

Pattern Avoiding Alternating Involutions

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ABSTRACT: We enumerate and characterize some classes of alternating and reverse alternating involutions avoiding a single pattern of length three or four. If on one hand the case of patterns of length three is trivial, on the other hand, the length four case is more challenging and involves sequences of combinatorial interest, such as Motzkin and Fibonacci numbers.

Keywords: alternating permutation; involution; permutation pattern

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1. Introduction

A permutation π avoids a pattern τ whenever π does not contain any subsequence order-isomorphic to τ .

The theory of permutation patterns goes back to the work of Knuth [9], who, in the 1970s, introduced the definition of pattern avoidance in connection to the stack sorting problem. The first systematic study of these objects appears in the paper by Simion and Schmidt ([16]). Nowadays, the theory is very rich and widely expanded, with hundreds of papers appearing in the last decades (see e.g. [8] and references therein).

More recently permutation patterns have been studied over a particular subset of the symmetric group. In particular pattern avoidance has been studied over involutions (see e.g. [1, 4, 7, 8]) and over alternating permutations (see e.g. [3, 5, 6, 8, 10–12, 22, 23]), i.e., permutations $\pi = \pi_1 \dots \pi_n$ such that $\pi_i < \pi_{i+1}$ if and only if i is odd.

The enumeration of alternating involutions is due to Stanley (see [18] and also his survey on alternating permutations [19]). However, to the best of our knowledge, pattern avoiding alternating involutions has not been studied so far.

In this paper, we consider alternating involutions that avoid some patterns of length three or four. If on one hand the case of patterns of length three is trivial, on the other hand, the length four case is more challenging and involves sequences of combinatorial interest, such as Motzkin and Fibonacci numbers.

2. Preliminaries

2.1 Permutations

Let S_n be the symmetric group over the symbols $\{1, 2, \dots, n\}$. We will often write permutations in S_n in one line notation as $\pi = \pi_1 \dots \pi_n$. An *involution* is a permutation π such that $\pi = \pi^{-1}$. The *reverse* of a permutation $\pi = \pi_1 \pi_2 \dots \pi_n$ is $\pi^r = \pi_n \pi_{n-1} \dots \pi_1$ and the *complement* of π is $\pi^c = n + 1 - \pi_1 \ n + 1 - \pi_2 \ \dots \ n + 1 - \pi_n$. The *reverse-complement* of π is $\pi^{rc} = (\pi^r)^c = (\pi^c)^r$. Notice that $(\pi^{-1})^{rc} = (\pi^{rc})^{-1}$. In particular the reverse-complement of an involution is an involution.

A *descent* in a permutation π is an index i such that $\pi_i > \pi_{i+1}$. Denote by $\text{Des}(\pi)$ the set of descents of π and by $\text{des}(\pi)$ its cardinality.

Recall that a *left-to-right minimum* (ltr minimum from now on) of a permutation π is a value $\pi(i)$ such that $\pi(j) > \pi(i)$ for every $j < i$. A *left-to-right maximum* (ltr maximum from now on) is a value $\pi(i)$ such that $\pi(j) < \pi(i)$ for every $j < i$. The definition of right-to-left (rtl) minimum and maximum is analogous.

Lemma 2.1. *In any involution π the ltr minima form an involution themselves, i.e., a ltr minimum m is at position i if and only if i is a ltr minimum. The same is true for rtl maxima.*

Proof. We will consider only the case of ltr minima, since the other case is analogous. Denote by m_1, m_2, \dots, m_k the left-to-right minima of π , and by i_1, i_2, \dots, i_k their respective positions. We want to show that

$$\{m_1, m_2, \dots, m_k\} = \{i_1, i_2, \dots, i_k\}.$$

Suppose on the contrary that there exists an index j such that i_j is not the value of a left-to-right minimum. Then, to the left of i_j in π there is a symbol a less than i_j . In other terms, there exist two integers a, h such that $\pi(h) = a$, $a < i_j$ and $h < m_j$. In this situation, m_j is not a left-to-right minimum, since it is preceded by h , and $h < m_j$, contradiction. \square

Example 2.1. *Consider the involution $\pi = 79435611028$. The ltr minima of π are 7,4,3,1 and they form an involution themselves. The same is true for the rtl maxima 8 and 10.*

Given a word $w = w_1 \dots w_j$ whose letters are distinct numbers, the *standardization* of w is the unique permutation π in S_j order isomorphic to w . If two words w and u have the same standardization we write $u \sim w$.

The *decomposition into connected components* of a permutation π is the finest way to write π as $\pi = w_1 w_2 \dots w_k$, where each w_i is a permutation of the symbols from $|w_1| + |w_2| + \dots + |w_{i-1}| + 1$ to $|w_1| + |w_2| + \dots + |w_{i-1}| + |w_i|$. Each w_i is called a *connected component* of π . A permutation is said to be *connected* if it is composed by only one connected component.

Example 2.2. *The decomposition into connected components of the permutation $\pi = 34215786$ is $\pi = w_1 w_2 w_3$, where $w_1 = 3421$, $w_2 = 5$, $w_3 = 786$.*

2.2 Alternating permutations

A permutation $\pi = \pi_1 \dots \pi_n$ is said to be *alternating* if $\pi_i < \pi_{i+1}$ if and only if i is odd and *reverse alternating* if and only if $\pi_i > \pi_{i+1}$ if and only if i is odd. Equivalently, a permutation π is alternating whenever $\text{Des}(\pi) = \{2, 4, 6, \dots\}$.

Example 2.3. *The permutation $\pi = 4615273$ is alternating, while $\sigma = 5372614$ is reverse alternating.*

Denote by S_n (I_n , A_n , RA_n , AI_n and RAI_n , respectively) the set of permutations (involutions, alternating permutations, reverse alternating permutations, alternating involutions, reverse alternating involutions, respectively) of length n .

The following lemma will be useful in the sequel.

Lemma 2.2. *i) Let $\pi = \pi_1 \dots \pi_n \in RAI_n$. Then 1 is in even position, n is in odd position, hence π_1 is even and π_n is odd.*

ii) Let $\pi = \pi_1 \dots \pi_n \in AI_n$. Then 1 is in odd position, n is in even position, hence π_1 is odd and π_n is even.

2.3 Pattern avoidance

A permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ *avoids* the pattern $\tau \in S_k$ if there are no indices i_1, i_2, \dots, i_k such that the subsequence $\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_k}$ is order isomorphic to τ .

Denote by $S_n(\tau)$ the set of permutations of length n avoiding τ and let $S(\tau) := \bigcup_n S_n(\tau)$. We will keep this notation also when S_n is replaced by other subsets of permutations, such as I_n , A_n , etc.

Notice that an involution avoids τ if and only if it avoids τ^{-1} .

The next trivial lemma will be useful in the sequel.

Lemma 2.3. *The reverse-complement map is a bijection between $AI_{2n+1}(\tau)$ and $RAI_{2n+1}(\tau^{rc})$, between $AI_{2n}(\tau)$ and $AI_{2n}(\tau^{rc})$ and between $RAI_{2n}(\tau)$ and $RAI_{2n}(\tau^{rc})$.*

2.4 Motzkin paths

A *Motzkin path* of length n is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, consisting of up steps U of the form $(1, 1)$, down steps D of the form $(1, -1)$, and horizontal steps H of the form $(1, 0)$, and lying weakly above the x -axis. As usual, a Motzkin path can be identified with a *Motzkin word*, namely, a word $w = w_1w_2 \dots w_n$ of length n in the alphabet $\{U, D, H\}$ with the constraint that the number of occurrences of the letter U is equal to the number of occurrences of the letter D and, for every i , the number of occurrences of U in the subword $w_1w_2 \dots w_i$ is not smaller than the number of occurrences of D . In the following we will not distinguish between a Motzkin path and the corresponding word. Denote by \mathcal{M}_n the set of Motzkin path of length n and by M_n its cardinality, the n -th Motzkin number (see sequence [A001006](#) in [17]). The *diod decomposition* of a Motzkin path m of even length $2n$ is the decomposition of m as $m = d_1d_2 \dots d_n$, where each d_i is a subword of m of length two. Each d_i in the diod decomposition of a Motzkin path is called a *diod* of m .

3. General results

In this section, we prove two general results that will be used in the paper.

Lemma 3.1. *Let τ be any permutation of $\{3, \dots, m\}$, $m \geq 3$, Then*

$$|AI_n(12\tau)| = |AI_n(21\tau)|.$$

Proof. We closely follow [15], where a similar result is proved for doubly alternating permutations, i.e., alternating permutations whose inverse is also alternating.

Our goal is to find a bijection between $AI_n(12\tau) = AI_n(12\tau, 12\tau^{-1})$ and $AI_n(21\tau) = AI_n(21\tau, 21\tau^{-1})$.

We will use the diagrammatic representation of a permutation, i.e., we will identify a permutation π in S_n with a $n \times n$ diagram with a dot at position (i, π_i) , for every i (notice that Ouchterlony [15] uses a slightly different definition for the diagram of a permutation).

A dot, d , in the diagram of a permutation π is called *active* if d is the 1 or 2 in any $12\tau, 12\tau^{-1}, 21\tau$ or $21\tau^{-1}$ pattern in π , or *inactive* otherwise. Also the pair of dots (d_1, d_2) is called an *active pair* if d_1d_2 is the 12 in a 12τ or $12\tau^{-1}$ pattern or the 21 in a 21τ or $21\tau^{-1}$ pattern.

We now define a Young diagram, λ_π , consisting of the part of the diagram of π which contains the active dots. For any two dots d_1, d_2 , let R_{d_1, d_2} be the smallest rectangle with bottom left coordinates $(1, 1)$, such that $d_1, d_2 \in R_{d_1, d_2}$. Define

$$\lambda_\pi = \bigcup R_{d_1, d_2},$$

where the union is overall active pairs (d_1, d_2) of π . It is clear from the definition that λ_π is indeed a Young diagram. Since π is an involution, its diagram is symmetric with respect to the main diagonal, and for every active dot in position (i, j) there is also an active dot in position (j, i) , hence λ_π is a Young diagram symmetric with respect to the main diagonal.

A *rook placement* of a Young diagram λ is placement of dots in its boxes, such that all rows and columns contain exactly one dot. If some of the rows or columns are empty we call it a *partial rook placement*. Furthermore, we say that a rook placement on λ avoids the pattern τ if no rectangle, $R \subseteq \lambda$, contains τ . Notice that the rook placement on λ_π induced by an involution π is symmetric.

Now, it has been proved by Jaggard [7, Theorem 4.2], that the number of symmetric rook placements on the self-conjugate shape μ avoiding the patterns 12τ and $12\tau^{-1}$ is equal to the number of symmetric rook placements on the self-conjugate shape μ avoiding the patterns 21τ and $21\tau^{-1}$.

We call two permutations π and σ of the same size *a-equivalent* if they have the same inactive dots, and write $\pi \sim_a \sigma$.

In the sequel, we will need the following three facts that are immediate consequences of Lemma 6.4, Lemma 6.3, and Lemma 6.2 in [15], respectively.

(I) $\pi \sim_a \sigma \Rightarrow \lambda_\pi = \lambda_\sigma$.

(II) If $\pi \in AI_n(12\tau, 12\tau^{-1}) \cup AI_n(21\tau, 21\tau^{-1})$ and $\pi \sim_a \sigma$ then σ is doubly alternating.

(III) If $\pi \in AI_n$ and $rp(\lambda_\pi)$ is the partial rook placement on λ_π induced by π , then

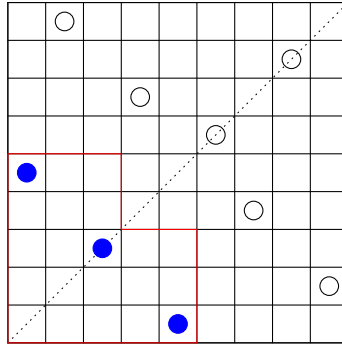
$$\pi \in AI_n(12\tau, 12\tau^{-1}) \Leftrightarrow rp(\lambda_\pi) \text{ is } 12\text{-avoiding and } \pi \in AI_n(21\tau, 21\tau^{-1}) \Leftrightarrow rp(\lambda_\pi) \text{ is } 21\text{-avoiding.}$$

Now we are ready to construct a bijection

$$\phi : AI_n(12\tau, 12\tau^{-1}) \rightarrow AI_n(21\tau, 21\tau^{-1}).$$

Let $\pi \in AI_n(12\tau, 12\tau^{-1})$, so that the restriction of π to λ_π is a partial 12-avoiding symmetric rook placement. By Jaggard's theorem (ignoring the empty rows and columns) and by point **(I)** there exists a unique 21-avoiding (partial) symmetric rook placement on λ_π with the same rows and columns empty, which we combine with the inactive dots of π to get $\phi(\pi)$. By point **(II)**, $\phi(\pi)$ is doubly alternating and, since its diagram is symmetric, it is an involution. By point **(III)** it avoids 21τ (and hence also $21\tau^{-1}$). It is also clear from Jaggard's theorem that ϕ is indeed a bijection. □

Example 3.1. Consider the permutation $\pi = 593716482 \in AI_9(12435)$. Notice that π contains the pattern 21435. Here $\tau = 435$. The diagram of π is the following



The blue dots are the active dots of π and the red Young diagram is λ_π . Now applying to λ_π the procedure described in the above proof we get $\phi(\pi) = 195736482 \in AI_9(21435)$.

It follows from the previous lemma and Lemma 2.3 that if τ is any permutation of $\{1, \dots, k - 2\}$ then

$$|AI_{2n}(\tau k - 1 k)| = |AI_{2n}(\tau k k - 1)| \text{ and } |RAI_{2n+1}(\tau k - 1 k)| = |RAI_{2n+1}(\tau k k - 1)|.$$

Notice that similar relations do not hold for RAI_{2n} , in fact, numerical computations show that, for general n , $|RAI_{2n}(1234)| \neq |RAI_{2n}(1243)|$ and $|RAI_{2n}(1234)| \neq |RAI_{2n}(2134)|$.

When τ is an increasing sequence it is possible to provide a more explicit bijection f between $|AI_{2n}(\tau k - 1 k)|$ and $|AI_{2n}(\tau k k - 1)|$. Such a bijection has been defined for the first time by J. West [21] for permutations and has been used by M. Bóna [3] to prove a conjecture by J. B. Lewis about alternating permutations. To this aim, Bóna proved that f preserves the alternating property when the length of the permutation is even. Here we recall the definition of the map $f : S_t(12 \dots k - 1 k) \rightarrow S_t(12 \dots k k - 1)$. Consider a permutation $\pi \in S_t(12 \dots k - 1 k)$ and define the *rank* of the element π_i to be the maximum length of an increasing subsequence ending at π_i . Since π avoids $12 \dots k - 1 k$, the maximal rank of an element is $k - 1$. Let R be the set of elements of π whose rank is $k - 1$, and P the set of their positions. The permutation $\rho = f(\pi)$ is obtained as follows.

- if $j \notin P$, $\rho_j = \pi_j$,
- if $j \in P$, ρ_j is the smallest unused element of R that is larger than the closest entry of rank $k - 2$ to the left of π_j .

Notice that, if $k = 3$, the map f reduces to the classic Simion-Schmidt bijection ([16], [21]).

In the following lemma, we prove that f preserves also the property of being an involution and we describe what happens in the odd case.

Lemma 3.2. The map f is a bijection between $AI_{2n}(12 \dots k - 1 k)$ and $AI_{2n}(12 \dots k k - 1)$. Moreover, f maps bijectively the subset of $AI_{2n+1}(1234)$ of those permutations having $2n + 1$ at position 2 to the set $AI_{2n+1}(1243)$.

Proof. First of all, we notice that, in an involution π , all the elements of a given rank form an involution. To prove this fact it is sufficient to observe that elements of π of rank 1 are the ltr minima of the permutation, which form an involution (see Lemma 2.1). We can consider the involution obtained by removing from π the elements of rank 1 and proceed inductively.

Now we want to prove that f sends involutions of length n to involutions of length n (for every length n). Also in this case we can obtain the result by induction on k . When $k = 3$, we noticed above that f coincides with the Simion-Schmidt map ([16]), and it is well-known that this map sends involutions to involutions. If $k > 3$, we can delete from π the elements of rank 1 and standardize, obtaining an involution π' which avoids $12 \dots k - 2 k - 1$. If we apply the map f , we obtain an involution avoiding $12 \dots k - 1 k - 2$ which can be obtained from $f(\pi)$ by removing the elements of rank 1 and standardizing. Hence also $f(\pi)$ is an involution. The fact that f preserves the alternating property (when the length is even) has been proved by Bóna [3, Theorem 1].

Now we turn our attention to the second assertion. We want to prove that f^{-1} maps bijectively the set $AI_{2n+1}(1243)$ onto the set $Y_{2n+1} = AI_{2n+1}(1234) \cap \{\pi \in S_{2n+1} \mid \pi(2) = 2n + 1\}$.

The fact that the injective map f^{-1} preserves the alternating property has been already observed by Bóna [3, Corollary 1]. Moreover, f^{-1} preserves the involutory property, as proved above.

Now consider $\pi \in AI_{2n+1}(1243)$. The symbol $2n + 1$ appears at even position a . Suppose that $a \geq 4$. Then $\pi_2 = 2n$, otherwise $\pi_1 \pi_2 2n + 1 2n \sim 1243$. As a consequence, $\pi_{2n+1} = a > 2 = \pi_{2n}$ which is impossible since π is alternating. Hence $a = 2$ and $\pi_2 = 2n + 1$.

Since for every $\pi \in AI_{2n+1}(1243)$, $\pi(2) = 2n + 1$, the symbols $2n + 1$ and 2 have rank 2. The map f^{-1} fixes the elements of rank $k - 2$ or less, hence it fixes the positions of $2n + 1$ and 2 .

With similar arguments one can show that $f(Y_{2n+1}) \subseteq AI_{2n+1}(1243)$ and conclude that $f(Y_{2n+1}) = AI_{2n+1}(1243)$. □

Example 3.2. Consider the permutation $\pi = 59710183624 \in AI_{10}(1234)$. The elements of rank 1 are 5 and 1, the elements of rank 2 are 9, 7, 3 and 2, the elements of rank 3 are 10, 8, 6 and 4. Then $f(\pi) = 59781103426 \in AI_{10}(1243)$.

4. Patterns of length three

The following proposition is an easy consequence of Proposition 3.1 in [15]. See also [14].

Proposition 4.1. *We have*

$$\begin{aligned} |AI_n(123)| &= |AI_n(213)| = |AI_n(231)| = |AI_n(321)| \\ &= |RAI_n(132)| = |RAI_n(231)| = |RAI_n(312)| = |RAI_n(321)| = 1, \\ |AI_n(132)| &= |RAI_n(213)| = \begin{cases} 1 & \text{if } n \text{ is even or } n = 1 \\ 0 & \text{otherwise,} \end{cases} \\ |AI_n(321)| &= |RAI_n(123)| = \begin{cases} 2 & \text{if } n \text{ is even and } n \geq 4 \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

5. Patterns 4321 and 1234

We recall the definition of a bijection Φ between the set $I_n(4321)$ of 4321-avoiding involutions of the symmetric group S_n and the set \mathcal{M}_n of Motzkin paths of length n (see [1]), that is essentially a restriction to the set $I_n(4321)$ of the bijection appearing in [2].

Consider an involution τ avoiding 4321 and determine the set $exc(\tau) = \{i \mid \tau(i) > i\}$ of its excedances. Start from the empty path and create the Motzkin path $\Phi(\tau)$ by adding a step for every integer $1 \leq i \leq n$ as follows:

- if $\tau(i) = i$, add a horizontal step at the i -th position;
- if $\tau(i) > i$, add an up step at the i -th position;
- if $\tau(i) < i$, add a down step at the i -th position.

The map Φ is a bijection whose inverse can be described as follows.

Given a Motzkin path M , let $A = (a_1, \dots, a_k)$ be the list of positions of up steps in M , written in increasing order, and let $B = (b_1, \dots, b_k)$ the analogous list of positions of down steps. Then $\Phi^{-1}(M)$ is the involution τ given by the product of the cycles (a_i, b_i) for $1 \leq i \leq k$.

Lemma 5.1. Φ maps the set $RAI_{2n}(4321)$ onto the set $\widehat{\mathcal{M}}_{2n}$ of Motzkin paths of length $2n$ whose diods are either of the forms UH , HD or UD .

Proof. Let $\tau \in RAI_{2n}(4321)$. For every integer $k = 1, 2, \dots, 2n - 1$, if k is odd, and hence a descent of τ , $\tau_k \tau_{k+1}$ is mapped by Φ to one of the following pairs: UH , UD , HD . □

Theorem 5.1. *We have*

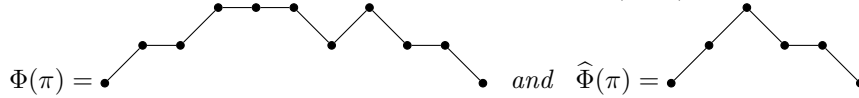
$$|RAI_{2n}(4321)| = M_n.$$

Proof. The previous lemma allows us to define a bijection $\hat{\Phi} : RAI_{2n}(4321) \rightarrow \mathcal{M}_n$. Let $m = d_1 d_2 \dots d_n$ be a Motzkin path (of even length) whose diods d_i are either of the form UH, UD or HD . Define $\Delta(m) := s_1 \dots s_n \in \mathcal{M}_n$ where

$$s_i = \begin{cases} U & \text{if } d_i = UH \\ D & \text{if } d_i = HD \\ H & \text{if } d_i = UD. \end{cases}$$

Set now $\hat{\Phi} = \Delta \circ \Phi$. □

Example 5.1. Consider $\pi = 62845110397 \in RAI_{10}(4321)$. Then



Theorem 5.2. We have

$$|RAI_{2n-1}(4321)| = M_n - M_{n-2}.$$

Proof. Consider a permutation $\pi \in RAI_{2n}(4321)$. By Lemma 2.2, π_{2n} is odd. Now consider the element $2n - 1$ in π . Notice that this element is at position $2n$ or $2n - 1$, otherwise π would contain the pattern $2n \ 2n - 1 \ \pi(2n - 1) \ \pi(2n) \sim 4321$ (notice that $2n$ appears before $2n - 1$ because π is an involution and $\pi_{2n-1} > \pi_{2n}$).

If $\pi_{2n} = 2n - 1$, the permutation π' obtained from π by removing the element $2n - 1$ and standardizing is an arbitrary element of $RAI_{2n-1}(4321)$ which ends with its maximum element.

If $\pi_{2n-1} = 2n - 1$, write π as $\pi = \begin{pmatrix} \dots & k & \dots & 2n-2 & 2n-1 & 2n \\ \dots & 2n & \dots & j & 2n-1 & k \end{pmatrix}$.

If $k > j$, consider the permutation π' obtained from π by removing the element $2n - 1$ and standardizing. π' is an arbitrary element of $RAI_{2n-1}(4321)$ which does not end with its maximum element.

If $k < j$, we must have $k = 2n - 3$ and $j = 2n - 2$, otherwise π would contain 4321 . Hence we can write $\pi = \pi' \ 2n \ 2n - 2 \ 2n - 1 \ 2n - 3$, where π' is an arbitrary permutation in $RAI_{2n-4}(4321)$.

As a consequence, $|RAI_{2n}(4321)| = |RAI_{2n-1}(4321)| + |RAI_{2n-4}(4321)|$. □

The preceding lemma implies that the sequence $(|AI_{2n-1}(4321)|)_{n \geq 1}$ coincides with sequence [A102071](#) in [17].

Theorem 5.3. We have

$$|AI_n(1234)| = |RAI_n(4321)| \text{ and } |RAI_n(1234)| = |AI_n(4321)|.$$

Proof. The *Robinson-Schensted* map RS (see e.g. [20]) associates an involution π with a standard Young Tableau whose descent set D is equal to the descent set of π . If we apply the inverse of the map RS to the transposed Tableau we get another involution π' whose descent set is $[n - 1] \setminus D$. In particular, π is alternating if and only if π' is reverse alternating. Moreover, by the properties of the map RS , π avoids 1234 if and only if the corresponding tableau has at most three columns, hence the transposed tableau has at most three rows and π' avoids 4321 . □

Theorems 5.1, 5.2 and 5.3 and Lemma 2.3 imply the following result.

Corollary 5.1. We have

$$|RAI_{2n}(4321)| = |AI_{2n}(1234)| = M_n$$

and

$$|RAI_{2n-1}(1234)| = |AI_{2n-1}(4321)| = |RAI_{2n-1}(4321)| = |AI_{2n-1}(1234)| = M_n - M_{n-2}.$$

We now give an explicit formula for the cardinalities of the sets $RAI_{2n}(1234)$ and $AI_{2n}(4321)$.

Theorem 5.4. We have

$$|RAI_{2n}(1234)| = |AI_{2n}(4321)| = M_{n+1} - 2M_{n-1} + M_{n-3}.$$

Proof. We proceed as in the proof of Theorem 5.2. Consider a permutation $\pi \in AI_{2n+1}(4321)$. The symbol $2n$ is either at position $2n$ or $2n + 1$, otherwise π would contain 4321 .

If $\pi_{2n+1} = 2n$, the permutation π' obtained from π by removing the element $2n + 1$ and standardizing is an arbitrary element of $AI_{2n}(4321)$ which ends with its maximum element.

If $\pi_{2n} = 2n$, set $\pi = \begin{pmatrix} \dots & k & \dots & 2n-1 & 2n & 2n+1 \\ \dots & 2n+1 & \dots & j & 2n & k \end{pmatrix}$.

If $j < k$, the permutation π' obtained from π by removing the element $2n$ and standardizing is an arbitrary element of $AI_{2n}(4321)$ which does not end with its maximum element.

If $j > k$, the element $j = 2n - 1$ must be fixed and $k = 2n - 2$, otherwise π would contain an occurrence of 4321. Removing from π the last four elements we get an arbitrary permutations in $AI_{2n-3}(4321)$.

Hence $|AI_{2n+1}(4321)| = |AI_{2n}(4321)| + |AI_{2n-3}(4321)|$ and the assertion now follows from Corollary 5.1. \square

6. Pattern 3412

Theorem 6.1. *We have*

$$|RAI_{2n}(3412)| = |AI_{2n+2}(3412)| = |AI_{2n+1}(3412)| = |RAI_{2n+1}(3412)| = M_n.$$

Proof. We recall that also the set $I_n(3412)$ corresponds bijectively to the set \mathcal{M}_n via a map Ψ whose definition coincides with the definition of the map Φ of Section 5 (see [1]). It is easily seen that Ψ maps the set $RAI_{2n}(3412)$ to the set of Motzkin paths in \mathcal{M}_{2n} whose diods are either of the form UH , HD , or UD . Hence the map $\widehat{\Psi} = \Delta \circ \Psi$ is a bijection between $RAI_{2n}(3412)$ and \mathcal{M}_n , where Δ is the map defined in the proof of Theorem 5.1.

We observe that any permutation in $AI_t(3412)$ begins with 1, hence, a permutation $\pi \in AI_{2n+1}(3412)$ can be written as $\pi = 1\pi'$, where π' is an arbitrary permutation in $RAI_{2n}(3412)$. Moreover, any permutation π in $AI_{2n+2}(3412)$ ends with its maximum element, hence it can be written as $\pi = 1\pi'2n+2$ where π' is an arbitrary permutation in $RAI_{2n}(3412)$.

The last equality follows from Lemma 2.3. \square

7. Patterns 2143, 2134 and 1243

Theorem 7.1. *We have*

$$|AI_{2n}(1243)| = |AI_{2n}(2143)| = |AI_{2n}(2134)| = M_n$$

and

$$|AI_{2n-1}(2134)| = |RAI_{2n-1}(1243)| = M_n - M_{n-2}.$$

Proof. It is an immediate consequence of Lemma 2.3, Lemma 3.1 and Corollary 5.1. \square

Theorem 7.2. *We have*

$$|AI_{2n+1}(1243)| = |AI_{2n+1}(2143)| = |RAI_{2n+1}(2143)| = |RAI_{2n+1}(2134)| = M_n.$$

Proof. By Lemmas 2.3, 3.1 and 3.2, it suffices to show that $|AI_{2n+1}(1234) \cap \{\sigma \in S_{2n+1} \mid \sigma(2) = 2n+1\}| = M_n$.

If π is a permutation in this last set, removing $2n+1$ and 2 from π and standardizing we get an arbitrary permutation $\widehat{\pi}$ in $I_{2n-1}(1234)$ with either $\text{Des}(\widehat{\pi}) = \{3, 5, 7, \dots, 2n-3\}$ or $\text{Des}(\widehat{\pi}) = \{1, 3, 5, 7, \dots, 2n-3\}$. In other words, this permutation is reverse alternating or it is reverse alternating only to the right of the second position. Hence we have

$$\begin{aligned} & |AI_{2n+1}(1234) \cap \{\sigma \in S_{2n+1} \mid \sigma(2) = 2n+1\}| \\ &= |I_{2n-1}(1234) \cap \{\sigma \in S_{2n-1} \mid \text{Des}(\sigma) = \{3, 5, 7, \dots, 2n-3\}\}| + |RAI_{2n-1}(1234)|. \end{aligned}$$

We want to show that the sum on the right-hand side of the previous equality is equal to M_n . We proceed by induction on n . The assertion is trivial when $n = 2$ or $n = 3$. Assume that the assertion is true for every $m < n$. We know by Corollary 5.1 that $|RAI_{2n-1}(1234)| = M_n - M_{n-2}$. Consider a permutation ρ in $I_{2n-1}(1234) \cap \{\sigma \in S_{2n-1} \mid \text{Des}(\sigma) = \{3, 5, 7, \dots, 2n-3\}\}$. Notice that $\rho(2) = 2n-2$ and $\rho(3) = 2n-1$, otherwise the permutation would contain the subsequence $\rho(1) \rho(2) 2n-2 2n-1 \sim 1234$. Removing $2, 3, 2n-2$ and $2n-1$ and standardizing we get an arbitrary permutation in $I_{2n-5}(1234) \cap \{\sigma \in S_{2n-5} \mid \text{Des}(\sigma) = \{3, 5, 7, \dots, 2n-7\}\}$ or in $RAI_{2n-5}(1234)$. By the inductive hypothesis we have

$$|I_{2n-5}(1234) \cup \{\sigma \in S_{2n-5} \mid \text{Des}(\sigma) = \{3, 5, 7, \dots, 2n-7\}\}| + |RAI_{2n-5}(1234)| = M_{n-2}.$$

This completes the proof. \square

Example 7.1. Consider $\pi = 5119101768342 \in AI_{11}(1234)$. If we remove 11 and 2 from π and standardize we get the involution $\widehat{\pi} = 489165723$ with $\text{Des}(\widehat{\pi}) = \{3, 5, 7\}$. Removing from $\widehat{\pi}$ the symbols 8, 9, 2 and 3 we get the involution $21435 \in RAI_5(1234)$.

Consider now the permutation $\pi = 9117851034162$. If we remove 11 and 2 from π and standardize we get the involution $\widehat{\pi} = 867492315 \in RAI_9(1234)$.

Theorem 7.3. *We have*

$$|RAI_{2n}(2143)| = M_{n-1}.$$

Proof. Let π be a permutation in $RAI_{2n}(2143)$. The maximum of π is in position 1, otherwise $21\pi_{2n-1}\pi_{2n}$ would be an occurrence of 2143 in π . Write π as $2n\hat{\pi}1$. Then the standardization of $\hat{\pi}$ is an arbitrary permutation in $AI_{2n-2}(2143)$. The assertion now follows from Theorem 7.1. \square

8. Patterns 3421 and 4312

Notice that these two patterns are inverse of each other. Hence $AI_n(3421) = AI_n(4312) = AI_n(3421, 4312)$ and the same is true for reverse alternating involutions. The following theorem shows that all these classes are enumerated by the Fibonacci numbers F_n (sequence [A000045](#) in [17]).

Theorem 8.1. *We have*

$$|AI_n(3421, 4312)| = |RAI_n(3421, 4312)| = F_{n-1},$$

where F_k is the k -th Fibonacci number.

Proof. Let $\pi \in RAI_{2n}(3421, 4312)$. The last element of π is odd and the position of $2n$ is odd by Lemma 2.2.

Notice that the symbol $2n - 1$ can be either at position $2n$ or $2n - 1$, otherwise π would contain the subsequence $2n\ 2n - 1\ \pi(2n - 2)\ \pi(2n - 1)$ whose standardization is 4312.

If $\pi_{2n} = 2n - 1$, then $\pi = \pi' 2n\ 2n - 1$, where π' is an arbitrary permutation in $RAI_{2n-2}(3421, 4312)$.

If $\pi_{2n-1} = 2n - 1$, let $y = \pi_{2n-2}$ and $x = \pi_{2n}$. If $x > y$, removing from π the symbol $2n - 1$ and standardizing we obtain an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$ which does not end with its maximum.

If $x < y$, notice that either $\pi(2n - 2) = 2n - 2$ or $\pi(2n - 3) = 2n - 2$ (otherwise π would contain the patterns). If $\pi(2n - 2) = 2n - 2$, then $\pi(2n - 3)$ is forced to be $2n$ (because π is alternating), so we can write $\pi = \tau 2n\ 2n - 2\ 2n - 1\ 2n - 3$, where $\tau \in RAI_{2n-4}(3421, 4312)$. We associate to π the permutation $\pi' = \tau 2n - 2\ 2n - 3\ 2n - 1$. Such π' is an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$ which ends with its maximum and in which $2n - 3$ precedes the maximum.

If $\pi(2n - 3) = 2n - 2$, we write $\pi = \sigma' 2n\ \sigma'' 2n - 2\ 2n - 3\ 2n - 1\ x$ and we associate to π the permutation $\pi' = \sigma' 2n - 2\ \sigma'' 2n - 3\ x\ 2n - 1$ which is an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$ which ends with its maximum and in which $2n - 3$ follows the maximum.

As a consequence

$$|RAI_{2n}(3421, 4312)| = |RAI_{2n-1}(3421, 4312)| + |RAI_{2n-2}(3421, 4312)|.$$

Let $\pi \in RAI_{2n+1}(3421, 4312)$. The last element of π is odd and the position of $2n + 1$ is odd by Lemma 2.2. Notice that the symbol $2n + 1$ is either at position $2n + 1$ or $2n - 1$, otherwise π would contain the subsequence $2n\ 2n + 1\ \pi(2n - 1)\ \pi(2n) \sim 3421$.

If $\pi_{2n+1} = 2n + 1$, then $\pi = \pi' 2n + 1$, where π' is an arbitrary permutation in $RAI_{2n}(3421, 4312)$.

If $\pi_{2n-1} = 2n + 1$, write $\pi = \tau 2n + 1\ x\ 2n - 1$. Then $\pi' = \tau 2n - 1\ x$ is an arbitrary permutation in $RAI_{2n}(3421, 4312)$ with $2n - 1$ as a fixed point. We observed above that such permutations are in bijection with $RAI_{2n-1}(3421, 4312)$.

As a consequence

$$|RAI_{2n+1}(3421, 4312)| = |RAI_{2n}(3421, 4312)| + |RAI_{2n-1}(3421, 4312)|. \tag{1}$$

Hence we can conclude that

$$|RAI_m(3421, 4312)| = F_{m-1}.$$

By Lemma 2.3, we have also $|AI_{2n+1}(3421, 4312)| = |RAI_{2n+1}(3421, 4312)|$.

Consider now $\pi \in AI_{2n}(3421, 4312)$. By Lemma 2.2, π_1 is odd and the symbol 1 is in odd position. Notice that either $\pi(1) = 1$ or $\pi(3) = 1$ otherwise π would contain the subsequence $\pi_1\ \pi_2\ 2\ 1$, whose standardization is 3421.

If $\pi_1 = 1$, the standardization of $\pi_2 \dots \pi_{2n}$ is an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$. If $\pi(3) = 1$, write $\pi = 3\ \pi_2\ 1\ \tau$. The standardization of the word $\pi_2\ 3\ \tau$ is an arbitrary permutation π' in $RAI_{2n-1}(3421, 4312)$ which fixes the symbol 2.

Observe that, if $\pi \in RAI_{2n-1}(3421, 4312)$, then $\pi(1) = 2$ or $\pi(2) = 2$, namely $RAI_{2n-1}(3421, 4312) = A \cup B$ where $A = RAI_{2n-1}(3421, 4312) \cap \{\pi \in S_{2n-1} \mid \pi(2) = 1\}$ and $B = RAI_{2n-1}(3421, 4312) \cap \{\pi \in S_{2n-1} \mid \pi(2) = 2\}$.

The set A corresponds bijectively to the set $RAI_{2n-3}(3421, 4312)$ by removing the first two elements, hence the set B corresponds bijectively to the set $RAI_{2n-2}(3421, 4312)$ by equation (1).

Hence,

$$|AI_{2n}(3421, 4312)| = |RAI_{2n-1}(3421, 4312)| + |RAI_{2n-2}(3421, 4312)|$$

and

$$|AI_m(3421, 4312)| = F_{m-1}.$$

□

Example 8.1. Consider the set $RAI_6(3421, 4312)$, whose elements are

$$\alpha = 216453 \quad \beta = 214365 \quad \gamma = 426153 \quad \delta = 423165 \quad \epsilon = 624351.$$

The permutations β and δ correspond to 2143 and 4231, namely, the elements of the set $RAI_4(3421, 4312)$, whereas the permutations α, γ and ϵ correspond to 21435, 42513 and 52341, respectively, namely, the elements of $RAI_5(3421, 4312)$.

9. Patterns 2431, 4132, 3241 and 4213.

Notice that the two patterns 2431 and 4132 are inverse of each other. Hence $AI_n(2431) = AI_n(4132) = AI_n(2431, 4132)$ and the same is true for reverse alternating involutions and for the pair of patterns (3241, 4213).

Theorem 9.1. We have

$$|RAI_{2n}(2431, 4132)| = |RAI_{2n}(3241, 4213)| = |RAI_{2n+1}(3241, 4213)| = |AI_{2n+1}(2431, 4132)| = 2^{n-1}.$$

Proof. Consider a permutation $\pi \in RAI_{2n}(2431, 4132)$. We want to show that $\pi = \tau' 2n \tau''$, where τ' is a permutation of the symbols $1, 2, \dots, i-1$ and τ'' is a permutation of the symbols $i, \dots, 2n-1$.

Suppose on the contrary that there exists an element b in τ' such that $b > i$, so that $\pi = \dots b \dots 2n \dots i$. Let $a = \pi_{2n-1}$. Since π is alternating, we have $a > i$, hence $2n-1$ follows $2n$ in π and the subsequence $b 2n 2n-1$ is order-isomorphic to 2431.

As a consequence, the word $2n \tau''$ is itself a reverse alternating involution avoiding 132, and there exists only one of such permutations, as observed in Section 4, while the word τ' is an arbitrary permutation in $RAI_j(2431, 4132)$, j even.

We can conclude that

$$|RAI_{2n}(2431, 4132)| = \sum_{j \text{ even}, j < 2n} |RAI_j(2431, 4132)| = 2^{n-1},$$

where the last equality follows from the previous one by induction.

The facts that

$$|RAI_{2n}(2431, 4132)| = |RAI_{2n}(3241, 4213)| \text{ and } |RAI_{2n+1}(3241, 4213)| = |AI_{2n+1}(2431, 4132)|$$

follow by Lemma 2.3.

Consider now a permutation π in $RAI_{2n+1}(4213, 3241)$.

Write $\pi = m_1 w_1 m_2 w_2 \dots m_k w_k$, where the m_i 's are the ltr maxima of π and the w_i 's are words. Since π is alternating and avoids the patterns 4213 and 3241, it follows that for every $i = 1, 2, \dots, k-1$ the word w_i with is non empty and that $w_i < w_{i+1}$, namely, every symbol in w_i is smaller than any symbol in w_{i+1} .

As a consequence $m_i w_i$ is an interval for every i . Obviously $m_k = 2n+1$ and its position is π_{2n+1} since π is an involution. Suppose that $\pi_{2n+1} \neq 2n+1$. Since $\pi_{2n} < \pi_{2n+1}$, the symbol $2n$ should appear to the left of $2n+1$ in π , but this contradicts the fact that $m_k w_k$ is an interval. Hence $2n+1$ is the last element of π and removing it from π we get an arbitrary permutation in $RAI_{2n}(4213, 3241)$, so we have

$$|RAI_{2n}(4213, 3241)| = |RAI_{2n+1}(4213, 3241)|.$$

□

Theorem 9.2. We have

$$|RAI_{2n+1}(2431, 4132)| = |AI_{2n+1}(3241, 4213)| = \left\lfloor \frac{2^{n-1} \cdot 5}{3} \right\rfloor.$$

To prove this theorem, we need the following lemma.

Lemma 9.1. The number of connected permutations in $RAI_{2l+1}(4132, 2431)$ is

$$\left\lfloor \frac{2l+1}{4} \right\rfloor.$$

Proof. Let $\tau \in RAI_{2l+1}(4132, 2431)$ be a connected permutations. Consider the position i of $2l + 1$ in τ . This position is odd by lemma 2.2 and $i \neq 2l + 1$ since τ is connected.

If $i = 2l - 1$, we can remove from τ the symbols $2l + 1$ and $2l - 1$ obtaining an arbitrary connected permutation τ in $RAI_{2l-1}(2431, 4132)$.

If $i < 2l - 1$, let $a = \tau_{2l}$, $b = \tau_{2l-1}$ and $c = \tau_{2l-2}$.

We can write $\tau = (\dots \begin{matrix} a & \dots & i & \dots & b & \dots & 2l-2 & 2l-1 & 2l & 2l+1 \\ w_1 & 2l & w_2 & 2l+1 & w_3 & 2l-1 & w_4 & c & b & a & i \end{matrix} \dots)$ where, since τ is alternating, $c < b > a < i$.

If $bai \sim 312$, since τ avoids 2431, the elements of w_1 and w_2 are smaller than the elements of $w_4 c b a i$ that, in turn, are smaller than the elements of w_3 . But now, if $a \neq i - 1$, we would have an occurrence of 4132. Hence $a = i - 1$, and this is impossible since τ is connected (w_1 would be a connected component of τ).

Hence $bai \sim 213$. Notice that we must have $cba \sim 231$ and we can write

$$\tau = \left(\dots \begin{matrix} a & \dots & c & \dots & b & \dots & i & \dots & 2l-2 & 2l-1 & 2l & 2l+1 \\ w_1 & 2l & w_2 & 2l-2 & w_2 & 2l-1 & w_3 & 2l+1 & w_4 & c & b & a & i \end{matrix} \dots \right),$$

where, as above, $w_1 < w_4 < w_3 < w_{2_2} < w_{2_1}$.

The word w_1 is empty, since τ is connected. If $i \neq 2l - 3$, we have $\tau_{2l-3} > \tau_{2l-2}$, since τ is alternating. Hence $2l - 3$ must follow $2l - 2$. So we can conclude that also w_{2_1} is empty and $c = 2$. Since $\tau_2 < \tau_3$, 3 must follows 2 in τ so $b = 3$ and w_{2_2} is also empty.

Iterating these arguments one can show that

$$\tau = 2l \ 2l - 2 \ 2l - 1 \ 2l - 4 \ 2l - 3 \ \dots \ l - 1 \ 2l + 1 \ l - 3 \ l - 2 \ \dots \ 2 \ 3 \ 1 \ l + 1.$$

In particular $i = l + 1$. Since i is odd, this implies that $2l + 1 \equiv 1 \pmod 4$.

As a consequence, the number of connected permutations in $RAI_{2l+1}(4132, 2431)$ is the same as the number of connected permutations in $RAI_{2l-1}(4132, 2431)$ if $2l + 1 \equiv 3 \pmod 4$ and increases by one if $2l + 1 \equiv 1 \pmod 4$.

As an example, when $2l + 1 = 21$ such permutations are

4 2 6 1 8 3 10 5 12 7 14 9 16 11 18 13 20 15 21 17 19
 8 6 7 4 10 2 3 1 12 5 14 9 16 11 18 13 20 15 21 17 19
 12 10 11 8 9 6 14 4 5 2 3 1 16 7 18 13 20 15 21 17 19
 16 14 15 12 13 10 11 8 18 6 7 4 5 2 3 1 20 9 21 17 19
 20 18 19 16 17 14 15 12 13 10 21 8 9 6 7 4 5 2 3 1 11

(notice that all of them but the last one have 21 at position 19 and the last one has 21 in position 11). When $2l + 1 = 23$ such permutations are

4 2 6 1 8 3 10 5 12 7 14 9 16 11 18 13 20 15 22 17 23 19 21
 8 6 7 4 10 2 3 1 12 5 14 9 16 11 18 13 20 15 22 17 23 19 21
 12 10 11 8 9 6 14 4 5 2 3 1 16 7 18 13 20 15 22 17 23 19 21
 16 14 15 12 13 10 11 8 18 6 7 4 5 2 3 1 20 9 22 17 23 19 21
 20 18 19 16 17 14 15 12 13 10 22 8 9 6 7 4 5 2 3 1 23 11 21

(all of them have the symbol 23 in position 21).

Since when $2l + 1 = 5$ there is only one of such permutations, the number of connected permutations in $RAI_{2l+1}(4132, 2431)$ is $\lfloor \frac{2l+1}{4} \rfloor$. □

Now we turn to the proof of Theorem 9.2.

Proof of Theorem 9.2. Let $\pi \in RAI_{2n+1}(2431, 4132)$. Consider the last connected component τ of π and write $\pi = \sigma\tau$.

Now, σ is an arbitrary permutation in $RAI_{2k}(2431, 4132)$, with $k \geq 0$ (notice that σ has even length, since τ is connected and π is alternating) and τ' , the standardization of τ , is an arbitrary connected permutation in $RAI_{2n+1-2k}(2431, 4132)$, with $l \geq 0$.

Hence, by Lemma 9.1 and by Theorem 9.1, we have

$$|RAI_{2n+1}(2431, 4132)| = 2^{n-1} + \sum_{k \geq 1}^{n-1} 2^{k-1} \left[\frac{2n+1-2k}{4} \right] + \left[\frac{2n+1}{4} \right],$$

where the first summand of the right-hand side corresponds to the case $\tau = 2n + 1$ and the third summand corresponds to the case $|\tau| = 2n + 1$.

Now it can be shown by induction that

$$|RAI_{2n+1}(2431, 4132)| = \left\lfloor \frac{2^{n-1} \cdot 5}{3} \right\rfloor.$$

The fact that $|AI_{2n+1}(3241, 4213)| = |RAI_{2n+1}(2431, 4132)|$ follows by Lemma 2.3.

Notice that $\{|RAI_{2n+1}(2431, 4132)|\}_n$ is sequence [A081254](#) in [17].

On the contrary, the sequence enumerating $AI_{2n}(3241, 4213)$ and $AI_{2n}(2431, 4132)$, whose first terms are 1, 2, 5, 9, 17, 31, 59, . . . , is not present in [17].

10. Patterns 2413 and 3142

The set $AI_n(2413, 3142)$ coincides with the set of alternating Baxter involutions of length n .

We recall that a *Baxter permutation* of length n is a permutation $\pi \in S_n$ such that, for every $1 \leq i \leq j \leq k \leq l \leq n$,

$$\text{if } \pi_i + 1 = \pi_l \text{ and } \pi_j > \pi_l \text{ then } \pi_k > \pi_l \quad \text{and} \quad \text{if } \pi_l + 1 = \pi_i \text{ and } \pi_k > \pi_i \text{ then } \pi_j > \pi_i.$$

In [15] the author shows that the set of doubly alternating Baxter permutations coincides with the set of doubly alternating permutations avoiding 2413 and 3142 = 2413⁻¹.

An involution is clearly doubly alternating. As a consequence, the set $AI_n(2413, 3142)$ coincides with the set of alternating Baxter involutions of length n . A recurrence relation for this sequence has been found in [13] and the corresponding sequence in [17] is [A347546](#).

Moreover we have

$$|AI_{2n+1}(2413, 3142)| = |RAI_{2n+1}(2413, 3142)| = |RAI_{2n}(2413, 3142)|,$$

where the first equality follows from Lemma 2.3 and the second one is a consequence of the fact that each permutation in $RAI_{2n+1}(2413, 3142)$ ends with its maximum (see [13]) and this maximum can be removed, hence obtaining any permutation in $RAI_{2n}(2413, 3142)$.

11. Patterns 4123, 2341

Since 4123 = 2341⁻¹ we have $AI_n(4123) = AI_n(2341) = AI_n(4123, 2341)$ and the same is true for reverse alternating involutions.

To enumerate these classes we need the following lemma.

Lemma 11.1. *The number of connected permutations in $RAI_n(4123, 2341)$ is*

$$\begin{cases} 1 & \text{if } n \text{ is odd and } n \neq 3 \\ 2 & \text{if } n \text{ is even and } n \geq 6 \\ 0 & \text{if } n = 3 \\ 1 & \text{if } n = 2, 4. \end{cases}$$

The number of connected permutations in $AI_n(4123, 2341)$ is

$$\begin{cases} 1 & \text{if } n \geq 4 \text{ or } n = 1 \\ 0 & \text{if } n = 2 \text{ or } 3 \end{cases}.$$

Proof. The small cases are trivial, so we can assume $n \geq 5$.

Let $\pi \in RAI_{2n}(4123, 2341)$ be connected. Write $\pi = w_k m_k w_{k-1} m_{k-1} \dots m_2 w_1 m_1$ where the m_i 's are the rtl maxima of π .

Since π is reverse alternating and avoids 4123, every w_i with $2 \leq i \leq k - 1$ must have length 1. For the same reason, w_1 must be empty (π ends with a descent) and the length of w_k is 0 or 2.

If $|w_k| = 0$,

$$\pi = 2n 2n - 2 2n - 1 2n - 4 2n - 3 \dots 4 5 2 3 1,$$

if $|w_k| = 2$,

$$\pi = 2n - 2 2n - 4 2n 2n - 6 2n - 1 2n - 8 2n - 3 \dots 2 7 1 5 3.$$

In a similar way, it is possible to prove that the only connected permutation in $RAI_{2n+1}(4123, 2341)$ is

$$\pi = 2n 2n - 2 2n + 1 2n - 4 2n - 1 2n - 6 2n - 3 \dots 9 4 7 2 5 1 3.$$

The classification of the connected permutations in $AI_n(4123, 2341)$ is fully analogous. □

Example 11.1. *There are two connected permutations in $RAI_{10}(4123, 2341)$, namely,*

$$86104927153 \text{ and } 10896745231,$$

while the only connected permutation in $RAI_{11}(4123, 2341)$ is 1081169472513. The only connected permutation in $AI_{10}(4123, 2341)$ is 91078563412 and the only connected permutation in $AI_{11}(4123, 2341)$ is 9117105836142.

In the following theorem we find the ordinary generating functions that enumerate reverse alternating involutions avoiding 4123 and 2341 of even and odd length (sequences [A052980](#) and [A193641](#) in [17], respectively). We also find the same generating function for the alternating even case. This last sequence does not appear in [17].

Theorem 11.1. *We have*

$$\begin{aligned} \sum_{n \geq 0} |AI_{2n+1}(4123, 2341)|x^{2n+1} &= \sum_{n \geq 0} |RAI_{2n+1}(4123, 2341)|x^{2n+1} = \frac{x^5 - x^3 + x}{1 - 2x^2 - x^6}, \\ \sum_{n \geq 0} |RAI_{2n}(4123, 2341)|x^{2n} &= \frac{1 - x^2}{1 - 2x^2 - x^6}, \\ \sum_{n \geq 0} |AI_{2n}(4123, 2341)|x^{2n} &= 1 + \frac{x^4}{1 - x^2} + \frac{(x^5 - x^3 + x)^2}{(1 - 2x^2 - x^6) \cdot (1 - x^2)}. \end{aligned}$$

Proof. The fact that $|RAI_{2n+1}(4123, 2341)| = |AI_{2n+1}(4123, 2341)|$ follows from Lemma 2.3.

Let $\pi \in RAI_{2n+1}(4123, 2341)$. Decompose π in connected components as $\pi = \tau_1\tau_2 \dots \tau_k$. Denote by $\widehat{\tau}_i$ the standardization of τ_i . The $\widehat{\tau}_i$'s are arbitrary connected permutations with $\widehat{\tau}_i \in \cup_{j \geq 1} RAI_{2j}(4123, 2341)$ for $1 \leq i \leq k - 1$ (because π is reverse alternating) and $\tau_k \in \cup_{j \geq 0} RAI_{2j+1}(4123, 2341)$. By the previous lemma it follows that the ordinary generating function that counts connected permutations in $\cup_{j \geq 1} RAI_{2j}(4123, 2341)$ (by length) is $F(x) = x^2 + x^4 + \frac{2x^6}{1-x^2}$ and the ordinary generating function that counts connected permutations in $RAI_{2j+1}(4123, 2341)$ is $G(x) = x + \frac{x^5}{1-x^2}$.

As a consequence

$$\sum_{n \geq 0} |RAI_{2n+1}(4123, 2341)|x^{2n+1} = \frac{G(x)}{1 - F(x)}.$$

Trivial algebraic manipulations lead to the desired generating function.

Similarly, given $\pi \in RAI_{2n}(4123, 2341)$ we can decompose π in connected components as $\pi = \tau_1\tau_2 \dots \tau_k$ where $\widehat{\tau}_i \in \cup_{j \geq 1} RAI_{2j}(4123, 2341)$ for $1 \leq i \leq k$.

As a consequence

$$\sum_{n \geq 0} |RAI_{2n}(4123, 2341)|x^{2n} = \frac{1}{1 - F(x)}.$$

The decomposition in connected components of a permutation $\pi \in AI_{2n}(4123, 2341)$ is more subtle than the above ones because these components are not necessarily only of even or odd length. However such a permutation π can only be

- the empty one,
- connected itself,
- of the form $\sigma\tau$, where σ is an arbitrary permutation in $\cup_{j \geq 0} AI_{2j+1}(4123, 2341)$ and τ is a connected permutation in $\cup_{j \geq 0} RAI_{2j+1}(4123, 2341)$.

By the above results and by the previous lemma we get

$$\sum_{n \geq 0} |AI_{2n}(4123, 2341)|x^{2n} = 1 + \frac{x^4}{1 - x^2} + \frac{x^5 - x^3 + x}{1 - 2x^2 - x^6} \cdot \frac{x^5 - x^3 + x}{1 - x^2}$$

where the three summand on the left hand side correspond to the three-dotted cases. □

12. Other patterns

The following three conjectures are based on numerical evidences. The first one covers the last open case about the patterns 1243 and 2134 and the second covers the patterns 1432 and 3214.

Conjecture 12.1. *We have*

$$|RAI_{2n}(1243)| = |RAI_{2n}(2134)| = M_n.$$

Conjecture 12.2. *We have*

$$\begin{aligned} |AI_{2n}(1432)| &= |AI_{2n}(3214)| = |RAI_{2n}(1432)| = |RAI_{2n}(3214)| = M_n, \\ |AI_{2n+1}(1432)| &= |RAI_{2n+1}(3214)| = M_n, \\ |AI_{2n-1}(3214)| &= |RAI_{2n-1}(1432)| = M_n - M_{n-2}. \end{aligned}$$

Conjecture 12.3. *Let τ be any permutation of $\{4, \dots, m\}$, $m \geq 4$. Then*

$$|AI_n(123\tau)| = |AI_n(321\tau)|.$$

A classical subject in pattern avoidance is the study of Wilf-equivalent patterns. Let σ and τ two patterns and P_n a given subset of S_n . Then σ and τ are said to be *Wilf-equivalent* on P_n if $|P_n(\sigma)| = |P_n(\tau)|$.

A proof of Conjectures 12.1 and 12.2 would conclude the Wilf-classification of alternating and reverse alternating involutions avoiding a pattern of length 4. In fact, trivial numerical experiments show that the only Wilf-equivalences among all the other patterns not mentioned in the paper are the trivial ones given by the reverse complement map and the inverse map.

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