

Catalan–Spitzer Permutations

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ABSTRACT: We study two classes of permutations intimately related to the visual proof of Spitzer’s lemma and Huq’s generalization of the Chung–Feller theorem. Both classes of permutations are counted by the Fuss–Catalan numbers. The study of one class leads to a generalization of results of Flajolet from continued fractions to continuants. The study of the other class leads to the discovery of a restricted variant of the Foata–Strehl group action.

Keywords: Continued fraction; Foata–Strehl group action; Fuss–Catalan number; Motzkin path; Raney number

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Introduction

A classical result of lattice path enumeration arising from tossing n fair coins is the Chung–Feller theorem [1]. It states that the Catalan number C_n counts not only the lattice paths consisting of unit northeast and southeast steps from $(0, 0)$ to $(2n, 0)$ that stay above the horizontal axis, but we can also prescribe the number r of northeast steps above the horizontal axis. For each $r \in \{0, 1, \dots, n\}$ we have the same Catalan number of lattice paths. Generalizations of this result are due to Spitzer [10, Theorem 2.1] as well as Huq in his dissertation [6, Theorem 2.1.1]. All these results may be shown using the following simple visual idea: if we slightly “tilt” the diagram of a lattice path (see Figure 2), all steps occur at different heights, and the relative order of these heights may be rotated cyclically by changing the designation of the first step in the lattice path. This simple idea was perhaps first used by Raney [9, Theorem 2.1], who observed that there is exactly one rotational equivalent of a sequence of $n + 1$ positive units and n negative units in which the partial sums are all positive. A question naturally arises: which permutations are relative orders of steps in such tilted pictures of lattice paths?

In this paper we partially answer this general question in two specific settings. Both are related to k -Catalan paths, defined as lattice paths consisting of unit up steps $(1, 1)$ and down steps $(1, -k + 1)$ which start and end on the horizontal axis but never go below it. The study of the relative order of all steps leads us to a generalization of some results of Flajolet from continued fractions to continuants. The study of the relative orders of the up steps leads us to the discovery of a restricted variant of the Foata–Strehl group action [3, 4].

Our paper is structured as follows. In the Preliminaries we review the Chung–Feller theorem [1], its generalizations by Spitzer [10, Theorem 2.1] and Huq [6, Theorem 2.1.1], and we point out a few connections between the two generalizations. In Section 2 we outline a visual proof of Huq’s results which inspires the definition of the permutations we intend to study. We introduce *Catalan–Spitzer permutations* (and their k -generalizations) in Section 3 as the relative orders of all steps in a Catalan path. Equivalently, these are obtained by labeling the steps in reverse lexicographic order and listing them in the order they occur along the path. Due to this labeling, a refined count of Catalan–Spitzer permutations amounts to enumerating all Catalan paths that have a given number of steps at a certain level. For the Catalan paths our formulas may be obtained using Flajolet’s result [2, Theorem 1] which provides a generalized continued fraction formula. We generalize these formulas

to k -Catalan paths by using continuants instead of continued fractions. In Section 4 we observe that the relative order of the up steps alone uniquely determines the Catalan paths. The resulting *short Catalan–Spitzer permutations* may be characterized in terms of the associated *Foata–Strehl trees*, first studied by Foata and Strehl [3, 4] who introduced a group action on the set of all permutations using these ordered $0-1-2$ trees. Finally, in Section 5 we study a restricted variant of the Foata–Strehl group action which takes each short Catalan–Spitzer permutation into another short Catalan–Spitzer permutation. The number of orbits on the set of C_n permutations is the Catalan number C_{n-1} . This is a consequence of a generating function formula that is applicable to any class of permutations that is closed under the restricted Foata–Strehl group action. In particular, for the set of all permutations the number of orbits is the same as the number of indecomposable permutations.

Our results inspire revisiting three classical topics: generalizations of the Chung–Feller theorem, Flajolet’s continued fraction approach to lattice path enumeration, and the Foata–Strehl group actions. They are likely the first to connect these three areas.

1. Preliminaries

This paper focuses on permutations that are associated with the *Chung and Feller theorem* [1] and some of its generalizations.

Theorem 1.1 (Chung–Feller). *Among the lattice paths from $(0, 0)$ to $(2n, 0)$ consisting of n up steps $(1, 1)$ and n down steps $(1, -1)$, the number of paths having $2r$ steps above the x axis is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$, independently of r , for each $r \in \{0, 1, \dots, n\}$.*

In the special case when $r = n$ the Chung–Feller theorem implies that the number of *Dyck paths*, that is, lattice paths of the above type that never go below the x axis from $(0, 0)$ to $(2n, 0)$ is the Catalan number C_n . This well-known special case has been generalized to k -Dyck paths (whose definition may be found in Lemma 1.1 below) by Raney; see [5, p. 361].

Lemma 1.1 (Raney). *The number of lattice paths from $(0, 0)$ to $(kn, 0)$ consisting of $(k-1)n$ up steps $(1, 1)$ and n down steps $(1, 1-k)$ that never go below the x -axis is the Fuss–Catalan number*

$$C_{n,k} = \frac{1}{kn+1} \binom{kn+1}{n} = \frac{1}{(k-1)n+1} \binom{kn}{n}. \quad (1)$$

Huq has generalized Theorem 1.1 to the lattice paths appearing in Lemma 1.1 by proving the following result [6, Theorem 2.1.1].

Theorem 1.2 (Huq). *Let (y_1, \dots, y_m) be any sequence of integers whose sum is 1. Then for each $r \in \{0, 1, \dots, m-1\}$ exactly one of the cyclic shifts*

$$(y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(m)}) \in \{(y_1, y_2, \dots, y_m), (y_2, \dots, y_m, y_1), \dots, (y_m, y_1, \dots, y_{m-1})\}$$

has the property that exactly r of the partial sums $y_{\sigma(1)} + y_{\sigma(2)} + \dots + y_{\sigma(k)}$ for $1 \leq k \leq m$ are positive.

Huq’s proof of Theorem 1.2 is a consequence of the following theorem of Spitzer [10, Theorem 2.1].

Theorem 1.3 (Spitzer). *Let $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ be any vector with real coordinates such that*

$$x_1 + x_2 + \dots + x_m = 0$$

but no shorter cyclic partial sum $x_{i+1} + x_{i+2} + \dots + x_j$ of the coordinates vanishes. Then for each $r \in \{0, 1, \dots, m-1\}$ exactly one of the cyclic shifts

$$(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)}) \in \{(x_1, x_2, \dots, x_m), (x_2, \dots, x_m, x_1), \dots, (x_m, x_1, \dots, x_{m-1})\}$$

has the property that exactly r of the partial sums $x_{\sigma(1)} + x_{\sigma(2)} + \dots + x_{\sigma(k)}$ for $1 \leq k \leq m$ are positive.

Indeed, introducing $x_i = y_i - 1/m$ for $i = 1, 2, \dots, m$, the resulting vector $(x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ satisfies the conditions of Theorem 1.3 as the sum $x_1 + x_2 + \dots + x_m$ is zero but no shorter partial sum $x_{i+1} + x_{i+2} + \dots + x_j$ of the coordinates, read cyclically, is an integer. Furthermore, any shorter sum $x_{i+1} + \dots + x_j$ is positive if and only if $y_{i+1} + \dots + y_j - 1 \geq 0$ holds, since $y_{i+1} + \dots + y_j$ is an integer and we have $y_{i+1} + \dots + y_j - 1 < x_{i+1} + \dots + x_j < y_{i+1} + \dots + y_j$.

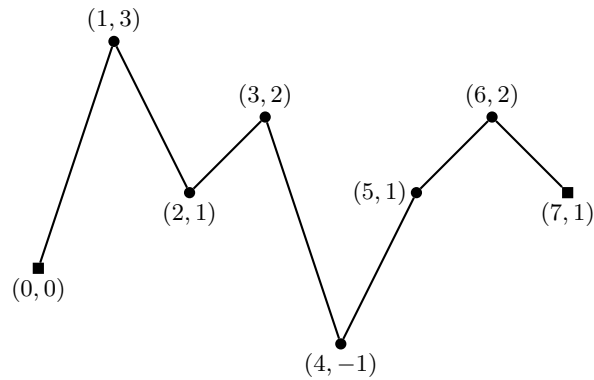


Figure 1: The lattice path associated with the vector $\mathbf{y} = (3, -2, 1, -3, 2, 1, -1)$.

Proof of Theorem 1.3. Introducing

$$z_i = x_1 + x_2 + \cdots + x_i, \tag{2}$$

all cyclically consecutive sums may be expressed as $x_{i+1} + x_{i+2} + \cdots + x_j = z_j - z_i$. This is clear when $i \leq j$, and it is easy to prove when $i > j$ using $z_m = 0$. Putting the numbers z_1, z_2, \dots, z_m in increasing order, for each $0 \leq r \leq m - 1$ there is exactly one z_i , the $(r + 1)$ st largest number, for which exactly r of the differences $z_j - z_i$ are positive. \square

As observed by Huq [6, Corollary 5.1.2], Theorem 1.2 has the following consequence.

Corollary 1.1 (Huq). *The number of lattice paths from $(0, 0)$ to $(kn, 0)$ consisting of $(k - 1)n$ up steps $(1, 1)$ and n down steps $(1, 1 - k)$ with exactly r up steps below the x -axis is independent of r for $r \in \{0, 1, \dots, (k - 1)n\}$ and is given by the Fuss–Catalan number $C_{n,k}$.*

Remark 1.1. The special instance of Spitzer’s theorem when $j = m - 1$ is often called Spitzer’s lemma; see [7, Lemma 10.4.3]. The special instance of Corollary 1.1 when $j = m - 1$ is also a special case of Raney’s theorem [9, Theorem 2.1]; see [5, p. 359].

As noted, Theorem 1.2 above is a consequence of Spitzer’s theorem 1.3, but the converse is also true.

Proposition 1.1. *Spitzer’s theorem 1.3 is a consequence of Huq’s theorem 1.2.*

Proof. Assume that the sum of all coordinates $(x_1, \dots, x_m) \in \mathbb{R}^m$ is zero, and each of the shorter cyclically consecutive sums of terms $x_{i+1} + \cdots + x_j$ is nonzero and has sign $\varepsilon_{i,j}$. We may assume that there is a fixed positive integer $k > 0$ such that all numbers x_i are rational of the special form

$$x_i = \frac{m \cdot y_i - 1}{m \cdot k} \text{ for some } y_i \in \mathbb{Z}. \tag{3}$$

Indeed, we may perturb the coordinates of (x_1, \dots, x_m) as long as they satisfy all $m(m - 1) - 2$ inequalities of the form

$$\varepsilon_{i,j} \cdot (x_{i+1} + \cdots + x_j) > 0, \tag{4}$$

together with the equation

$$x_1 + \cdots + x_m = 0 \tag{5}$$

in \mathbb{R}^m . The inequalities (3) define an open subset of the hyperplane defined by (5). This subset is not empty as it contains the vector we began with. Points whose coordinates are of the form given in (3) form a dense subset in the hyperplane defined by (5), hence we may replace (x_1, \dots, x_m) with a vector whose coordinates are of the form given in (3) and that satisfy the same inequalities. Similar to the other implication, Theorem 1.3 now follows from Theorem 1.2 after observing that each shorter sum $x_{i+1} + \cdots + x_j$ satisfies the inequality $y_{i+1} + \cdots + y_j - 1 \leq k \cdot (x_{i+1} + \cdots + x_j) < y_{i+1} + \cdots + y_j$. \square

2. A lattice path visualization of Huq’s result

In the spirit of Krattenthaler [7, Remark 10.4.4] and also of Graham, Knuth and Patashnik [5, p. 360], we may visualize a self contained proof of Theorem 1.2, using lattice paths, as follows. This visualization makes the

result and its proof a generalization of Raney's lemma 1.1 and its geometric proof given in [5, p. 359–360]. If we generalize the notion of lattice paths to connect vertices with non-integer second coordinates, our visualization also includes the proof of Theorem 1.3.

Let us extend the vector \mathbf{y} to an infinite vector $(\dots, y_{-1}, y_0, y_1, y_2, \dots)$ by setting $y_i = y_j$ for $i \equiv j \pmod{m}$, and consider the associated infinite lattice path with steps $\dots, (1, y_{-1}), (1, y_0), (1, y_1), (1, y_2), \dots$, containing the lattice point $(0, v_0) = (0, 0)$ and satisfying $(i + 1, v_{i+1}) = (i, v_i) + (1, y_{i+1})$ for all integers i ; see Figure 2. The

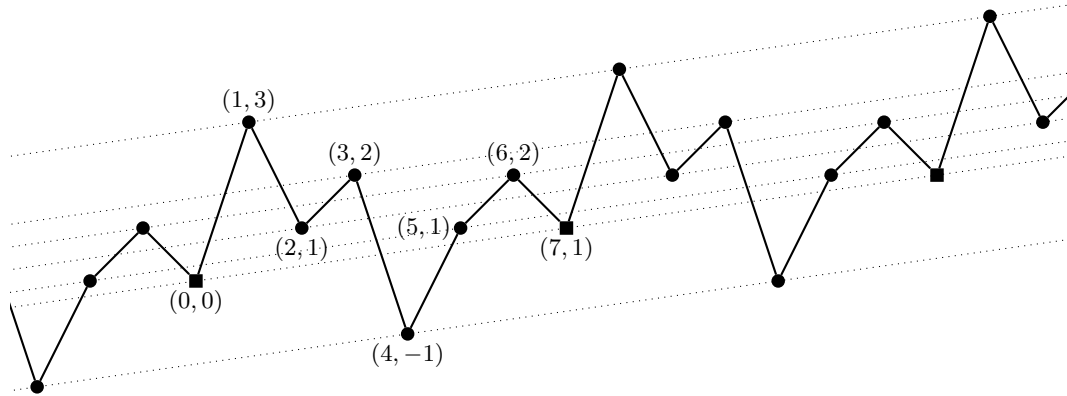


Figure 2: The infinite lattice path associated to the lattice path in Figure 1. The dotted lines are the lines where the functional $F(u, v) = v - 1/m \cdot u$ is constant. The functional F yields the linear order $(4, -1)$, $(0, 0)$, $(5, 1)$, $(2, 1)$, $(6, 2)$, $(3, 2)$ and $(1, 3)$ on the points on the path.

finite path p_i occurs in the infinite path as a subpath starting at (i, v_i) and ending at $(i + m, v_i + 1)$. Introducing $x_i = y_i - 1/m$ and ordering the z_i s defined in (2) amounts to the following. Consider the linear functional $F(u, v) = v - 1/m \cdot u$ defined on the plane, and consider its level curves which are lines with slope $1/m$. For any $i \in \{1, 2, \dots, m\}$ we have $z_i = F(i, v_i)$ and we may extend this observation to all $i \in \mathbb{Z}$ keeping in mind that $z_m = 0$. Thus we may set $z_i = z_j$ if $i \equiv j \pmod{m}$. Ordering $z_1, z_2, \dots, z_m = z_0$ in increasing order amounts to ordering the m lattice points $(0, v_0)$ through $(m - 1, v_{m-1})$ according to the linear functional F . If (i, v_i) is the $(r + 1)$ st largest lattice point in this order, then there are exactly r lattice points (i, v_i) above this level.

3. Catalan–Spitzer permutations

In this section we investigate the restriction of Theorem 1.2 and its proof to k -Catalan paths. In particular, we describe the permutations of partial sums that appear in the proof of Theorem 1.2, when we prove it by reducing it to Spitzer's theorem 1.3.

We define an *augmented* k -Catalan path of order n as a lattice path consisting of $(k - 1)n + 1$ up steps $(1, 1)$ and n down steps $(1, -k + 1)$ that begins with an up step and never goes below the line $y = 1$ after the initial up step. The sum of the second coordinates of these steps is 1, hence Theorem 1.2 is applicable. In this special case the proof of this theorem calls for replacing each step $(1, y)$ by $(1, y - 1/(kn + 1))$. Hence up steps become $(1, kn/(kn + 1))$ and down steps become $(1, -k((k - 1)n + 1)/(kn + 1))$ and the transformed path goes from $(0, 0)$ to $(kn + 1, 0)$. Between its endpoints it remains strictly above the line $y = 0$. The transformed path is not a lattice path, but we can easily transform it into one by multiplying all y -coordinates by the factor $(kn + 1)/k$. This vertical stretch does not change the relative vertical order of the y -coordinates of the endpoints of the steps.

Definition 3.1. A k -Catalan–Spitzer path of order n is a lattice path consisting of $(k - 1)n + 1$ up steps $(1, n)$ and n down steps $(1, -((k - 1)n + 1))$ from $(0, 0)$ to $(kn + 1, 0)$ that remains strictly above the line $y = 0$.

There is a natural bijection between augmented k -Catalan paths and k -Catalan–Spitzer paths of order n : we associate to each augmented k -Catalan path $(0, 0), (1, z_1), \dots, (kn, z_{kn}), (kn + 1, 1)$ the k -Catalan–Spitzer path $(0, 0), (1, z'_1), \dots, (kn, z'_{kn}), (kn + 1, 0)$ in which up steps and down steps follow in the same order. Hence the number of k -Catalan–Spitzer paths of order n is also the Fuss-Catalan number $C_{n,k}$. The second coordinates z'_i of the lattice points in a k -Catalan–Spitzer path pairwise differ and may be easily computed from the second coordinates z_i of the corresponding augmented k -Catalan path as follows. Since a k -Catalan–Spitzer path is obtained from the corresponding augmented k -Catalan path by first decreasing the second coordinate of each

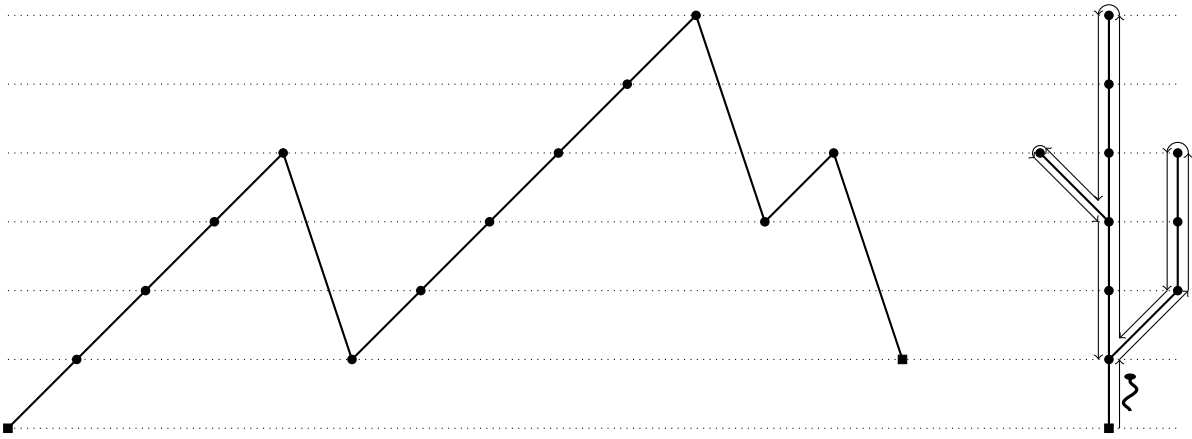


Figure 3: The augmented 4-Catalan path whose associated 4-Catalan–Spitzer permutation is 3, 5, 8, 11, 2, 4, 7, 10, 12, 13, 6, 9, 1.

step by $1/(kn + 1)$ and then performing a vertical stretch by a factor of $(kn + 1)/k$, we obtain that

$$z'_i = \frac{(kn + 1) \cdot z_i - i}{k} \text{ holds for } i = 1, 2, \dots, kn. \quad (6)$$

The next proposition describes the relative position of these lattice points in a k -Catalan–Spitzer path in terms of the positions of lattice points in the corresponding augmented k -Catalan path.

Proposition 3.1. *Consider an augmented k -Catalan path $(0, 0), (1, z_1), \dots, (kn, z_{kn}), (kn + 1, 1)$ of order n , and let $(0, 0), (1, z'_1), \dots, (kn, z'_{kn}), (kn + 1, 0)$ be the corresponding k -Catalan–Spitzer path. Then for some $i \neq j$ the inequality $z'_i < z'_j$ holds if and only if $(-i, z_i) < (-j, z_j)$ in the reverse lexicographic order, where coordinates are compared right to left.*

Proof. By (6) we have

$$z'_j - z'_i = \frac{(kn + 1) \cdot (z_j - z_i) + (i - j)}{k}. \quad (7)$$

Observe first that in the case when $z_i \neq z_j$, the sign of $z'_j - z'_i$ is the same as the sign of $z_j - z_i$. Indeed, $(kn + 1) \cdot (z_j - z_i)$ in (7) above is a nonzero integer multiple of $(kn + 1)$, whereas $|i - j| < kn$. On the other hand, in the case when $z_i = z_j$, by (7) we have $z'_i < z'_j$ if and only if $-i < -j$ holds. \square

Proposition 3.1 inspires the following definition.

Definition 3.2. *A k -Catalan–Spitzer permutation of order n is the relative order of the numbers z'_1, \dots, z'_{kn} in a k -Catalan–Spitzer path $(0, 0), (1, z'_1), \dots, (kn, z'_{kn}), (kn + 1, 0)$ of order n . Equivalently it is the relative order of the lattice points $(1, z_1), \dots, (kn, z_{kn})$ in the corresponding augmented k -Catalan path of order n , where we order the lattice points first by the second coordinate in increasing order and then by the first coordinate in decreasing order. In the case when $k = 2$ we will use the term Catalan–Spitzer permutation.*

Example 3.1. An example of an augmented 4-Catalan path and its labeling giving rise to the associated 4-Catalan–Spitzer permutation is shown in Figure 3. There are 3 lattice points at level one, numbered right to left, followed by 2 lattice points at the next level, and so on. Modifying an idea presented in [11], we may visualize an augmented k -Catalan path as a description of the movement of a worm crawling around a rooted plane tree in counterclockwise order, shown on the right of Figure 3. The plane tree is rooted with a root edge at level 0. Each up step in the lattice path corresponds to the worm moving up one level and each down step corresponds to the worm moving down $(k - 1)$ levels. We can think of the worm moving down $(k - 1)$ times faster than up. The set of all rooted plane trees with $n + 1$ vertices is in bijection with the set of all augmented Catalan paths with $(n + 1)$ up steps and n down steps. Only a subset of the set of rooted plane trees with $(k - 1) \cdot n + 2$ vertices corresponds bijectively to the set of $(k - 1)$ -Catalan paths with $(k - 1) \cdot n + 1$ up steps and n down steps. The numbering of the lattice points corresponds to the labeling of the points where the worm begins or ends a move.

In order to describe the finer structure of k -Catalan–Spitzer permutations, we make the following definition.

Definition 3.3. Let (i_1, \dots, i_r) be a vector with nonnegative integer coordinates. We say that a k -Catalan-Spitzer permutation and the corresponding augmented k -Catalan path has type (i_1, i_2, \dots, i_r) if the augmented k -Catalan path has i_j lattice points at level j . We denote the number of k -Catalan-Spitzer permutations having type $\bar{i} = (i_1, i_2, \dots, i_r)$ by $t_k(\bar{i})$.

Note that this definition of type is not unique in the sense that an augmented k -Catalan path has type (i_1, i_2, \dots, i_r) if and only if it has type $(i_1, i_2, \dots, i_r, 0)$. In other words, we may add as many zero coordinates to the type of an augmented k -Catalan path as we wish. We exclude the empty lattice path from consideration as we consider it non-augmented. Hence i_1 must be positive.

Let e_i denote the i th unit vector. For S a finite subset of positive integers, let e_S denote the sum $e_S = \sum_{i \in S} e_i$ and x_S denote the product $x_S = \prod_{i \in S} x_i$. Furthermore, we also use the notation $x^{\bar{i}} = x_1^{i_1} x_2^{i_2} \dots x_r^{i_r}$. We write $[n] = \{1, 2, \dots, n\}$ and $[i, j] = \{i, i + 1, \dots, j\}$.

Lemma 3.1. The numbers $t_k(\bar{i})$, where $\bar{i} = (i_1, \dots, i_{r-1}, i_r)$, are determined by the initial condition $t_k(i_1) = \delta_{i_1, 1}$ where $\delta_{k_1, 1}$ is the Kronecker delta and if there is an index $j \in [r - k + 1, r - 1]$ such that $i_j < i_r$ then $t_k(\bar{i}) = 0$. When $r \geq k$ the following recurrence holds:

$$t_k(\bar{i}) = \binom{i_{r-k+1} - 1}{i_r} \cdot t_k(\bar{i} - i_r \cdot e_{[r-k+1, r]}).$$

Proof. In an augmented k -Catalan path exactly one lattice point must be at level 1 if the lattice path never hits level 2. This yields the initial condition. Notice that any lattice point at level r in an augmented k -Catalan path of type \bar{i} is a peak immediately preceded by a run of $k - 1$ up steps $(1, 1)$ and immediately followed by a down step $(1, 1 - k)$. By removing these steps, each k -Catalan path of type \bar{i} may be uniquely reduced to an augmented k -Catalan path of type

$$(i_1, \dots, i_{r-k}, i_{r-k+1} - i_r, \dots, i_{r-1} - i_r, i_r - i_r) = \bar{i} - i_r \cdot e_{[r-k+1, r]}.$$

Conversely, given an augmented k -Catalan path of type $\bar{i} - i_r \cdot e_{[r-k+1, r]}$, there are

$$\binom{i_{r-k+1} - i_r}{i_r} = \binom{i_{r-k+1} - 1}{i_r}$$

ways to select the place to reinsert i_r runs of $k - 1$ up steps $(1, 1)$ immediately followed by a down step $(1 - k, 1)$, after one of the $i_{r-k+1} - i_r$ lattice points at level $r - k + 1$ of the reduced lattice path, where $\binom{n}{j} = \binom{n+j-1}{j}$ denotes the number of j element multisubsets of an n -set. \square

Using Lemma 3.1 we obtain the following recurrence for the associated generating functions.

Lemma 3.2. The generating functions for the k -Catalan-Spitzer permutations of type \bar{i} , that is,

$$T_k(x_1, \dots, x_r) = \sum_{\bar{i} \in \mathbb{P} \times \mathbb{N}^{r-1}} t_k(\bar{i}) \cdot x^{\bar{i}}$$

are given by the initial conditions $T_k(x_1) = T_k(x_1, x_2) = \dots = T_k(x_1, \dots, x_{k-1}) = x_1$ and for $r \geq k$ by the recurrence

$$T_k(x_1, x_2, \dots, x_r) = T_k\left(x_1, x_2, \dots, x_{r-k}, \frac{x_{r-k+1}}{1 - x_{[r-k+1, r]}}, x_{r-k+2}, \dots, x_{r-1}\right). \tag{8}$$

Observe that the function on the left-hand side of (8) is r -ary, whereas the function on the right-hand side is $(r - 1)$ -ary.

Proof of Lemma 3.2. The initial conditions are straightforward to verify. Using the recurrence stated in Lemma 3.1, we obtain

$$\begin{aligned} T_k(x_1, \dots, x_r) &= \sum_{\substack{\bar{i} \in \mathbb{P} \times \mathbb{N}^{r-1} \\ i_r \leq i_{r-k+1}, i_{r-k+2}, \dots, i_{r-1}}} t_k(\bar{i} - i_r \cdot e_{[r-k+1, r]}) \cdot \binom{i_{r-k+1} - 1}{i_r} \cdot x^{\bar{i}} \\ &= \sum_{\substack{\bar{i} \in \mathbb{P} \times \mathbb{N}^{r-1} \\ i_r \leq i_{r-k+1}, i_{r-k+2}, \dots, i_{r-1}}} t_k(\bar{i} - i_r \cdot e_{[r-k+1, r]}) \cdot x^{\bar{i} - i_r \cdot e_{[r-k+1, r]}} \cdot \binom{i_{r-k+1} - 1}{i_r} \cdot x_{[r-k+1, r]}^{i_r}. \end{aligned} \tag{9}$$

Introduce $\bar{\ell}$ to be the index vector $\bar{\ell} - i_r \cdot e_{[r-k+1,r]}$ with the last zero removed, that is, we set $\ell_j = i_j$ for $1 \leq j \leq r-k$ and $\ell_j = i_j - i_r$ for $r-k+1 \leq j \leq r-1$. The sum (9) is now

$$\begin{aligned} T_k(x_1, \dots, x_r) &= \sum_{\bar{\ell} \in \mathbb{P} \times \mathbb{N}^{r-2}} t_k(\bar{\ell}) \cdot x^{\bar{\ell}} \cdot \sum_{0 \leq i_r} \binom{i_r + \ell_{r-k+1} - 1}{i_r} x_{[r-k+1,r]}^{i_r} \\ &= \sum_{\bar{\ell} \in \mathbb{P} \times \mathbb{N}^{r-2}} t_k(\bar{\ell}) \cdot x^{\bar{\ell}} \cdot \frac{1}{(1 - x_{[r-k+1,r]})^{\ell_{r-k+1}}}. \quad \square \end{aligned}$$

In order to give an explicit rational expression for these generating functions, we define the denominator polynomial as follows.

Definition 3.4. Given any positive integer $k \geq 2$ and any interval $[r, s]$ of consecutive positive integers, we define the k -Catalan denominator polynomial $Q_k(x_r, x_{r+1}, \dots, x_s)$ as the signed sum

$$Q_k(x_r, x_{r+1}, \dots, x_s) = \sum_S (-1)^{|S|/k} \cdot x_S,$$

where S ranges over all subsets of $[r, s]$ that arise as a disjoint union of sets consisting of k consecutive integers. The empty set is included in the sum and contributes the term 1.

Theorem 3.1. For $k \geq 2$ we have

$$T_k(x_1, x_2, \dots, x_r) = \frac{x_1 \cdot Q_k(x_2, \dots, x_r)}{Q_k(x_1, \dots, x_r)}.$$

Proof. For $r < k$ the sets $[r]$ and $[2, r]$ do not contain any subset of k consecutive integers, hence we have $Q_k(x_1, \dots, x_r) = Q_k(x_2, \dots, x_r) = 1$ and the identity holds. We proceed by induction for $r \geq k$. By Lemma 3.2 the generating function $T_k(x_1, x_2, \dots, x_r)$ is obtained from $T_k(x_1, x_2, \dots, x_{r-1})$ by substituting $x_{r-k+1}/(1 - x_{[r-k+1,r]})$ into x_{r-k+1} . This substitution turns the stated formula for $T_k(x_1, x_2, \dots, x_{r-1})$ into a four-level fraction which may be transformed into a quotient of two polynomials by multiplying the numerator and the denominator by $1 - x_{[r-k+1,r]}$. This operation leaves all monomials x_S where $S \subseteq [r-1]$ containing a factor of x_{r-k+1} unchanged, as replacing x_{r-k+1} with $x_{r-k+1}/(1 - x_{[r-k+1,r]})$ and then multiplying by $1 - x_{[r-k+1,r]}$ amounts to no change at all. On the other hand, each monomial x_S satisfying $r-k+1 \notin S$ is replaced with $x_S - x_S \cdot x_{[r-k+1,r]}$.

Assuming the induction hypothesis, the terms x_S appearing in the denominator of $T_k(x_1, \dots, x_{r-1})$ are indexed exactly by those subsets $S \subseteq [r-1]$ which arise as a disjoint union of sets consisting of k consecutive integers. Each such set is also a subset of $[r]$, and by our recurrence the corresponding term x_S remains in the denominator with the same coefficient. The additional new terms in the denominator of $T_k(x_1, x_2, \dots, x_{r-1})$ are exactly the terms $x_S \cdot x_{[r-k+1,r]}$ where S ranges through all terms of $Q_k(x_1, \dots, x_{r-1})$ that do not contain x_{r-k+1} as a factor. Note that $r-k+1 \notin S \subseteq [r-1]$ implies $S \subseteq [1, r-k]$ as the interval $[r-k+2, r-1]$ contains less than k consecutive integers. The converse is also true. Hence the monomial $x_S \cdot x_{[r-k+1,r]}$ is square-free, and the underlying set is obtained by adding the disjoint set $[r-k+1, r]$ consisting of k consecutive integers to S . Hence we are adding exactly those terms of $Q_k(x_1, \dots, x_r)$ to the denominator which do not appear in $Q_k(x_1, \dots, x_{r-1})$, and the sign of $x_S \cdot x_{[r-k+1,r]}$ is the opposite of x_S , consistent with the definition of $Q_k(x_1, \dots, x_r)$. Similar reasoning may be used to show that the numerator of $T_k(x_1, x_2, \dots, x_r)$ is $x_1 \cdot Q_k(x_2, \dots, x_r)$. \square

Example 3.2. As an example of Theorem 3.1, we obtain for $k = 3$ and $r = 6$

$$T_3(x_1, x_2, \dots, x_6) = \frac{x_1 \cdot (1 - x_{[2,4]} - x_{[3,5]} - x_{[4,6]})}{1 - x_{[1,3]} - x_{[2,4]} - x_{[3,5]} - x_{[4,6]} + x_{[1,6]}}.$$

Theorem 3.1 may be restated in a more compact form in terms of the following generalization of continuants.

Definition 3.5. The n th k -continuant $K_{k,n}(x_1, x_2, \dots, x_n)$ is defined recursively by the initial condition $K_{k,n} = x_1 x_2 \cdots x_n$ for $n < k$ and for $n \geq k$ by the recurrence

$$K_{k,n}(x_1, x_2, \dots, x_n) = K_{k,n-k}(x_1, x_2, \dots, x_{n-k}) + K_{k,n-1}(x_1, x_2, \dots, x_{n-1}) \cdot x_n.$$

Note that for $k = 2$ Definition 3.5 is the classical definition of the continuants. The verification of the following facts for k -continuants are completely analogous to the proof in the $k = 2$ case, and are left to the reader.

Proposition 3.2. The k -continuant $K_{k,n}(x_1, x_2, \dots, x_n)$ can be computed by taking the sum of all possible products of x_1, x_2, \dots, x_n in which any number of disjoint sets of k consecutive variables are deleted from the product $x_1 x_2 \cdots x_n$.

Proposition 3.3. *Introducing the $k \times k$ matrix*

$$M_k(x) = \begin{bmatrix} x & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},$$

we may write

$$\begin{bmatrix} K_{k,n}(x_1, x_2, \dots, x_n) \\ K_{k,n-1}(x_1, x_2, \dots, x_{n-1}) \\ \vdots \\ K_{k,n-k+1}(x_1, x_2, \dots, x_{n-k+1}) \end{bmatrix} = M_k(x_n)M_k(x_{n-1}) \cdots M_k(x_1) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proposition 3.4. *The number of terms in $K_{k,n}(x_1, \dots, x_n)$, that is, $K_{k,n}(1, \dots, 1)$, is recursively obtained by*

$$K_{k,n}(1, \dots, 1) = \begin{cases} 1 & \text{for } 0 \leq n \leq k-1, \\ K_{k,n-k}(1, \dots, 1) + K_{k,n-k}(1, \dots, 1) & \text{for } n \geq k. \end{cases} \quad (10)$$

For $k = 2$ the sequence $K_{2,n}(1, \dots, 1)$ is the Fibonacci number F_n . For $k = 3$ the sequence $\{K_{3,n}(1, \dots, 1)\}_{n \geq 0}$ is sequence A000930 in [8], also known as Narayana's cow sequence. The same page also contains information regarding the general sequence $\{K_{k,n}(1, \dots, 1)\}_{n \geq 0}$.

As a direct consequence of Definition 3.4 and Proposition 3.2, we have the following corollary.

Corollary 3.1. *Given any positive integer $k \geq 2$, let ζ denote a primitive $(2k)$ th root of unity. Then for any interval $[r, s]$ of consecutive positive integers, the k -Catalan denominator polynomial $Q_k(x_r, x_{r+1}, \dots, x_s)$ is given by*

$$Q_k(x_r, x_{r+1}, \dots, x_s) = \zeta^{s-r+1} \cdot x_r x_{r+1} \cdots x_s \cdot K_{k,s-r+1} \left(\frac{1}{\zeta x_r}, \frac{1}{\zeta x_{r+1}}, \dots, \frac{1}{\zeta x_s} \right).$$

Substituting Corollary 3.1 into Theorem 3.1, after simplifying by $x_1 x_2 \cdots x_r$, we obtain the formula

$$T_k(x_1, x_2, \dots, x_r) = \frac{1}{\zeta} \cdot \frac{K_{k,r-1} \left(\frac{1}{\zeta x_2}, \dots, \frac{1}{\zeta x_r} \right)}{K_{k,r} \left(\frac{1}{\zeta x_1}, \dots, \frac{1}{\zeta x_r} \right)}. \quad (11)$$

We conclude this section by having a closer look at the Catalan case. Using the well-known relation between continuants and continued fractions, equation (11) may be rewritten as

$$T_2(x_1, x_2, \dots, x_r) = \frac{1}{\mathbf{i}} \cdot \frac{1}{\frac{1}{\mathbf{i}x_1} + \frac{1}{\frac{1}{\mathbf{i}x_2} + \cdots + \frac{1}{\mathbf{i}x_r}}} = \frac{1}{x_1 - \frac{1}{x_2 - \cdots - \frac{1}{x_r}}}, \quad (12)$$

where \mathbf{i} is the square root of -1 .

Remark 3.1. Equation (12) is also a consequence of Flajolet's result [2, Theorem 1] which provides a generating function formula for *Motzkin paths*, starting at $(0, 0)$ and ending on the x axis consisting of up steps $(1, 1)$, down steps $(1, -1)$ and horizontal steps $(1, 0)$ that never go below the x -axis. To obtain Equation (12), we must set $c_i = 0$ and $a_i = b_i = x_{i+1}$ for all $i \geq 0$ in Flajolet's formula and multiply the resulting generating function by x_1 . The additional factor of x_1 is induced by the fact that we consider augmented Catalan paths. Thus we obtain the generating function

$$\frac{x_1}{1 - \frac{x_1 x_2}{1 - \frac{x_2 x_3}{1 - \frac{x_3 x_4}{\ddots}}}}.$$

It is straightforward to see that we obtain the generating function that is the limit, as r goes to infinity, of the function given in (12).

4. Short k -Catalan–Spitzer permutations

The k -Catalan–Spitzer permutations defined in the previous section contain redundant information. In this section we show that we may restrict our attention to the lattice points which are the lower ends of the up steps. The resulting permutations have a particularly nice representation when we consider the Foata–Strehl group action [3, 4].

Definition 4.1. A short k -Catalan–Spitzer permutation of order n is the relative order of the numbers $z'_1, \dots, z'_{i_{(k-1)n}}$ in a k -Catalan–Spitzer path $(0, 0), (1, z'_1), \dots, (kn, z'_{kn}), (kn+1, 0)$ of order n , where $\{i_1, i_2, \dots, i_{(k-1)n}\}$ is the set of indices $i_j \geq 1$ satisfying $z'_{i_j} < z'_{i_j+1}$. In the case when $k = 2$ we use the term short Catalan–Spitzer permutation.

In analogy to Definition 3.2, a short Catalan–Spitzer permutation may be also defined in terms of k -Catalan paths as follows.

Proposition 4.1. The set of short k -Catalan–Spitzer permutations of order n is the set of all permutations arising by the following procedure. Take a k -Catalan path of order n and number its up steps so that the values increase from right to left at the same level and upward between different levels. Record the numbers along the lattice path.

We will say that a short k -Catalan–Spitzer permutation σ is induced by a k -Catalan path P if the procedure described in Proposition 4.1 applied to P yields the permutation σ .

A short k -Catalan–Spitzer permutation associated to a k -Catalan path may be computed directly from the corresponding (full) k -Catalan–Spitzer permutation using the notions of an *ascents* and *patterns*. Recall that the index $i \in \{1, \dots, n-1\}$ is an *ascent* of a permutation $\pi(1)\pi(2) \cdots \pi(n)$ if $\pi(i) < \pi(i+1)$ holds. Furthermore, given an ordered alphabet X with n letters, a *permutation* of X is a word $w_1w_2 \cdots w_n$ containing each letter of X exactly once. The *pattern* of the permutation w is the permutation $\pi(1)\pi(2) \cdots \pi(n)$ of the set $\{1, 2, \dots, n\}$ satisfying $\pi(i) < \pi(j)$ if and only if $w_i < w_j$ for each $1 \leq i < j \leq n$.

The following lemma follows directly from the definitions.

Lemma 4.1. Let $\pi(1)\pi(2) \cdots \pi(kn)$ and $\sigma(1)\sigma(2) \cdots \sigma((k-1)n)$ be a k -Catalan–Spitzer, respectively a short k -Catalan–Spitzer, permutation of order n associated to the same k -Catalan–Spitzer path. Then $\sigma(1)\sigma(2) \cdots \sigma((k-1)n)$ is the pattern of $\pi(i_1)\pi(i_2) \cdots \pi(i_{(k-1)n})$ where $\{i_1, i_2, \dots, i_{(k-1)n}\}$ is the set of ascents of $\pi(1)\pi(2) \cdots \pi(kn)$.

Example 4.1. Consider the 4-Catalan–Spitzer permutation 3, 5, 8, 11, 2, 4, 7, 10, 12, 13, 6, 9, 1 of order 3 discussed in Example 3.1. Its ascent set is $\{1, 2, 3, 5, 6, 7, 8, 9, 11\}$. The pattern of the word 3, 5, 8, 2, 4, 7, 10, 12, 6 is 2, 4, 7, 1, 3, 6, 8, 9, 5.

As shown in Proposition 4.2 below, the operation assigning to each k -Catalan–Spitzer permutation (equivalently, each k -Catalan path) the corresponding short k -Catalan–Spitzer permutation is injective. We will prove this by considering the *Foata–Strehl trees* of short k -Catalan–Spitzer permutations. Recall that a *plane* $0-1-2$ tree is a rooted plane tree, in which each vertex has degree at most 2. (It is not unusual to call plane $0-1-2$ trees plane binary trees. However, a plane binary tree in the strict sense cannot contain a vertex with a single child.)

Definition 4.2. Let $w_1w_2 \cdots w_n$ be a word with letters from an ordered alphabet containing no repeated letters. The Foata–Strehl tree $\mathcal{FS}(w_1w_2 \cdots w_n)$ of this word is the plane $0-1-2$ tree defined recursively as follows. The root of the tree is $w_i = \min(w_1, w_2, \dots, w_n)$ whose left child is $\min(w_1, w_2, \dots, w_{i-1})$ and whose right child is $\min(w_{i+1}, w_2, \dots, w_n)$. There is no left child if $i = 1$ and no right child if $i = n$. The subtree of the left child is $\mathcal{FS}(w_1w_2 \cdots w_{i-1})$, and the subtree of the right child is $\mathcal{FS}(w_{i+1}w_{i+2} \cdots w_n)$.

Clearly, the correspondence between permutations of $\{1, 2, \dots, n\}$ and Foata–Strehl trees with n vertices is a bijection.

Lemma 4.2. Let $\sigma = \sigma(1)\sigma(2) \cdots \sigma((k-1)n)$ be a short k -Catalan–Spitzer permutation induced by a k -Catalan path P of order n . If $\sigma(i)$ is the label of an up step that is immediately followed by a down step in P then $\sigma(i)$ has no right child. If $\sigma(i)$ is the label of an up step that is immediately followed by an up step in P then $\sigma(i)$ has a right child $\sigma(j)$ and the level of the up step labeled by $\sigma(j)$ is one more than the level of the up step labeled by $\sigma(i)$.

Proof. If the up step labeled $\sigma(i)$ is immediately followed by a down step, then the level of the next up step is not greater than that of the up step labeled $\sigma(i)$. In this case $\sigma(i+1) < \sigma(i)$ holds and the right subtree of $\sigma(i)$ in $\mathcal{FS}(\sigma)$ is empty. Assume from now on that the up step labeled $\sigma(i)$ is immediately followed by an up step. As we follow the up steps along P after the up step labeled $\sigma(i)$, all have a larger label than $\sigma(i)$ until P returns

to the level of $\sigma(i)$. The label of the next up step is less than the label of $\sigma(i)$. Hence the labels belonging to the right subtree of $\sigma(i)$ in $\mathcal{FS}(\sigma)$ are exactly the up steps belonging to the part P' of P that begins with the up step labeled $\sigma(i)$ and ends with the first return of P to the same level. The labels in this subtree belong to up steps whose level is greater than the level of the up step labeled $\sigma(i)$. The level of the last up step in P' is one more than that of the up step labeled $\sigma(i)$ and it is the rightmost among all up steps of P' at this level. Hence its label is the least element of the right subtree of $\sigma(i)$. \square

Inspired by Lemma 4.2, we define the *level* of each letter in a permutation as follows.

Definition 4.3. *Let T be a plane $0-1-2$ tree. We define the level of each vertex of T as follows.*

1. *The level of the root is zero.*
2. *For any other vertex v , the level of v is the number of right steps in the unique path in T from the root to the vertex v .*

Given any permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ we define the level of $\sigma(i)$ as the level of the vertex labeled $\sigma(i)$ in the Foata–Strehl tree $\mathcal{FS}(\sigma)$.

Using Definition 4.3 we may rephrase Lemma 4.2 as follows.

Corollary 4.1. *Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma((k-1)n)$ be a short k -Catalan–Spitzer permutation induced by a k -Catalan path P of order n . Then for each i the level of the up step labeled $\sigma(i)$ is the same as the level of $\sigma(i)$.*

An important consequence of Corollary 4.1 is the following.

Proposition 4.2. *The operation associating to each k -Catalan path P its induced k -Catalan–Spitzer permutation is injective.*

Proof. Assume the k -Catalan–Spitzer permutation σ is induced by the k -Catalan path P . It suffices to show that P may be uniquely reconstructed from the Foata–Strehl tree $\mathcal{FS}(\sigma)$ of σ . By Corollary 4.1 the level of each up step may be read from $\mathcal{FS}(\sigma)$. Note finally that the number of down steps between the up step labeled $\sigma(i)$ and the next up step labeled $\sigma(i+1)$ is the difference between the level of $\sigma(i)$ and the level of $\sigma(i+1)$. \square

Definition 4.3 allows us to define the level of each letter in any permutation. The next definition allows us to identify the short k -Catalan–Spitzer permutations by looking at their Foata–Strehl trees.

Definition 4.4. *Let T be a plane $0-1-2$ tree with n vertices numbered from 1 to n . We say that T is levelwise numbered if the labeling of its vertices satisfies the following criteria:*

1. *If the level of the vertex labeled i is less than the level of the vertex labeled j then $i < j$.*
2. *If the vertex labeled j is in the left subtree of the vertex labeled i then $i < j$.*
3. *If the vertex labeled i and the vertex labeled j have the same level, but there is a $k < i, j$ such that vertex labeled i (respectively j) is in the right (respectively left) subtree of the vertex labeled k then $i < j$.*

Proposition 4.3. *Each plane $0-1-2$ tree has a unique levelwise numbering.*

Proof. By Definition 4.4 vertices at the same level must be numbered consecutively. We only need to show that the second and third rules of the definition uniquely determine the labeling of the vertices at the same level. We will show this by considering for each vertex v the unique path from the root to v . This path may be encoded by an rl -word $RL(v)$ defined as follows. As we move along the path from the root to a vertex v , we record a letter r each time we move to the right child and a letter ℓ each time to a left child. For example, $r\ell\ell$ represents the left child of the left child of the right child of the root. Clearly the level of v is the number of letters r in its rl -word. It suffices to show that for vertices at the same level, the second and third rules amount to ordering their rl -words in the left-to-right lexicographic order as follows: the letter r precedes the letter ℓ and each word is succeeded by all words obtained by appending any number of letters ℓ at their right end.

Consider the rl -words $RL(u)$ and $RL(v)$ encoding the vertices u and v at the same level. Let us compare their letters left to right and stop where we see the first difference. Two possibilities arise:

1. One of the two words, say $RL(u)$, is an initial segment of the other. Since u and v are at the same level, in this case we must have $RL(v) = RL(u)\ell^k$ for some k . In this case v is in the left subtree of u and by the second rule v must have a higher number than u .

2. Neither of the two words is an initial segment of the other one, and (without loss of generality) the leftmost letter that is different is r in $RL(u)$ and ℓ in $RL(v)$. In other words, we have $RL(u) = RL(w)rx$ and $RL(v) = RL(w)\ell y$ for some $r\ell$ -words x and y and a vertex w whose $r\ell$ -word is the longest common initial segment of $RL(u)$ and $RL(v)$. In this case, u is in the right subtree of w , v is in the left subtree of w and the third rule is applicable.

Note that the above two cases, as well as the premises of the second and third rules in Definition 4.4, are mutually exclusive and represent a complete enumeration of all possibilities. We have shown that the rules in Definition 4.4 are logically equivalent to defining the above lexicographic order on the $r\ell$ -words of the vertices at the same level. \square

Proposition 4.4. *The Foata–Strehl tree of a short k -Catalan–Spitzer permutation of order n is levelwise numbered.*

Proof. By Lemma 4.2 the Foata–Strehl tree of a short k -Catalan–Spitzer permutation σ satisfies the first condition of Definition 4.4. Consider two letters $\sigma(i)$ and $\sigma(j)$ at the same level. By Lemma 4.2 there is a k -Catalan path P inducing σ in which $\sigma(i)$ and $\sigma(j)$ label up steps at the same level. If the vertex labeled $\sigma(j)$ is in the left subtree of the vertex labeled $\sigma(i)$ then $\sigma(j)$ precedes $\sigma(i)$ in σ , that is, we have $j < i$ and $\sigma(i) < \sigma(j)$ must hold as up steps at the same level are numbered in the right to left order. Similarly, if there is a vertex labeled $\sigma(k)$ such that the vertex labeled $\sigma(i)$, respectively $\sigma(j)$, is in its right, respectively left subtree, then we must have $j < k < i$ and $\sigma(i) < \sigma(j)$ must hold. \square

We conclude this section with a theorem that completely describes short k -Catalan–Spitzer permutations in terms of their Foata–Strehl trees.

Theorem 4.1. *A permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ is a short Catalan–Spitzer permutation if and only if its Foata–Strehl tree $\mathcal{FS}(\sigma)$ is levelwise numbered. It is also a short k -Catalan–Spitzer permutation if and only if its Foata–Strehl tree $\mathcal{FS}(\sigma)$ also has the following additional property: in each longest path containing only edges between parent and right child, the number of vertices is a multiple of $k - 1$.*

Proof. By Proposition 4.2 the number of short Catalan–Spitzer permutations of order n is the Catalan number C_n . By Proposition 4.4 the Foata–Strehl tree $\mathcal{FS}(\sigma)$ of each short Catalan–Spitzer permutation must be levelwise numbered. By Proposition 4.3 each plane $0 - 1 - 2$ tree has exactly one levelwise numbering, and the number of plane $0 - 1 - 2$ trees on n vertices is also the Catalan number C_n . Therefore the set of Foata–Strehl trees of all short Catalan–Spitzer permutations of order n must equal the set of all levelwise numbered plane $0 - 1 - 2$ trees on n vertices.

To prove the second statement, observe that each k -Catalan path of order n may be transformed into a Catalan path of order $(k - 1)n$ by replacing each down step $(1, 1 - k)$ with a run of $k - 1$ consecutive down steps $(1, -1)$. Under this correspondence k -Catalan paths of order n bijectively correspond to those Catalan paths of order $(k - 1)n$ in which the length of each longest run of consecutive down steps is a multiple of $k - 1$. Using this correspondence it is easy to see that a short Catalan–Spitzer permutation of order n is also a short k -Catalan–Spitzer permutation of order n if and only if for each $i \in \{1, 2, \dots, n - 1\}$ the difference between the level of $\sigma(i)$ and the level of $\sigma(i + 1)$ is a multiple of $k - 1$. This condition is equivalent to the condition stated in the theorem. The details are left to the reader. \square

5. A restricted Foata–Strehl group action and its enumerative consequences

Foata–Strehl trees were first defined [3, 4] to introduce the *Foata–Strehl group action* on permutations of order n . This \mathbb{Z}_2^{n-1} action is generated by $n - 1$ commuting involutions $\phi_1, \phi_2, \dots, \phi_{n-1}$, where ϕ_i swaps the left and right subtrees of the vertex labeled i in the Foata–Strehl tree of each permutation. (Both subtrees may be empty.) Theorem 4.1 provides a characterization of short k -Catalan–Spitzer permutations in terms of their Foata–Strehl trees. Unfortunately, in most cases the Foata–Strehl action destroys the levelwise numbering, except for some special situations. In this section we focus on such a special situation, introduce a subgroup of the Foata–Strehl group action, and observe how it may be used in proving identities in enumerative combinatorics beyond the world of Catalan objects.

Definition 5.1. *Let X be an ordered alphabet, $x \in X$ and w a permutation of X . We say that w is x -decomposable if it may be written in the form $w = w_1w_2w_3$ such that the following hold:*

1. All letters of w_1 and w_3 are less than x ;
2. all letters of w_2 are greater than or equal to x ;

3. the letter x is either the first or the last letter of w_2 .

Under the above circumstances, we call the decomposition $w = w_1w_2w_3$ the x -decomposition of w . An x -flip consists of moving the letter x from one end of the subword w_2 to its other end.

When $x = \max(X)$ is the largest element of the alphabet X , any permutation is *trivially x -decomposable* since w_2 must consist of the single letter x . For this x , the x -flip is the identity map. For all other $x \in X$ an x -decomposition must be *nontrivial* as w_2 must contain all letters larger than x . The application of an x -flip results in a different word. The verification of the following equivalent description is left to the reader.

Proposition 5.1. *For $x < \max(X)$ the permutation w is x -decomposable if and only if its Foata–Strehl tree $\mathcal{FS}(w)$ satisfies the following:*

1. exactly one of the right and left subtrees of x is empty;
2. the set of descendants of x contains all letters larger than x .

If w has an x -decomposition, an x -flip is precisely the application of the operation ϕ_x of the Foata–Strehl group action.

An important consequence of Proposition 5.1 is that x -flips and y -flips commute the same way the generators of the Foata–Strehl group action commute.

Corollary 5.1. *If a word w is simultaneously x -decomposable and y -decomposable then the same holds for $\phi_x(w)$, $\phi_y(w)$ and $\phi_x\phi_y(w)$. Furthermore, $\phi_x\phi_y(w) = \phi_y\phi_x(w)$*

Definition 5.2. *Given an ordered alphabet X , we extend the x -flip operations ϕ_x to all permutations w of X by setting $\phi_x(w) = w$ whenever w is not x -decomposable. We call the group action generated by the operations $\{\phi_x \mid x < \max(X)\}$ the restricted Foata–Strehl group action on the permutations of X .*

Our interest in x -decompositions and x -flips is due to the following result.

Theorem 5.1. *If a short Catalan–Spitzer permutation σ of order n is i -decomposable for some $i < n$ then its i -flip $\phi_i(\sigma)$ is also a short Catalan–Spitzer permutation.*

Proof. By Proposition 5.1 the set of descendants of i is the set $\{i + 1, i + 2, \dots, n\}$ and all of them are in the subtree of $i + 1$, which is the only child of i . If $i + 1$ is the right child of i then the level of $i + 1$ is one more than the level of i , i is the largest letter at its level and $i + 1$ is the smallest letter at its level. Moving $i + 1$ to the left of i results in merging the levels of i and $i + 1$ into a single level: the level of the labels larger than $i + 1$ uniformly decreases by one. For all other labels, the sets of labels at the same level remains unchanged. Consider two vertices u and v whose label belongs to the merged level. If the labels of u and v are both greater than i , then the shortest path leading to both contains the vertex $i + 1$: moving $i + 1$ to the left induces changing a letter r into a letter ℓ in the same position in both $RL(u)$ and $RL(v)$. Such a change does not affect the relative order of the two $r\ell$ -words in the lexicographic order. If the labels of u and v are both at most i , then $RL(u)$ and $RL(v)$ remain unchanged after moving $i + 1$ to the left of i . Consider finally the case when the label of u is at most i and the label of v is less than i . When the vertex labeled $i + 1$ is the right child of the vertex labeled i then the label of v is more than the label of u because they are at different levels. When we move the vertex labeled $i + 1$ to the left, the vertex v becomes a vertex in the left subtree of the vertex labeled i , hence its label must be still greater than i and also greater than the label of the vertex u (whose $r\ell$ -word remains unchanged).

We have shown that applying an i -flip to an i -decomposable Catalan–Spitzer permutation that moves $i + 1$ from the right to the left results in a Catalan–Spitzer permutation. The proof of the converse is analogous and left to the reader. \square

As a consequence of Theorem 5.1 the set of all Catalan–Spitzer permutations of order n may be partitioned into orbits of the restricted Foata–Strehl group action. A question naturally arises, namely, what is the number of such orbits. We answer this in the greatest generality. A *class of permutations* \mathcal{P} is a rule assigning to each finite ordered set X a subset $\mathcal{P}_{|X|}(X)$ of its permutations in such a way that membership of w in $\mathcal{P}_{|X|}(X)$ depends only on the pattern of w . For brevity, we will say that a permutation w belongs to the class \mathcal{P} if w is an element of $\mathcal{P}_{|X|}(X)$ for some finite set X .

Definition 5.3. *A class of permutations is compatible with the restricted Foata–Strehl group action if it satisfies the following: for each $x \in X$ and for each x -decomposable $w \in \mathcal{P}_{|X|}(X)$ the following holds:*

1. The permutation $\phi_x(w)$ belongs to the class \mathcal{P} .
2. If $w_1w_2w_3$ is the x -decomposition of w then w_2 and w_1w_3 also belong to the class \mathcal{P} .

For a class of permutations \mathcal{P} that is compatible with the restricted Foata–Strehl group action, let P_n , respectively O_n , be the number of permutations in $\mathcal{P}(\{1, 2, \dots, n\})$, respectively number of orbits of the restricted Foata–Strehl group action on the set $\mathcal{P}(\{1, 2, \dots, n\})$. We introduce the ordinary generating functions

$$P(x) = \sum_{n \geq 1} P_n \cdot x^n \quad \text{and} \quad Q(x) = \sum_{n \geq 1} Q_n \cdot x^n. \quad (13)$$

We consider these generating functions as formal power series from $\mathbb{Q}[[x]]$. Our first general result is the following.

Theorem 5.2. *The generating functions $P(x)$ and $O(x)$ satisfy*

$$P(x) = \frac{O(x)}{1 - O(x)} \quad (14)$$

equivalently,

$$O(x) = \frac{P(x)}{1 + P(x)} \quad (15)$$

Proof. We only need to show (14) as equation (15) is algebraically equivalent. For each $\sigma \in \mathcal{P}(\{1, 2, \dots, n\})$ denote the orbit of σ under the restricted Foata–Strehl group action by $[\sigma]$. To each orbit $[\sigma]$ we may associate a set $I([\sigma]) \subseteq \{1, 2, \dots, n-1\}$ such that each permutation in the orbit is i -decomposable if and only if i is an element of $I([\sigma])$. The size of the orbit will be $2^{|I([\sigma])|}$. We say that σ is a *distinguished orbit representative* if for each $i \in I$, the letter $i+1$ is to the right of i in σ . Equivalently, in the Foata–Strehl tree $\mathcal{FS}(\sigma)$ of σ , each $i \in I([\sigma])$ has a right child and not a left child. Clearly, there is exactly one distinguished orbit representative in each orbit. For all permutations $\sigma \in \mathcal{P}(\{1, 2, \dots, n\})$ that are not distinguished orbit representatives, there is a unique smallest $k \in I([\sigma])$ such that $k+1$ is the left child of k and $i+1$ is a right child for all $i \in I([\sigma])$ satisfying $i < k$. The removal of the left subtree of $k+1$ results in the Foata–Strehl tree of a distinguished orbit representative in $\mathcal{P}(\{1, 2, \dots, k\})$, whereas the left subtree of k is the Foata–Strehl tree of a permutation in $\mathcal{P}(\{k+1, k+2, \dots, n\})$. The two permutations may be selected independently and determine σ uniquely. This observation justifies the formula

$$P_n = O_n + \sum_{k=1}^{n-1} O_k \cdot P_{n-k}.$$

The stated formula for the generating functions follows immediately. \square

Example 5.1. If \mathcal{P} is the class of short Catalan–Spitzer permutations then $P(x) = C(x) - 1$ where

$$C(x) = \sum_{n \geq 0} C_n \cdot x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the generating function of the Catalan numbers. Equation (15) gives $O(x) = \frac{C(x)-1}{C(x)} = x \cdot C(x)$. Hence $O_n = C_{n-1}$.

Example 5.2. If \mathcal{P} is the class of all permutations then $P(x) = \sum_{n \geq 1} n!x^n$. Equation (15) gives the ordinary generating function of the indecomposable permutations. The numbers of these are listed as sequence A003319 in [8].

To refine Theorem 5.2 let $P_{n,k}$ denote the number of permutations belonging to an orbit of size 2^k of the restricted Foata–Strehl group action on $\mathcal{P}(\{1, 2, \dots, n\})$ and let $O_{n,k}$ be the number of orbits of size 2^k . Clearly, we have

$$O_{n,k} = \frac{P_{n,k}}{2^k}. \quad (16)$$

We introduce the generating functions

$$P(x, y) = \sum_{n \geq 1, k \geq 0} P_{n,k} x^n y^k \quad \text{and} \quad O(x, y) = \sum_{n \geq 1, k \geq 0} O_{n,k} x^n y^k.$$

Theorem 5.3. *The generating functions $P(x, y)$ and $O(x, y)$ are given by*

$$P(x, y) = \frac{P(x)}{1 - 2(y-1)P(x)}, \quad (17)$$

$$O(x, y) = \frac{P(x)}{1 - (y-2)P(x)}. \quad (18)$$

Proof. We use the notation $I([\sigma])$ introduced in the proof of Theorem 5.2. Given any $\sigma \in \mathcal{P}(\{1, 2, \dots, n\})$ and any element i_1 of $I([\sigma])$, the permutation σ may be written as a concatenation of words

$$\sigma = \sigma_0 \sigma_1 \sigma'_0, \quad (19)$$

where $\sigma_0 i_1 \sigma'_0$ is an element of $\mathcal{P}(\{1, 2, \dots, i_1\})$ and σ_1 is an element of $\mathcal{P}(\{i_1, i_1 + 1, \dots, n\})$ containing i_1 as the first or the last letter. At the level of Foata–Strehl trees, $\mathcal{FS}(\sigma)$ may be obtained by selecting a levelwise labeled plane $0 - 1 - 2$ tree with i_1 vertices and then adding any Foata–Strehl tree with $n - i_1$ as the right or left subtree of i_1 . Iterating the procedure, for any subset $\{i_1, i_2, \dots, i_k\}$ of $I([\sigma])$ satisfying $i_1 < i_2 < \dots < i_k$ we may decompose $\mathcal{FS}(\sigma)$ into a sequence of Foata–Strehl trees (T_0, T_1, \dots, T_k) such that

1. $T_1 = \mathcal{FS}(\sigma_0 i_1 \sigma'_0)$ for some $\sigma_0 i_1 \sigma'_0 \in \mathcal{P}(\{1, 2, \dots, i_1\})$;
2. for $j = 2, 3, \dots, k - 1$ the labeled tree $T_j = \mathcal{FS}(\sigma_{j-1} i_j \sigma'_{j-1})$ for some $\sigma_{j-1} i_j \sigma'_{j-1} \in \widehat{\mathcal{P}}(\{i_{j-1} + 1, i_{j-1} + 2, \dots, i_j\})$;
3. $T_k = \mathcal{FS}(\sigma_{k-1} i_k \sigma'_{k-1})$ for some $\sigma_{k-1} i_k \sigma'_{k-1} \in \mathcal{P}(\{i_{k-1} + 1, i_{k-1} + 2, \dots, i_k\})$, or it may be empty;
4. for $j = 2, \dots, k$ the labeled tree T_j is the right or left subtree of i_j .

Conversely, if $\mathcal{FS}(\sigma)$ is decomposed into into a sequence of Foata–Strehl trees (T_0, T_1, \dots, T_k) satisfying the above criteria then $\{i_1, i_2, \dots, i_k\}$ must be a subset of $I([\sigma])$. Introducing the variable y to mark the selected elements of $I([\sigma])$, we obtain the formula

$$P(x, 1 + y) = \sum_{n \geq 1, j \geq 0} P_{n,j} x^n (1 + y)^j = (1 + P(x)) \cdot \sum_{k \geq 0} (2y)^k P(x)^k = \frac{1 + P(x)}{1 - 2yP(x)}.$$

Equation (17) follows by replacing y with $y - 1$ in the last equation. To obtain (18), by equation (16), we only need to substitute $y/2$ into y in (17). \square

Note that Theorem 5.2 may be obtained by substituting $y = 1$ in (18).

Example 5.3. For Catalan–Spitzer permutations we have $P(x) = C(x) - 1$ and

$$\begin{aligned} O(x, y) &= \frac{C(x) - 1}{1 - (y - 2)(C(x) - 1)} \\ &= x + yx^2 + (y^2 + 1)x^3 + (y^3 + 2y + 2)x^4 + (y^4 + 3y^2 + 4y + 6)x^5 \\ &\quad + (y^5 + 4y^3 + 6y^2 + 13y + 18)x^6 + \dots \end{aligned}$$

Substituting $y = 1$ and $y = 2$ gives $O(x) = xC(x)$ and $P(x) = C(x) - 1$, respectively. The substitutions $y = 3, 4, 5, 6$ are listed as sequences A001700, A049027, A076025, A076026 in [8]. The generating functions listed for these sequences are all substitutions into y in $\frac{1 - (y-2)xC(x)}{1 - (y-1)xC(x)}$ which is easily seen to be equal to $1 + O(x, y)$.

Example 5.4. For all permutations we have $P(x) = \sum_{n \geq 1} n!x^n$ and

$$\begin{aligned} O(x, y) &= \frac{P(x)}{1 - (y - 2)P(x)} \\ &= x + yx^2 + (y^2 + 2)x^3 + (y^3 + 4y + 8)x^4 + (y^4 + 6y^2 + 16y + 48)x^5 \\ &\quad + (y^5 + 8y^3 + 24y^2 + 100y + 328)x^6 + \dots \end{aligned}$$

Substituting $y = 1$ and $y = 2$ gives $O(x)$ and $P(x)$, respectively. The substitution $y = 3$ is listed as sequence A051296 in [8]. The presence of this substitution is not surprising: $O(x, 3) = P(x)/(1 - P(x))$ is always the generating function for the ordered collections of permutations in the class \mathcal{P} .

6. Concluding remarks

Are there any other results on the distribution of the quantity α for more for general paths from the origin $(0, 0)$ to $(s, 0)$? For instance, what can be said if we have one type of up step, but two types of down steps?

Are there other Fuss–Catalan structures that belong to a larger set of structures of cardinality $\binom{kn+1}{n}$ with a uniformly distributed statistic on the set $\{0, 1, \dots, kn\}$ and the Fuss–Catalan structure is the fiber of one particular value of this statistic?

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