1 \cdot e^{ln\left(\frac{1}{1-z}\right)} = \sum_{n\geq 0} n! \frac{z^n}{n!}.$$

6. Other Types of Riordan Arrays

We have introduced both ordinary Riordan arrays and exponential Riordan arrays. In this section, we will now generalize these ideas. Given a sequence $\mathbf{r} = (r_0, r_1, ...)$, we will define generating functions $g(z; \mathbf{r})$ and $f(z; \mathbf{r})$. The Riordan array defined by these generating functions will be called an $\mathbf{r}$-Riordan array and the sequence $\mathbf{r}$ will be called a denominator sequence. Much of Section 6.1 can be found in T. Wang and W. Wang [51] and L. Woodson [52].

6.1 r - Riordan Arrays

Let $\mathbf{r} = (r_0, r_1, r_2, ...)$, where $r_0 = 1, r_n \neq 0$ for $n \geq 1$. Then we define

$$g(z; \mathbf{r}) = a_0 + a_1 \frac{z}{r_1} + a_2 \frac{z^2}{r_2} + a_3 \frac{z^3}{r_3} + \cdots = \sum_{n=0}^{\infty} a_n \frac{z^n}{r_n}$$

and

$$f(z; \mathbf{r}) = b_1 \frac{z}{r_1} + b_2 \frac{z^2}{r_2} + b_3 \frac{z^3}{r_3} + \cdots = \sum_{n=1}^{\infty} b_n \frac{z^n}{r_n}.$$

Define the $\mathbf{r}$ - Riordan matrix M by

$$M = (g(z;\mathbf{r}), f(z;\mathbf{r})) = (g, f;\mathbf{r}) = \begin{bmatrix} | & | & | & | & \\ g & g\frac{f}{r_1} & g\frac{f^2}{r_2} & g\frac{f^3}{r_3} & ... \\ | & | & | & | & \end{bmatrix}$$

For a fixed sequence $\mathbf{r}$, we have the following property: the set of $(g, f;\mathbf{r})$ forms a group when $a_0 \neq 0$ and $b_1 \neq 0$. In the case of ordinary Riordan arrays we have $\mathbf{r} = (1, 1, 1, ...)$ and when $\mathbf{r} = (n!)_{n=0}^{\infty}$ we get the exponential Riordan arrays. Many of the subgroups given in the ordinary Riordan group are defined similarly for r-Riordan arrays. For example, the Appell subgroup is the set of elements of the form $(g, z;\mathbf{r})$ and the Associated subgroup is the set of elements of the form $(1, f;\mathbf{r})$.

Example 6.1. *Let* $r_n = \frac{(2n)!}{2^n}$.

$$\left(sec^2(\sqrt{z}), tan^2(\sqrt{z}); \tfrac{(2n)!}{2^n}\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 4 & 10 & 1 & & & \\ 34 & 154 & 35 & 1 & & \\ 496 & 3520 & 1344 & 84 & 1 & \\ 11056 & 112816 & 63580 & 6468 & 165 & 1 \\ & & \cdots & & & \end{bmatrix}$$

The decomposition of this matrix into the product of an Associated matrix with an Appell matrix is

$$\left(sec^2(\sqrt{z}), z; (2n)!/2^n\right) * \left(1, tan^2(\sqrt{z}); (2n)!/2^n\right), \ i.e.$$

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 4 & 6 & 1 & & & \\ 34 & 60 & 15 & 1 & & \\ 496 & 952 & 280 & 28 & 1 & \\ 11056 & 22320 & 7140 & 840 & 45 & 1 \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 4 & 1 & & & \\ 0 & 34 & 20 & 1 & & \\ 0 & 496 & 504 & 56 & 1 & \\ 0 & 11056 & 16960 & 3108 & 120 & 1 \\ & & \cdots & & & \end{bmatrix}$$

6.1.1 Generalized Binomial Coefficients

In this subsection, we will give the definition of the generalized binomial coefficient. These coefficients will be used later when we discuss summation functions. Throughout this subsection, $\mathbf{r}$ will be a fixed sequence.
Definition. $[x+y]^n = \sum_{k=0}^{n} \frac{r_n}{r_k r_{n-k}} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k}_{\mathbf{r}} x^k y^{n-k}$, where $\binom{n}{k}_{\mathbf{r}} = \frac{r_n}{r_k r_{n-k}}$.

Theorem 6.1. *If*

$$A(z) = \sum_{n\geq 0} a_n \frac{z^n}{r_n} \text{ and } B(z) = \sum_{n\geq 0} b_n \frac{z^n}{r_n},$$

then

$$A(z)B(z) = \sum_{n\geq 0} d_n \frac{z^n}{r_n}, \text{ where } d_n = \sum_{k=0}^{\infty} \binom{n}{k}_{r_n} a_k b_{n-k}.$$

Note that if $\mathbf{r} = \{n!\}$, then $\binom{n}{k}_{\mathbf{r}} = \binom{n}{k}$ and $[x+y]_{\mathbf{r}}^n = (x+y)^n$. When $\mathbf{r} = \{1\}$, then

$$[x+y]_{\mathbf{r}}^n = \sum_{k=0}^{n} x^k y^{n-k} = x^n + x^{n-1}y + ... + xy^{n-1} + y^n.$$

6.1.2 Summation Functions

Define the summation function $E(z;\mathbf{r}) = E(z) = \sum_{n=0}^{\infty} \frac{z^n}{r_n}$. Also, define the linear functional $\sigma_{\mathbf{r}}$ by

$$\sigma_{\mathbf{r}} A(z) = \sum_{n\geq 0} a_{n+1} \frac{z^n}{r_n}, \text{ where } A(z) = \sum_{n\geq 0} a_n \frac{z^n}{r_n}.$$

Then, $\sigma_{\mathbf{r}} \cdot E(z;\mathbf{r}) = E(z, \mathbf{r})$.

For example, when $\mathbf{r} = \{n!n!\}, \sigma_{\mathbf{r}} = DzD$, where D is the differential operator.

$$E(z,n!n!)=\sum_{n=0}^{\infty}\frac{z^n}{n!n!} \implies DzD\sum_{n=0}^{\infty}\frac{z^n}{n!n!}=Dz\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!(n-1)!}=D\sum_{n=1}^{\infty}\frac{z^n}{n!(n-1)!}$$
$$=\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!(n-1)!}=\sum_{n=0}^{\infty}\frac{z^n}{n!n!}=E(z,n!n!).$$

A classical well-known example using summation functions is

$$\begin{bmatrix} 1 &&&&& \\ 1 & 1 &&&& \\ 1 & 2 & 1 &&& \\ 1 & 3 & 3 & 1 && \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \\ && \cdots &&& \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdots \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \\ \cdots \end{bmatrix}$$

where the entries are represented either by $\left(\frac{1}{1-z},\frac{z}{1-z};1\right)*\frac{1}{1-z}=\frac{1}{1-2z}$ or by $(e^z,z;n!)*e^z=[e^z,z]*e^z=e^{2z}$.

The summation function $E(z,\mathbf{r})$ is also the row sum function for Riordan arrays. That is, the Riordan array $(g,f;\mathbf{r})$ has row sum $g\cdot E(f,\mathbf{r})$. Where $(E,z;\mathbf{r})$ is the generalized Pascal triangle.

$$\begin{bmatrix} \binom{0}{0}_{\mathbf{r}} &&&& \\ \binom{1}{0}_{\mathbf{r}} & \binom{1}{1}_{\mathbf{r}} &&& \\ \binom{2}{0}_{\mathbf{r}} & \binom{2}{1}_{\mathbf{r}} & \binom{2}{2}_{\mathbf{r}} && \\ \binom{3}{0}_{\mathbf{r}} & \binom{3}{1}_{\mathbf{r}} & \binom{3}{2}_{\mathbf{r}} & \binom{3}{3}_{\mathbf{r}} & \\ \binom{4}{0}_{\mathbf{r}} & \binom{4}{1}_{\mathbf{r}} & \binom{4}{2}_{\mathbf{r}} & \binom{4}{3}_{\mathbf{r}} & \binom{4}{4}_{\mathbf{r}} \\ && \cdots && \end{bmatrix}$$

Note that, for a summation function E, $(E,z;\mathbf{r})*E=E^2$.

Here are a few examples.

1. $(\frac{1}{1-z},z;1)*\frac{1}{1-z}=(\frac{1}{1-z})^2$
2. $(e^z,z,n!)*e^z=(e^z)^2=e^{2z}$
3. $(I_0(2\sqrt{z}),z;(n!)^2)*I_0(2\sqrt{z})=I_0^2(2\sqrt{z})$ where $I_0(z)$ is the modified Bessel function of order 0.

To find the entries of $(I_0(2\sqrt{z}),z;(n!)^2)$, we look at the generating functions for the first few columns:
$I_0(2\sqrt{z})=1+z+\frac{z^2}{2!2!}+\frac{z^3}{3!3!}+\frac{z^4}{4!4!}+...$
$zI_0(2\sqrt{z})=z+\frac{4z^2}{2!2!}+\frac{9z^3}{3!3!}+\frac{16z^4}{4!4!}+\frac{25z^5}{5!5!}+...$
$\frac{z^2}{2!2!}I_0(2\sqrt{z})=\frac{z^2}{2!2!}+\frac{9z^3}{3!3!}+\frac{36z^4}{4!4!}+\frac{100z^5}{5!5!}+...$
$\frac{z^3}{3!3!}I_0(2\sqrt{z})=\frac{z^3}{3!3!}+\frac{16z^4}{4!4!}+\frac{100z^5}{5!5!}+\frac{400z^6}{6!6!}+...$

$$(I_0(2\sqrt{z}),z;(n!)^2)=\begin{bmatrix} 1 &&&&&& \\ 1 & 1 &&&&& \\ 1 & 4 & 1 &&&& \\ 1 & 9 & 9 & 1 &&& \cdots \\ 1 & 16 & 36 & 16 & 1 && \\ 1 & 25 & 100 & 100 & 25 & 1 & \end{bmatrix}$$

Note that $I_0^2(2\sqrt{z})=1+2z+\frac{6z^2}{2!2!}+\frac{20z^3}{3!3!}+\frac{70z^4}{4!4!}+...$, which we see below:

$$\begin{bmatrix} 1 &&&&& \\ 1 & 1 &&&& \\ 1 & 4 & 1 &&& \\ 1 & 9 & 9 & 1 && \\ 1 & 16 & 36 & 16 & 1 & \\ 1 & 25 & 100 & 100 & 25 & 1 \\ && \cdots &&& \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdots \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 6 \\ 20 \\ 70 \\ 252 \\ \cdots \end{bmatrix}$$

Thus, we have confirmed that $(I_0(2\sqrt{z}), z; (n!)^2) * I_0(2\sqrt{z}) = I_0^2(2\sqrt{z})$. Without the generating functions we have the famous identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

6.1.3 Appell Riordan Arrays

Riordan arrays of the form $(g, z; \mathbf{r})$ are Appell Riordan arrays. We have the following results:

1. $(g(z), z; \mathbf{r}) + (h(z), z; \mathbf{r}) = ((g + h)(z), z; \mathbf{r})$
2. $a(g(z), z; \mathbf{r}) = (ag(z), z; \mathbf{r})$ for any constant a
3. $(z, z; \mathbf{r})^n = (z^n, z; \mathbf{r}), n \in \mathbb{Z}^+$. We define $(z, z; \mathbf{r})^0 = (1, z; \mathbf{r})$

Note that some of the matrices from equations $1, 2$, and 3 above are not necessarily Riordan.

Theorem 6.2. *Let $g(z; \boldsymbol{r}) = \sum_{n\geq 0} g_n \frac{z^n}{r_n}$ and $A = (z, z; \boldsymbol{r})$. Then, $g(A) = (g(z), z; \boldsymbol{r})$.*

Proof.

$$g(A) = \sum_{n\geq 0} g_n \frac{A^n}{r_n} = \sum_{n\geq 0} \frac{g_n}{r_n}(z, z; \mathbf{r})^n = \sum_{n\geq 0} \left(g_n \frac{z^n}{r_n}, z; \mathbf{r}\right) = \left(\sum_{n\geq 0} g_n \frac{z^n}{r_n}, z; \mathbf{r}\right) = (g(z), z; \mathbf{r})$$

□

An immediate corollary of this theorem is that an Appell matrix $(g(z), z; \mathbf{r})$ is diagonally reducible to $(E(z, r), z; \mathbf{r})$. That is to say, the matrix is the diagonally-enhanced generalized Pascal triangle.

$$\begin{bmatrix}
\binom{0}{0}_{\mathbf{r}} a_0 & & & & \\
\binom{1}{0}_{\mathbf{r}} a_1 & \binom{1}{1}_{\mathbf{r}} a_0 & & & \\
\binom{2}{0}_{\mathbf{r}} a_2 & \binom{2}{1}_{\mathbf{r}} a_1 & \binom{2}{2}_{\mathbf{r}} a_0 & & \\
\binom{3}{0}_{\mathbf{r}} a_3 & \binom{3}{1}_{\mathbf{r}} a_2 & \binom{3}{2}_{\mathbf{r}} a_1 & \binom{3}{3}_{\mathbf{r}} a_0 & \\
\binom{4}{0}_{\mathbf{r}} a_4 & \binom{4}{1}_{\mathbf{r}} a_3 & \binom{4}{2}_{\mathbf{r}} a_2 & \binom{4}{3}_{\mathbf{r}} a_1 & \binom{4}{4}_{\mathbf{r}} a_0 \\
 & & \cdots & &
\end{bmatrix}$$

The following example looks at an element in the Bell subgroup:

$$(e^z, ze^z; n!) = \begin{bmatrix}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 4 & 1 & & & & \\
1 & 12 & 9 & 1 & & & \\
1 & 32 & 54 & 16 & 1 & & \\
1 & 80 & 270 & 160 & 25 & 1 & \\
1 & 192 & 1215 & 1280 & 375 & 36 & 1 \\
 & & & \cdots & & &
\end{bmatrix}$$

Note that for $\mathbf{r} \equiv 1 = (1, 1, 1, ...)$, the matrix is not Riordan. The column-generating functions of this matrix are

$$\begin{bmatrix}
| & | & | & | & | \\
\frac{1}{1-z} & \frac{z}{(1-2z)^2} & \frac{z^2}{(1-3z)^3} & \frac{z^3}{(1-4z)^4} & \cdots \\
| & | & | & | & |
\end{bmatrix}.$$

So, the coefficients of the columns are $1, \binom{n}{1}2^{n-1}, \binom{n}{2}3^{n-2}, \binom{n}{3}4^{n-3}, ...$ A combinatorial interpretation will be the following: let the zeroth column be 1. For the first column, you must choose an executive committee of 1 person and form all possible committees. The committees can be empty. For the second column, you must choose an executive committee of 2 people and forming all possible committees and subcommittees of the committees. The third column will represent choosing an executive committee of 3 people and form all possible committees, subcommittees, sub-subcommittees, and so on for the following columns.

6.2 Double Riordan Array

In a Riordan array, we use one multiplier function. Consider the case where we use two multiplier functions. So, if g gives column zero and f_1 and f_2 are the multiplier functions, then the first column is gf_1, the second is gf_1f_2, the third is $gf_1f_2f_1$, and so on. The set of double Riordan arrays is not closed under multiplication. However, if we require that g be an even function and f_1 and f_2 odd functions, then we can develop an analog of FTRA and obtain a group structure.

Definition 6.1. *Let $g(z) = \sum_{k=0}^{\infty} g_{2k}z^{2k}$, $f_1(z) = \sum_{k=0}^{\infty} f_{1,2k+1}z^{2k+1}$, and $f_2(z) = \sum_{k=0}^{\infty} f_{2,2k+1}z^{2k+1}$, with $g_0 \neq 0$, $f_{1,1} \neq 0$, and $f_{2,1} \neq 0$. Then the double Riordan matrix (or array) of g, f_1 and f_2, denoted by $(g; f_1, f_2)$, is given by*

$$(g; f_1, f_2) = (g,\ gf_1,\ gf_1f_2,\ gf_1^2f_2,\ gf_1^2f_2^2,\ \cdots).$$

The set of all double Riordan matrices is denoted as $\mathcal{DR}$. Note that all of these matrices are aerated.

Example 6.2. *Let $C(z) = \frac{1-\sqrt{1-4z}}{2z}$. This is the generating function for the Catalan numbers. The following is a double Riordan array.*

$$\text{Let } (g, f_1, f_2) = \Big(C(z^2),\ zC^2(z^2),\ z\Big)$$

$$\begin{array}{c}
\begin{array}{ccccccccc} g & gf_1 & gf_1f_2 & gf_1(f_1f_2) & g(f_1f_2)^2 & gf_1(f_1f_2)^2 & g(f_1f_2)^3 & gf_1(f_1f_2)^3 & \cdots \end{array} \\
\left[\begin{array}{ccccccccc}
1 & & & & & & & & \\
0 & 1 & & & & & & & \\
1 & 0 & 1 & & & & & & \\
0 & 3 & 0 & 1 & & & & & \\
2 & 0 & 3 & 0 & 1 & & & & \ddots \\
0 & 9 & 0 & 5 & 0 & 1 & & & \\
5 & 0 & 9 & 0 & 5 & 0 & 1 & & \\
0 & 28 & 0 & 20 & 0 & 7 & 0 & 1 & \\
14 & 0 & 28 & 0 & 20 & 0 & 7 & 0 & 1 \\
 & & & & \cdots & & & &
\end{array}\right]
\end{array}$$

Theorem 6.3. ***(The Fundamental Theorem of Double Riordan Arrays):*** *Let $g(z) = \sum_{k=0}^{\infty} g_{2k}z^{2k}$, $f_1(z) = \sum_{k=0}^{\infty} f_{1,2k+1}z^{2k+1}$, and $f_2(z) = \sum_{k=0}^{\infty} f_{2,2k+1}z^{2k+1}$.*

Case 1: If $A(z) = \sum_{k=0}^{\infty} a_{2k}z^{2k}$ and $B(z) = \sum_{k=0}^{\infty} b_{2k}z^{2k}$, and $A = (a_0, 0, a_2, 0, \cdots)^T$ and $B = (b_0, 0, b_2, 0, \cdots)^T$ are column vectors. Then $(g, f_1, f_2)A = B$ if and only if $B(z) = g(z)A(\sqrt{f_1(z)f_2(z)})$.

Case 2: If $A(z) = \sum_{k=0}^{\infty} a_{2k+1}z^{2k+1}$ and $B(z) = \sum_{k=0}^{\infty} b_{2k+1}z^{2k+1}$ with $(g, f_1, f_2)A = B$, then $B(z) = g(z)\sqrt{f_1/f_2}A(\sqrt{f_1(z)f_2(z)})$.

Definition 6.2. *Let (g, f_1, f_2) and (G, F_1, F_2) be elements of $\mathcal{DR}$. Then*
$(g; f_1, f_2)(G; F_1, F_2) = (gG(\sqrt{f_1f_2}); \sqrt{f_1/f_2}F_1(\sqrt{f_1f_2}), \sqrt{f_2/f_1}F_2(\sqrt{f_1f_2}))$.

The following theorem is analogous to Theorem 2.

Theorem 6.4. *$(\mathcal{DR}, *)$ is a group.*

Proof. The matrix $(1; z, z)$ is the identity. Matrix multiplication is associative.
Let $(g; f_1, f_2)$ be in $\mathcal{DR}$ and let $h = \sqrt{f_1f_2}$ and also denote by $\bar{h}$ the compositional inverse of h. Then $((1/g(\bar{h}); z\bar{h}/f_1(\bar{h}), z\bar{h}/f_2(\bar{h}))$ is the inverse of $(g; f_1, f_2)$. $\square$

Theorem 6.5. *Let $\mathcal{A}$=$\{(g; f_1, f_2) \in \mathcal{DR} : g = 1\}$ and*
$\mathcal{B}_1$=$\{(g; f_1, f_2) \in \mathcal{DR} : f_1 = zg\}$ and $\mathcal{B}_2$=$\{(g; f_1, f_2) \in \mathcal{DR} : f_2 = zg\}$. Then $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{A}$ are subgroups of $\mathcal{DR}$.

Theorem 6.6. *$\{(g; z, z) \in \mathcal{R}\}$ is a normal subgroup of $\mathcal{DR}$ and $\mathcal{DR}$ is the semidirect product of $\{(g; z, z) \in \mathcal{R} : f = z\}$ and $\mathcal{A}$.*

Theorem 6.7. *The following are subgroups.*

1. *$\{(f', f, cf) \in \mathcal{DR} : c \in \mathbb{R}, c > 0\}$ and $\{(f', cf, f) \in \mathcal{DR} : c \in \mathbb{R}, c > 0\}$*
2. *$\{(g, czg, f) \in \mathcal{DR} : c \in \mathbb{R}, c > 0\}$ and $\{(g, f, czg) \in \mathcal{DR} : c \in \mathbb{R}, c > 0\}$*

The subgroups are called the derivative subgroups and the C-Bell Subgroups, see [10]. We see that $\mathcal{DR}$ has some of the same subgroup properties as $\mathcal{R}$. For more about the Double Riordan Array, see [17].

A Riordan array has one $Z-$sequence and one $A-$sequence. It is interesting to note that there are three known ways to row construct elements in $\mathcal{DR}$. One uses two $Z-$sequences and one $A-$sequence and the other two use one $Z-$sequence and two $A-$sequences. One found by He is given in the following theorem, see [24].

Theorem 6.8. *Let $D = (g; f_1, f_2) = (d_{n,k})$ be an infinite lower triangular matrix. Then D is a Double Riordan array if and only if there exist three sequences $A_1 = \{a_{1,i}\}_{i=0}^{\infty}$, $A_2 = \{a_{2,i}\}_{i=0}^{\infty}$ and $Z = \{z_i\}_{i=0}^{\infty}$ with $a_{1,0} \neq 0$, $a_{2,0} \neq 0$, and $z_0 \neq 0$ such that for each n*

$$d_{n,2k-1} = \sum_{j=0}^{\infty} a_{1,j} d_{n-1,2k+2(j-1)}; k = 0, 1, 2, ...,$$

$$d_{n,2k} = \sum_{j=0}^{\infty} a_{2,j} d_{n-2,2k+2(j-1)}; k = 0, 1, 2, ..., \text{ and}$$

$$d_{n,0} = \sum_{j=0}^{\infty} z_j d_{n-2,2j}.$$

The following theorem is similar to Theorem 4 in Section 1. It allows us to find the $A-$ and $Z-$sequences using the generating functions that define the Double Riordan Array.

Corollary 6.1. *Let $D = (g; f_1, f_2)$ be an element of DR. Let $A_1(t) = \sum_{k=0}^{\infty} a_{1,k} t^{2k}$, $A_2(t) = \sum_{k=0}^{\infty} a_{2,k} t^{2k}$, and $Z(t) = \sum_{k=0}^{\infty} z_k t^{2k}$ respectively be the generating functions for the A_1-sequence, A_2-sequence, and $Z-$sequence. Let $h = \sqrt{f_1 f_2}$. Then*

$$A_1(t) = \frac{f_1(\overline{h})}{\overline{h}}$$

$$A_2(t) = \frac{t^2}{\overline{h}^2}$$

$$Z(t) = \frac{1}{\overline{h}^2} \cdot \left(1 - \frac{g(0)}{g(\overline{h})}\right).$$

The next two theorems give other row constructions of Double Riordan Arrays, see [10] and [18].

Theorem 6.9. *Let $D = (g; f_1, f_2) = (d_{n,k})$ be an infinite lower triangular matrix. Then D is a Double Riordan array if and only if there exist three sequences $A = \{a_i\}_{i=0}^{\infty}$, $Z_0 = \{z_{0,i}\}_{i=0}^{\infty}$ and $Z_1 = \{z_{1,i}\}_{i=0}^{\infty}$ with $a_0 \neq 0$, $z_{0,0} \neq 0$, and $z_{1,0} \neq 0$ such that for each n*

$$d_{n,k} = \sum_{j=0}^{\infty} a_j d_{n-2,k+2(j-1)}; k = 2, ...,$$

$$d_{n,1} = \sum_{j=0}^{\infty} z_{1,j} d_{n-2,2j+1}, \text{ and}$$

$$d_{n,0} = \sum_{j=0}^{\infty} z_{0,j} d_{n-2,2j}.$$

Corollary 6.2. *Let $D = (g; f_1, f_2)$ be an element of DR. Let $A(t) = \sum_{k=0}^{\infty} a_k t^{2k}$, $Z_0(t) = \sum_{k=0}^{\infty} z_{0,k} t^{2k}$, and $Z_1(t) = \sum_{k=0}^{\infty} z_{1,k} t^{2k}$ respectively be the generating functions for the $A-$sequence, Z_0-sequence, and Z_1-sequence. Let $h = \sqrt{f_1 f_2}$. Then*

$$A(t) = \frac{t^2}{\overline{h}^2}$$

$$Z_0(t) = \frac{1}{\overline{h}^2} \cdot \left(1 - \frac{g(0)}{g(\overline{h})}\right).$$

$$Z_1(t) = \frac{1}{(\overline{h})^2}\left(1 - \frac{f_{1,1}\overline{h}}{k(\overline{h})}\right).$$

Theorem 6.10. *Let $D = (g; f_1, f_2) = (d_{n,k})$ be an infinite lower triangular matrix. Then D is a Double Riordan array if and only if there exists three sequences $A_1 = \{a_{1,i}\}_{i=0}^{\infty}$, $A_2 = \{a_{2,i}\}_{i=0}^{\infty}$ and $Z = \{z_i\}_{i=0}^{\infty}$ with $a_{1,0} \neq 0$, $a_{2,0} \neq 0$, and $z_0 \neq 0$ such that for each n*

$$d_{n,2k-1} = \sum_{j=0}^{\infty} a_{1,j} d_{n-1,2k+2(j-1)}; k = 0, 1, 2, ...,$$

$$d_{n,2k} = \sum_{j=0}^{\infty} a_{2,j} d_{n-1,2k+2(j-1)}; k = 0, 1, 2, ..., \textit{and}$$
$$d_{n,0} = \sum_{j=0}^{\infty} z_j d_{n-1,2j}.$$

Corollary 6.3. *Let $D = (g; f_1, f_2)$ be an element of DR. Let $A_1(t) = \sum_{k=0}^{\infty} a_{1,k} t^{2k}$, $A_2(t) = \sum_{k=0}^{\infty} a_{2,k} t^{2k}$, and $Z(t) = \sum_{k=0}^{\infty} z_k t^{2k}$ respectively be the generating functions for the A_1-sequence, A_2-sequence, and $Z-$sequence. Let $h = \sqrt{f_1 f_2}$. Then*

$$A_1(t) = \frac{t}{f_1 \overline{h}}$$
$$A_2(t) = \frac{f_1(\overline{h})}{\overline{h}}$$
$$Z(t) = \frac{1}{\overline{h}} \cdot \left(\frac{1}{f_1} - \frac{g(0)}{g(\overline{h}) f_1} \right).$$

In **Section 1**, we discussed the production matrix for Riordan Arrays. For Double Riordan Arrays two production matrices were found, see [18].

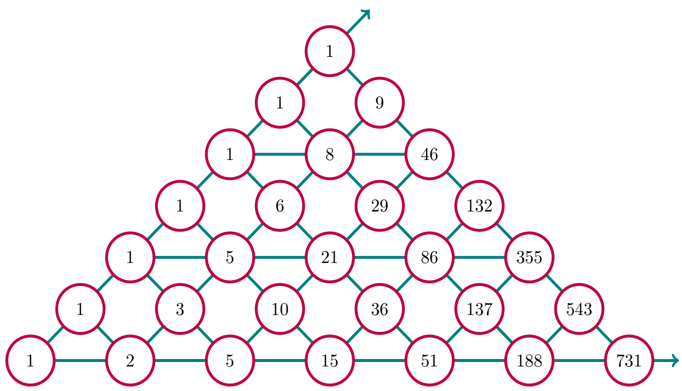
Definition 6.3. *Let $(g; f_1, f_2)$ be a double Riordan array. The* production matrix of the first kind *P_1 is given by:*
$$P_1 = (g; f_1, f_2)^{-1} \cdot \overline{(g; f_1, f_2)},$$
where $\overline{(g; f_1, f_2)}$ is the truncated double Riordan array with the first row omitted.

Definition 6.4. *Let $(g; f_1, f_2)$ be a double Riordan array. The* production matrix of the second kind *P_2 is given by:*
$$P_2 = (g; f_1, f_2)^{-1} \cdot \overline{\overline{(g; f_1, f_2)}},$$
where $\overline{\overline{(g; f_1, f_2)}}$ is the truncated double Riordan array with the first two rows omitted.

Theorem 6.11. *Let $D = (g; f_1, f_2)$ be a double Riordan array. Then the production matrix of the second kind P_2 for D is of the form $\begin{pmatrix} Z_0 & zZ_1 & A & zA & z^2A & \dots \end{pmatrix}$, where Z_0 is the first $Z-$sequence, Z_1 is the second $Z-$sequence, and A is the $A-$sequence for D from **Corollary 2**.*

Theorem 6.12. *Let $D = (g; f_1, f_2)$ be a double Riordan array. Then the production matrix of the first kind P_1 for D is of the form $\begin{pmatrix} Z & A_1 & zA_2 & z^2A_1 & z^3A_2 & z^4A_1 \dots \end{pmatrix}$, where Z is the $Z-$sequence, A_1 is the first $A-$sequence, and A_2 is the second $A-$sequence for D from **Corollary 3**.*

Example 6.3. *For a combinatorial example, consider Schröder paths with no level steps at odd heights. See the below grid.*



Arranging these numbers as a lower triangular array we get the following DR matrix.

$$D = \begin{bmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 2 & 0 & 1 &&&&&& \\ 0 & 3 & 0 & 1 &&&&& \\ 5 & 0 & 5 & 0 & 1 &&&& \vdots \\ 0 & 10 & 0 & 6 & 0 & 1 &&& \\ 15 & 0 & 21 & 0 & 8 & 0 & 1 && \\ 0 & 36 & 0 & 29 & 0 & 9 & 0 & 1 & \\ &&&& \cdots &&&& \end{bmatrix} \Longrightarrow D^{-1} = \begin{bmatrix} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ -2 & 0 & 1 &&&&&& \\ 0 & -3 & 0 & 1 &&&&& \\ 5 & 0 & -5 & 0 & 1 &&&& \vdots \\ 0 & 8 & 0 & -6 & 0 & 1 &&& \\ -13 & 0 & 19 & 0 & -8 & 0 & 1 && \\ 0 & -21 & 0 & 25 & 0 & -9 & 0 & 1 & \\ &&&& \cdots &&&& \end{bmatrix}$$

$$\overline{D} = \begin{bmatrix} 0 & 1 &&&&&&&& \\ 2 & 0 & 1 &&&&&&& \\ 0 & 3 & 0 & 1 &&&&&& \\ 5 & 0 & 5 & 0 & 1 &&&&& \\ 0 & 10 & 0 & 6 & 0 & 1 &&&& \vdots \\ 15 & 0 & 21 & 0 & 8 & 0 & 1 &&& \\ 0 & 36 & 0 & 29 & 0 & 9 & 0 & 1 && \\ 51 & 0 & 86 & 0 & 46 & 0 & 11 & 0 & 1 & \\ 0 & 137 & 0 & 132 & 0 & 57 & 0 & 12 & 0 & \\ &&&& \cdots &&&&& \end{bmatrix}$$

$$\overline{\overline{D}} = \begin{bmatrix} 2 & 0 & 1 &&&&&&& \\ 0 & 3 & 0 & 1 &&&&&& \\ 5 & 0 & 5 & 0 & 1 &&&&& \\ 0 & 10 & 0 & 6 & 0 & 1 &&&& \\ 15 & 0 & 21 & 0 & 8 & 0 & 1 &&& \vdots \\ 0 & 36 & 0 & 29 & 0 & 9 & 0 & 1 && \\ 51 & 0 & 86 & 0 & 46 & 0 & 11 & 0 & 1 & \\ 0 & 137 & 0 & 132 & 0 & 57 & 0 & 12 & 0 & \\ 188 & 0 & 355 & 0 & 235 & 0 & 80 & 0 & 14 & \\ &&&& \cdots &&&&& \end{bmatrix}$$

Now we compute P_1 and P_2.

$$P_1 = D^{-1} * \overline{D},$$

$$P_2 = D^{-1} * \overline{\overline{D}}.$$

As a result we get,

$$P_1 = \begin{bmatrix} 0 & 1 &&&&&&&& \\ 2 & 0 & 1 &&&&&&& \\ 0 & 1 & 0 & 1 &&&&&& \\ -1 & 0 & 2 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 1 & 0 & 1 &&&& \vdots \\ 1 & 0 & -1 & 0 & 2 & 0 & 1 &&& \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 && \\ -1 & 0 & 1 & 0 & -1 & 0 & 2 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \\ &&&& \cdots &&&&& \end{bmatrix}$$

$$Z - seq : (0, 2, 0, -1, 0, 1, 0, -1, ...)$$

$$A_1 - seq : (1, 0, 1) \qquad A_2 - seq : (1, 0, 2, 0, -1, 0, 1, ...)$$

$$P_2 = \begin{bmatrix} 2 & 0 & 1 & & & & & & & \\ 0 & 3 & 0 & 1 & & & & & & \\ 1 & 0 & 3 & 0 & 1 & & & & & \\ 0 & 1 & 0 & 3 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & 3 & 0 & 1 & & & \vdots \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & \\ & & & & \cdots & & & & & \end{bmatrix}$$

$$Z_1 - seq : (2, 0, 1) \qquad Z_2 - seq : (3, 0, 1)$$
$$A - seq : (1, 0, 3, 0, 1)$$

The question if we can define a Riordan array and Riordan group of matrices with more than two multiplier functions was affirmed in [17] without giving an explicit definition. In [24], He gave the following definition for matrix multiplication of matrices with k multiplier functions, such matrices we call k-Riordan arrays. In the k-Riordan group, we let $g(t) = \sum_{n=0}^{\infty} g_{kn} t^{kn}$ and for each $1 \le i \le k$, $f_i(t) = \sum_{n=0}^{\infty} f_{i,kn+1} t^{kn+1}$. For k-Riordan arrays $(g, f_1, f_2, ..., f_k)$ and $(G, F_1, F_2, ..., F_k)$, multiplication is defined as follows. Let $h(t) = \prod_{i=1}^{k} f_i(t)$. Then

$$(g, f_1, f_2, ..., f_k) \cdot (G, F_1, F_2, ..., F_k) =$$

$$\left(g(t) \cdot G\Big(\sqrt[k]{h(t)}\Big),\ \sqrt[k]{\frac{f_1^k(t)}{h(t)}} \cdot F_1\Big(\sqrt[k]{h(t)}\Big),\ \cdots,\ \sqrt[k]{\frac{f_k^k(t)}{h(t)}} \cdot F_k\Big(\sqrt[k]{h(t)}\Big) \right).$$

7. Involutions and Pseudo-involutions

In this section, after some definitions, we look at a few famous examples, then a few less famous examples. Then we give five methods of finding pseudo-involutions.

Definition 7.1. *Let G be a group with identity e. Then an element $g \in G$ is an involution $\iff$ $g^2 = e$.*

Definition 7.2. *Let $M = (1, -z)$. Then an element A in the Riordan group is a pseudo-involution (PI) $\iff$ AM or MA is an involution.*

Recall that the Bell subgroup of the Riordan group are the elements of the form (g, zg) or alternatively of the form $\left(\frac{f}{z}, f\right)$. More generally, there is the $k-$Bell subgroup with elements of the form $\left(g, zg^k\right)$ where k is a positive integer.

Among the examples of pseudo-involutions given are two in the Bell subgroup, one in the 3-Bell subgroup, three more in the 2-Bell subgroup, and one which is not in any k-Bell subgroup. Many of the g functions are well known in the arena of enumerative combinatorics. Displayed are the first five or so rows of each example along with some commentary.

The first example is Pascal's matrix

$$P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \end{bmatrix}$$

There is no need to discuss the importance of this matrix. This indeed is almost the prototype for Riordan group theory. It is in the Bell subgroup.

The RNA matrix

$$(g, zg) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \\ 2 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 6 & 4 & 1 & 0 & 0 & \\ 8 & 13 & 13 & 10 & 5 & 1 & 0 & \\ 17 & 28 & 30 & 24 & 15 & 6 & 1 & \end{bmatrix}$$

Here $g = 1 + zg + z^2 g\,(g-1)$. One combinatorial interpretation is as Motzkin paths with no peaks. The g sequence counts the number of possible RNA secondary structures for a chain of length n. Once again this matrix is in the Bell subgroup.

The Catalan companion

$$\left(C, zC^3\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \\ 5 & 14 & 7 & 1 & 0 & 0 & 0 & \cdots \\ 14 & 48 & 35 & 10 & 1 & 0 & 0 & \\ 42 & 165 & 154 & 65 & 13 & 1 & 0 & \\ 132 & 572 & 637 & 350 & 104 & 16 & 1 & \end{bmatrix}$$

This is an element in the 3-Bell subgroup.
Here are three pseudo-involutions in the 2-Bell subgroup:

$$\left(r, zr^2\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 & \\ 6 & 6 & 1 & 0 & 0 & \cdots \\ 22 & 30 & 10 & 1 & 0 & \\ 90 & 146 & 70 & 14 & 1 & \\ & & \cdots & & & \end{bmatrix}, \quad \left(B, zB^2\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 & \\ 6 & 6 & 1 & 0 & 0 & \cdots \\ 20 & 30 & 10 & 1 & 0 & \\ 70 & 140 & 70 & 14 & 1 & \\ & & \cdots & & & \end{bmatrix},$$

$$\left(t_3, zt_3^2\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 & \\ 6 & 6 & 1 & 0 & 0 & \cdots \\ 24 & 30 & 10 & 1 & 0 & \\ 110 & 152 & 70 & 14 & 1 & \\ & & \cdots & & & \end{bmatrix}$$

The three generating functions are the big Schröder numbers, r, the central binomial coefficients, B, and the ternary numbers doubled t_3. Since $r = 1 + z\left(r + r^2\right)$ it has as it's palindrome $\gamma = x + x^2$ with darga$(\gamma) = 1 + 2 = 3$. The darga and γ function will be discussed in Corollary E.

For B we have $B = 1 + 4z \cdot \frac{B^2}{1+B}$ so $\gamma = \frac{4x^2}{1+x}$ and the darga is $(2+2) - (0+1) = 3$. For the third example we start with $T_3 = 1 + zT_3^3$ substitute $T_3 = \frac{t_3+1}{2}$ and find that $t_3 = 1 + z\frac{t_3^3+3t_3^2+3t_3+1}{4}$ giving us $\gamma = \frac{x^3+3x^2+3x+1}{4}$ with darga$(\gamma) = 3 + 0 = 3$.

We comment that the row sums of the first example (r, zr^2) are the Central Delannoy numbers (see more in [1], [49]).

The Fibonacci numbers provide a non-Bell example of a pseudo-involution:

$$F = \left(\frac{1}{1-z-z^2}, \frac{1-\sqrt{\frac{1-5z-5z^2}{1-z-z^2}}}{2}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & \\ 3 & 14 & 7 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 50 & 35 & 10 & 1 & 0 & 0 & \\ 8 & 190 & 160 & 65 & 13 & 1 & 0 & \\ 13 & 778 & 720 & 360 & 104 & 16 & 1 & \end{bmatrix}$$

For any of these examples, a set of questions arise. Given a generating function g, presumably of combinatorial interest, how is f found? What is a combinatorial interpretation for f? What is the combinatorial meaning of the other entries in the matrix other than the leftmost column? How are the $A-$sequences, $Z-$sequences, and $B-$ sequences related? In this paper, we will answer some of these questions for a large number of examples involving k-Bell arrays and their relatives.

Here are a few ways to construct pseudo-involutions. The following construction is due to Candice Marshall, see [31] and [32]. Given a function $g = 1+g_1z+g_2z^2+g_3z^3+...$, we want to find a function f such that (g, f) is a pseudo-involution. We will say that f is a companion to g and we will abbreviate pseudo-involution to PI. Here is one method; Let $G = g - 1$. Then $f = -\overline{G}\left(\frac{1-g}{g}\right)$. As an example let g be the GF for the ternary numbers so that $g = 1 + zg^3 = 1 + z + 3z^2 + 12z^4 + \cdots = \sum_{n\geq 0} \frac{1}{3n+1}\binom{3n+1}{n}z^n$. Then $G = g - 1 = zg^3 = z(G + 1)^3$ and substituting $\overline{G}$ for z we find that $z = \overline{G}(1 + z)^3$ so that $\overline{G}(z) = \overline{G} = z/(1 + z)^3$. Since $\frac{1-g}{g} = -\frac{zg^3}{g} = -zg^2$ we have that

$$f = -\overline{G}\left(-zg^2\right) = -\frac{-zg^2}{(1 - zg^2)^3} = zg^2 \cdot g^3 = zg^5$$

Here is first part of the PI $\left(g, zg^5\right)$

$$\left(g, zg^5\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ 3 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 12 & 33 & 11 & 1 & 0 & 0 & \\ 55 & 182 & 88 & 16 & 1 & 0 & \\ 273 & 1020 & 627 & 168 & 21 & 1 & \end{bmatrix}$$

The same reasoning gives the more general result that if $g = 1 + zg^k$ then $\left(g, zg^{2k+1}\right)$ is a PI. When $k = 1$ we have the Pascal matrix and when $k = 2$ the Catalan PI $\left(C, zC^3\right)$ is the result.

Here is how this formula for f is derived. If f is the PI companion then $-f$ is an involution and thus

$$\begin{aligned} (g, -f)(g, -f) &= (1, z) = (g \cdot g(-f), -f(-f)) \\ \text{So } g \cdot (g(-f)) &= 1 \\ \text{and } g(-f) &= g \circ (-f) = \frac{1}{g} \\ G(-f) &= (g - 1) \circ (-f) = \frac{1}{g} - 1 = \frac{1-g}{g} \\ \text{Hence } -f &= \overline{G}\left(\frac{1-g}{g}\right) \end{aligned}$$

7.1 The Master Theorem and Five Corollaries

The master theorem uses the simple fact that a conjugate of an involution is an involution. Recall also that if $M = (1, -z)$, the identity matrix with alternating signs, then Z is a PI if and only if ZM is an involution, as is MZ. The corollaries have been proven previously but now we present a unifying approach.

Theorem 7.1. *Let Z be a PI. Then if $X = (g, f)$ is any element in the Riordan group then $XZMX^{-1}M$ is also a PI.*

Proof.

ZM is an involution.
$X(ZM)X^{-1}$ is also an involution.
Thus $XZMX^{-1}M$ is a PI.

□

It is convenient to denote $MX^{-1}M$ as $\widehat{X}$ and with this notation $XZ\widehat{X}$ is a PI. For generating functions it is also convenient to define $\widehat{f} = -\overline{f}(-z) = (-z) \circ f \circ (-z)$.

Corollary 7.1. *If X is any Riordan group element, then $X\widehat{X}$ and $\widehat{X}X$ are PIs.*

Proof. Let Z = I. □

Example 7.1. *A classic example has the Stirling numbers of the first kind as*

$$X = \left[\frac{1}{1-z}, \ln\left(\frac{1}{1-z}\right)\right]$$

whose inverse yields the Stirling numbers of the second kind. Then $MX^{-1}M = \widehat{X}$ *are the unsigned Stirling numbers of the second kind. The product* $X\widehat{X}$ *has various names: the Lah numbers, preferential arrangements, ordered partitions, or more informally, a horse race with possible ties. Here, in matrix form are the first few rows*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 6 & 11 & 6 & 1 & 0 \\ 24 & 50 & 35 & 10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 \\ 24 & 36 & 12 & 1 & 0 \\ 120 & 240 & 120 & 20 & 1 \end{bmatrix}$$

Let us examine the row 6,6,1. There are 3 horses in a race. Without ties, there are $3! = 6$ *possibilities. There are 3 ways to have two of the horses tie for first place and another 3 ways to have a tie for second place. There is just one way for all three horses in a tie for first.*

The PI $\widehat{X}X$ *is also interesting. Here*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 6 & 11 & 6 & 1 & 0 \\ 24 & 50 & 35 & 10 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 \\ 26 & 36 & 12 & 1 & 0 \\ 150 & 250 & 120 & 20 & 1 \end{bmatrix}$$

Here the left-hand column now counts horse races where not only can you have ties but one or more horses can drop out. For instance with 3 horses there are 13 ways with no dropouts, $3 \cdot 3 = 9$ *with one dropout, 3 with 2 horses dropping out, and 1 where, tragically, all three horses had to drop out. The total is* $13 + 9 + 3 + 1 = 26$, *the* $(3, 0)$ *entry.*

Here we have

$$X\widehat{X} = \left[\frac{1}{1-z}, \ln\left(\frac{1}{1-z}\right)\right][e^z, e^z - 1] = \left[\frac{1}{(1-z)^2}, \frac{z}{1-z}\right]$$

for the Lah case while

$$[e^z, e^z - 1]\left[\frac{1}{1-z}, \ln\left(\frac{1}{1-z}\right)\right] = \left[\frac{e^z}{2-e^e}, \ln\left(\frac{1}{2-e^z}\right)\right]$$

Note that all these matrices are in the derivative subgroup.

Corollary 7.2. *Let* $h(z) = \sum_{k=1}^{\infty} h_k z^k$ *with* $h_1 \neq 0$. *Then* $\left(\frac{1}{1-h}, \overline{h}\left(\frac{h}{1-h}\right)\right)$ *is a PI.*

Proof. Let $X = (1, h)$ and take $Z = P = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)$. □

This version is quite convenient and is discussed with many examples in [25] and [12].

There is the folklore heuristic principle that

$$\frac{1}{1 - \textit{Connected}\text{ part GF}} = \text{"}\textit{Total case GF}\text{"}$$

As a prime example, we want to find the PI involving the Fibonacci numbers. The generating function is

$$g = F = \frac{1}{1 - z - z^2}$$

so we take

$$h(z) = z + z^2$$

There is a vague duality between the Fibonacci and Catalan numbers. One way to express it is to note that

$$\widehat{zC} = z + z^2$$

so that

$$F = \frac{1}{1-z} \circ \widehat{zC}.$$

Using $X = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ we have

$$Z X \widehat{Z} \quad = \quad (1, z + z^2)\left(\frac{1}{1-z}, \frac{z}{1-z}\right)(1, zC)$$

$$
\begin{aligned}
&= \left(\frac{1}{1-z-z^2}, \frac{z+z^2}{1-z-z^2}\right)(1, zC) \\
&= \left(\frac{1}{1-z-z^2}, \frac{z+z^2}{1-z-z^2} \cdot C\left(\frac{z+z^2}{1-z-z^2}\right)\right) \\
&= (F, (F-1)\, C\,(F-1))
\end{aligned}
$$

The companion function

$$
\begin{aligned}
(zC) \circ (F-1) &= \frac{1}{2}\left(1-\sqrt{\frac{1-5z-5z^2}{1-z-z^2}}\right) \\
&= z+3z^2+9z^3+32z^4+126z^5+538z^6+2429z^7+11\,412z^8+O\left(z^9\right)
\end{aligned}
$$

Looking at $(zC) \circ (F-1)$ we see that that we obtain only positive integer coefficients.

The first few rows of this pseudo-involution are

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 4 & 1 & 0 & 0 & 0 \\
3 & 14 & 7 & 1 & 0 & 0 \\
5 & 50 & 35 & 10 & 1 & 0 \\
8 & 190 & 160 & 65 & 13 & 1
\end{bmatrix}.
$$

If we let $h = 2z + z^2$ then we obtain the Pell numbers $1, 2, 5, 12, 29, 70, \ldots$ with

$$
g = \frac{1}{1-2z-z^2}.
$$

The PI companion is

$$
\begin{aligned}
F &= \frac{2-\sqrt{4-4H}}{2} = 1-\sqrt{1-H} \\
&= 1-\sqrt{1-\frac{h}{1-h}} = 1-\sqrt{\frac{1-2h}{1-h}} \\
&= 1-\sqrt{\frac{1-4z-2z^2}{1-2z-z^2}}.
\end{aligned}
$$

The surprise comes starting with the coefficient of z^8.

$$
\begin{aligned}
F &= 1-\sqrt{\frac{1-4z-2z^2}{1-2z-z^2}} \\
&= z+3z^2+9z^3+28z^4+90z^5+299z^6+1025z^7+\frac{7233}{2}z^8+\frac{26\,183}{2}z^9+\ldots
\end{aligned}
$$

An open question is when does the companion have integer coefficients?

Corollary 7.3. *Let A and B both be PI. Then ABA is also a PI.*

Note that usually, AB is not a PI.

If X is a PI, then $X^{-1} = MXM$ or equivalently $X = MX^{-1}M = \widehat{X}$.

Thus if A and B are both PIs then so is $AB\widehat{A} = ABA$. Then it is easy to establish that any palindromic expression in PI is also a PI. This includes $AIA = AA, ABCBA, ABCDDCBA, \ldots$

There are many examples in [25] and here is one more.

Example 7.2. *Let $A = \left(\frac{1+z}{1-z}, z\right)$ and $B = \left(1, \frac{z}{1-z}\right)$. The product $ABA = \left(\frac{1+z}{1-3z+2z^2}, \frac{z}{1-z}\right)$. Here are the first few rows of this PI.*

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 \\
10 & 5 & 1 & 0 & 0 \\
22 & 15 & 6 & 1 & 0 \\
46 & 37 & 21 & 7 & 1
\end{bmatrix}
$$

The leftmost column counts rows of n squares where the leftmost squares can be red or green, followed possibly by some purple squares finishing with zero or one brown square. The 10 possibilities when $n = 2$ are RR,RG,GR,GG,RP,GP,PP,RB,GB, and PB.

For successive columns add one new color per column.

The sequence $1, 4, 10, 22, 46, 94, \ldots$ has the OEIS reference A033484 where more information can be found.

Here are two examples using the main theorem itself.

Example 7.3. *Let* $X = \left(\frac{1}{1-Az}, z\right)$ *so that* $\widehat{X} = (1 + Az, z)$. *For our seed PI we use* $Z = \left(1, \frac{z}{1-z}\right)$ *and then*

$$\begin{aligned} XZ\widehat{X} &= \left(\frac{1}{1-Az}, z\right)\left(1, \frac{z}{1-z}\right)(1 + Az, z) \\ &= \left(\frac{1}{1-Az}, \frac{z}{1-z}\right)(1 + Az, z) \\ &= \left(\frac{1}{1-Az} \cdot \frac{1 + (A-1)\,z}{1-z}, \frac{z}{1-z}\right). \end{aligned}$$

For instance if $A = 3$ *we get the following PI.*

$$\left(\frac{1+2z}{1-3z} \cdot \frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 \\ 21 & 7 & 1 & 0 & 0 \\ 66 & 28 & 8 & 1 & 0 \\ 201 & 94 & 36 & 9 & 1 \end{bmatrix}$$

A visit to the OEIS leads to the fact that the sequence in the leftmost column counts the number of maximal cliques in the series of Hanoi graphs. This is easily verified and it is a rather pretty sequence of graphs.

Example 7.4. *We again use* $Z = \left(1, \frac{z}{1-z}\right)$ *and this time let* $X = (C\,(-z)\,, z)$. *The resulting PI is*

$$\left(\frac{C\,(-z)}{C\left(\frac{z}{1-z}\right)}, \frac{z}{1-z}\right).$$

Reasonable enough? Here are the first eight rows of the PI.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & -10 & 0 & 1 & 1 & 0 & 0 & 0 \\ -92 & -5 & -10 & 1 & 2 & 1 & 0 & 0 \\ 4 & -97 & -15 & -9 & 3 & 3 & 1 & 0 \\ -1117 & -93 & -112 & -24 & -6 & 6 & 4 & 1 \end{bmatrix}$$

Reasonable assumptions do not save us from a combinatorial nightmare.

Corollary 7.4. *We recall that if* (g, f) *is a PI,* $g = 1 + g_1 z + g_2 z^2 + \cdots, g_1 \neq 0,$ *and* $G = g - 1$. *Then the companion is* $f = -\overline{G}\left(\frac{1-g}{g}\right)$.

We can rephrase this as

$$\begin{aligned} (g, f) &= (g, z)\,(1, f) \\ &= (g, z)\,(1, G)\left(1\frac{z}{1+z}\right)(1, -z)\left(1, \overline{G}\right)(1, -z) \\ &= (g, z)\,(1, G)\left(1, \frac{z}{1+z}\right)\widehat{(1, G)}. \end{aligned}$$

Thus

$$\begin{aligned} f &= (-z) \circ \overline{G} \circ (-z) \circ \frac{z}{1+z} \circ G \\ &= \widehat{G} \circ \frac{z}{1+z} \circ G. \end{aligned}$$

Corollary 7.5. *Suppose* $g = 1 + z\gamma\,(g)\,z + 1$. *Then the* f *companion is*

$$f = z\frac{\gamma\,(g)}{g\gamma\,(1/g)}.$$

Proof. Let $G = g - 1 = z\gamma(g) = z\gamma(G+1)$ so that $z = \overline{G}\gamma(z+1)$. Then $\widehat{G} = \frac{z}{\gamma(z-1)}$ and

$$\begin{aligned} f &= \widehat{G} \circ \frac{z}{1+z} \circ G \\ &= \frac{z}{\gamma(z-1)} \circ \frac{z}{1+z} \circ z\gamma(g) \\ &= \frac{z}{\gamma(z-1)} \circ \frac{G}{1+G} \\ &= \frac{G}{1+G} \frac{1}{\gamma\left(\frac{G}{1+G} - 1\right)} \\ &= \frac{g-1}{g} \frac{1}{\gamma(1/g)} = z\frac{\gamma(g)}{g\gamma(1/g)}. \end{aligned}$$

□

Corollary 7.6. *Suppose in addition that $\gamma(x)$ is a palindrome of darga d then $f = zg^{d-1}$ i.e. a member of the $(d-1)$-Bell subgroup. The darga is the sum of the degrees of the initial and final non-zero terms of the palindrome.*

First, we look at an example, then the proof.

Example 7.5. *Let $r = 1 + z(r + r^2)$. Then $\gamma(x) = x^1 + x^2$ is a palindrome of darga $1 + 2 = 3$. Thus $(r, zr^{3-1}) = (r, zr^2)$ is a pseudo-involution. This is a well-known example as r is the generating function for the large Schröder numbers.*

Proof. Since we have a palindrome of darga d it follows that $x^d\gamma(1/x) = \gamma(x)$. Hence

$$z\frac{\gamma(g)}{g\gamma(1/g)} = z\frac{g^{d-1}\gamma(g)}{g^d\gamma(1/g)} = z\frac{g^{d-1}\gamma(g)}{\gamma(g)} = zg^{d-1}.$$

□

See [12] for more examples.

8. Conclusion

The first applications of Riordan arrays were in providing quick proofs for somewhat involved combinatorial identities and the ability to invert combinatorial identities using the group structure. Since then, many new connections to other parts of mathematics have been found. These include

- Riordan Lie theory (Cheon, Luzón, Morón, Prieto-Martinez, Song [16])
- connections with the Banach fixed point theorem (Luzón, Morón [28])
- A-matrices (Barry, Merlini, Rogers, Sprugnoli, Verri [4, 33])
- summation methods (He [23])
- orthogonal polynomial theory (Barry, Mwafise, Viennot [3, 5, 7, 50])
- the RNA secondary structure (Evans, Nkwanta [20])
- directed animals (Barcucci, Del Lungo, Pergola, Pinzani [2])
- Riordan graphs and posets (Cheon, Curtis, Kwon, Mwafise [14])
- the Riemann hypothesis (Cheon, Kim [15])
- pattern avoidance (Burstein, Lankham, Merlini, Sprugnoli [11, 34])
- Somos sequences and elliptic functions (Barry, Mwafise [4, 8, 35])
- total positivity (Chen, Liang, Mao, Mu, Slowik, Wang [13, 30, 44]),

to name a few.

We hope this invitation will interest and equip the reader to go further. Despite, or because of, the simplicity of the concepts, much has developed since the first paper in 1991 [42]. Their ability to represent and manipulate sequences, generate combinatorial identities, and model intricate phenomena demonstrates their significance in advancing research and applications across diverse disciplines. As mathematicians and researchers delve deeper into Riordan arrays' rich and unexplored territory, we can anticipate even more profound insights and practical implications. For those who want to learn more about Riordan arrays, there is an annual international conference on Riordan arrays, an undergraduate textbook written by Paul Barry, [6], a Springer monograph by Shapiro, Sprugnoli, Barry, et al [41], and two Youtube presentations by Melkamu Zeleke [53, 54].

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