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A Formula Relating Bell Polynomials and Stirling Numbers of the First Kind

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Abstract: In this paper, we prove a general convolution formula involving the Bell polynomials and the Stirling numbers of the first kind. Our proof of the formula is algebraic and establishes an equivalent identity involving the associated exponential generating function, where we make use of induction, manipulation of finite sums and several identities to demonstrate the latter. A bijective proof that draws upon a sign-changing involution on the related combinatorial structure is given for a special case of the formula.

Keywords: Bell number; combinatorial identity; combinatorial proof; Stirling number

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1. Introduction

Let $s(n,k)$ and $S(n,k)$ denote the Stirling numbers of the first and second kind, respectively; see, e.g., [5, Section 6.1] or A008275 and A008277 in [14]. Recall that $|s(n,k)|=(-1)^{n-k}s(n,k)$ enumerates the permutations of $[n]=\{1,\dots,n\}$ that have exactly k cycles whereas $S(n,k)$ counts the partitions of $[n]$ with k blocks. The n-th Bell number [14, A000110] is given by $B_n=\sum_{k=0}^{n}S(n,k)$ and enumerates all partitions of size n. The complementary Bell number [14, A000587], which we will denote here by B_n^*, was introduced by Rao Uppuluri and Carpenter in [12] and is defined by the relation

$$\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}^*=0, \qquad n\geq 1,$$

with $B_0=B_0^*=1$. Note that B_n^* has exponential generating function (egf) formula

$$\sum_{n\geq 0}B_n^*\frac{x^n}{n!}=e^{1-e^x}.$$

The r-Bell number $B_n^{(r)}$ (see, e.g., [8]) enumerates the partitions of $[n+r]$ in which the elements of $[r]$ lie in distinct blocks and reduces to B_n when $r=0$.

Gould and Quaintance [4] found the following formulas relating the Bell numbers to the Stirling numbers of the first kind:

$$\sum_{m=0}^{p}B_{n+m}s(p,m)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}p^k$$

and

$$B_n=\sum_{k=0}^{n}(-p)^{n-k}\binom{n}{k}\sum_{m=0}^{p}B_{m+k}s(p,m).$$

Other related formulas involving the Stirling and Bell numbers appear in [1,11,13,15]. See also [12] for identities between B_n and B_n^*. Further, there is the following simple formula for $B_n^{(r)}$ in terms of the Bell and Stirling numbers of the first kind:

$$B_n^{(r)}=\sum_{j=0}^{r}s(r,j)B_{n+j}. \tag{1}$$

In this paper, we show the following formula relating the sequences $s(n,k)$, B_n, B_n^* and $B_n^{(r)}$:

$$\sum_{j=0}^{r}\sum_{i=0}^{n+j-1} s(r,j)\binom{n+j}{i}B_iB_{n+j-i-1}^* = \sum_{i=0}^{n-1}\binom{n}{i}B_i^{(r)}B_{n-i-1}^* - \sum_{j=0}^{r-1}(-1)^{r-j}(r-j-1)!j^n, \tag{2}$$

where $n,r\geq 0$. We obtain this result as a special case of a more general formula involving Bell polynomials; see Theorem 1.1 below.

Let $B_n(y)=\sum_{k=0}^{n}S(n,k)y^k$, where y is an indeterminate, denote the n-th Bell polynomial (also called a Touchard polynomial); see, e.g., [3] or [6]. The $B_n(y)$ are given explicitly by the Dobinski-like formula

$$B_n(y)=e^{-y}\sum_{i\geq 0}\frac{i^n}{i!}y^i, \qquad n\geq 0.$$

Note that $B_n(y)$ reduces to B_n when $y=1$ and to B_n^* when $y=-1$. The r-Bell polynomials $B_n^{(r)}(y)$ (see, e.g., [9] or [10]) reduce to the Bell polynomials when $r=0$ and have egf formula

$$\sum_{n\geq 0}B_n^{(r)}(y)\frac{x^n}{n!}=e^{y(e^x-1)+rx}, \qquad r\geq 0.$$

Recall

$$B_n^{(r)}(y)=\sum_{k=0}^{n}S_r(n,k)y^k, \qquad n,r\geq 0,$$

where $S_r(n,k)$ is the r-Stirling number of the second kind (see [2] or [7]) which enumerates the partitions of $[n+r]$ having $k+r$ blocks wherein the elements of $[r]$ all belong to different blocks.

Let $x^{\overline{n}}=x(x+1)\cdots(x+n-1)$ for $n\geq 1$ denote the rising factorial, where $x^{\overline{0}}=1$. In the next section, we prove the following general identity relating Bell polynomials, r-Bell polynomials and Stirling numbers of the first kind.

Theorem 1.1. *Let $n,r\geq 0$ and $\ell\geq 1$, with $r\geq \ell-1$. Then we have*

$$\begin{aligned}
&\sum_{j=0}^{r}\sum_{i=0}^{n+j-\ell} s(r,j)\binom{n+j}{i}B_i(y)B_{n+j-i-\ell}(-y)\\
&\quad=\sum_{k=1}^{\ell}\sum_{i=0}^{n-k}\sum_{p=\ell-k}^{r} s(p,\ell-k)\binom{r}{p}\binom{n}{i}y^{r-p}B_i^{(r-p)}(y)B_{n-i-k}(-y)\\
&\qquad+\sum_{j=1}^{r-\ell+1}\sum_{i=0}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!\binom{j-1}{i}\binom{r-j}{p}(-1)^{j-1}p^{\overline{i}}y^{r-j-p}(r-j-p)^n.
\end{aligned} \tag{3}$$

Note that (2) above corresponds to the $y=\ell=1$ case of (3). To prove (3), we establish an equivalent functional identity relating the formulas for the egf's of both sides. Our argument is inductive in the r variable (starting with the case $r=\ell-1$, which requires a separate treatment) and makes use of various identities between the relevant sequences as well as several manipulations involving finite sums. In the final section, we provide a combinatorial proof of the $\ell=1$ case of (3), which entails defining an appropriate sign-changing involution on a structure whose sum of (signed) weights is given by the left-hand side.

2. Proof of Theorem 1.1

Upon considering the egf in n (to be marked by the variable x) of both sides of (3), we prove the equivalent functional identity:

$$\begin{aligned}
&\sum_{j=0}^{r} s(r,j)D^j\left(e^{y(e^x-1)}\cdot I_\ell\left(e^{y(1-e^x)}\right)\right)\\
&\quad=\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r} s(p,\ell-k)\binom{r}{p}y^{r-p}e^{y(e^x-1)+(r-p)x}\cdot I_k\left(e^{y(1-e^x)}\right)\\
&\qquad+\sum_{j=1}^{r-\ell+1}\sum_{i=0}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!\binom{j-1}{i}\binom{r-j}{p}(-1)^{j-1}p^{\overline{i}}y^{r-j-p}e^{(r-j-p)x},
\end{aligned} \tag{4}$$

where D^j denotes the j-th derivative (in x) and

$$I_\ell(f(x)) = \int_0^x \int_0^{x_{\ell-1}} \cdots \int_0^{x_1} f(x_0) d_{x_0} d_{x_1} \cdots d_{x_{\ell-1}}, \qquad \ell \geq 1,$$

for a function $f(x)$.

To show (4), we proceed by induction on $r \geq \ell - 1$, where $\ell \geq 1$ is fixed. We first establish the $r = \ell - 1$ case of (4) directly, noting that it reduces to

$$\begin{aligned}\sum_{j=0}^{r} & s(r,j) D^j \left(e^{y(e^x-1)} \cdot I_{r+1}\left(e^{y(1-e^x)}\right)\right) \\ &= \sum_{k=1}^{r+1} \sum_{p=r-k+1}^{r} s(p, r-k+1) \binom{r}{p} y^{r-p} e^{y(e^x-1)+(r-p)x} \cdot I_k\left(e^{y(1-e^x)}\right), \qquad r \geq 0. \end{aligned} \tag{5}$$

To show (5), we expand the left-hand side using the product rule to get

$$\begin{aligned}\sum_{j=0}^{r} & s(r,j) D^j \left(e^{y(e^x-1)} \cdot I_{r+1}\left(e^{y(1-e^x)}\right)\right) \\ &= \sum_{k=1}^{r+1} I_k\left(e^{y(1-e^x)}\right) \sum_{j=r-k+1}^{r} s(r,j) \binom{j}{r-k+1} D^{j-(r-k+1)} \left(e^{y(e^x-1)}\right) \\ &= \sum_{k=1}^{r+1} I_k\left(e^{y(1-e^x)}\right) \sum_{j=r-k+1}^{r} s(r,j) \binom{j}{r-k+1} \sum_{i=0}^{j-r+k-1} d(j-r+k-1, i) y^i e^{y(e^x-1)+ix},\end{aligned}$$

where the array $d(m,i)$ for $0 \leq i \leq m$ is determined by

$$D^m \left(e^{y(e^x-1)}\right) = \sum_{i=0}^{m} d(m,i) y^i e^{y(e^x-1)+ix}.$$

By induction on m, we have $d(m,i) = S(m,i)$ and thus we get

$$\begin{aligned}\sum_{k=1}^{r+1} & I_k\left(e^{y(1-e^x)}\right) \sum_{j=r-k+1}^{r} s(r,j) \binom{j}{r-k+1} \sum_{i=0}^{j-r+k-1} S(j-r+k-1, i) y^i e^{y(e^x-1)+ix} \\ &= \sum_{k=1}^{r+1} I_k\left(e^{y(1-e^x)}\right) \sum_{i=0}^{k-1} y^i e^{y(e^x-1)+ix} \sum_{j=i+r-k+1}^{r} s(r,j) S(j-r+k-1, i) \binom{j}{r-k+1} \\ &= \sum_{k=1}^{r+1} I_k\left(e^{y(1-e^x)}\right) \sum_{i=0}^{k-1} s(r-i, r-k+1) \binom{r}{i} y^i e^{y(e^x-1)+ix},\end{aligned}$$

where we have used in the last equality the identity

$$\sum_{j=i+\ell}^{r} s(r,j) S(j-\ell, i) \binom{j}{\ell} = s(r-i, \ell) \binom{r}{i},$$

which we were unable to find in the literature but can be shown by induction on r. Replacing i with $r-p$ in the last expression above then implies (5).

We now show that the $(r+1)$-case of (4) follows from the r-case, where $r \geq \ell - 1$. To do so, first note that upon separating the terms for which $i = 0$ in the second sum on the right-hand side, we have that (4) may be written equivalently as

$$M_r(x) = M_r^{(1)}(x) + M_r^{(2)}(x) + M_r^{(3)}(x), \qquad r \geq \ell - 1,$$

where

$$M_r(x) = \sum_{j=0}^{r} s(r,j) D^j \left(e^{y(e^x-1)} \cdot I_\ell\left(e^{y(1-e^x)}\right)\right),$$

$$M_r^{(1)}(x) = \sum_{j=1}^{r-\ell+1} \sum_{p=\ell-1}^{r-j} s(p, \ell-1)(j-1)! \binom{r-j}{p} (-1)^{j-1} y^{r-j-p} e^{(r-j-p)x},$$

$$M_r^{(2)}(x) = \sum_{k=1}^{\ell} \sum_{p=\ell-k}^{r} s(p, \ell-k) \binom{r}{p} y^{r-p} e^{y(e^x-1)+(r-p)x} \cdot I_k\left(e^{y(1-e^x)}\right)$$

and

$$M_r^{(3)}(x) = \sum_{j=2}^{r-\ell+1} \sum_{i=1}^{j-1} \sum_{p=\ell-1}^{r-j} s(p, \ell-1)(j-i-1)! \binom{j-1}{i} \binom{r-j}{p} (-1)^{j-1} p^{\bar{i}} y^{r-j-p} e^{(r-j-p)x}.$$

Now observe

$$\begin{aligned} M_{r+1}(x) &= \sum_{j=0}^{r+1} s(r+1, j) D^j \left(e^{y(e^x-1)} \cdot I_\ell\left(e^{y(1-e^x)}\right)\right) \\ &= \sum_{j=0}^{r+1} (s(r, j-1) - r s(r, j)) D^j \left(e^{y(e^x-1)} \cdot I_\ell\left(e^{y(1-e^x)}\right)\right) \\ &= \left(\frac{d}{dx} - r\right) M_r(x). \end{aligned}$$

Thus, to complete the induction, it suffices to show

$$\sum_{i=1}^{3} \left(\frac{d}{dx} - r\right) M_r^{(i)}(x) = \sum_{i=1}^{3} M_{r+1}^{(i)}(x), \tag{6}$$

for then we would have

$$M_{r+1}(x) = \left(\frac{d}{dx} - r\right) M_r(x) = \sum_{i=1}^{3} \left(\frac{d}{dx} - r\right) M_r^{(i)}(x) = \sum_{i=1}^{3} M_{r+1}^{(i)}(x),$$

where the second equality in the preceding line follows from the induction hypothesis.

Let

$$D_r^{(i)}(x) = \left(\frac{d}{dx} - r\right) M_r^{(i)}(x) - M_{r+1}^{(i)}(x), \qquad 1 \le i \le 3.$$

To establish (6), we must show

$$\sum_{i=1}^{3} D_r^{(i)}(x) = 0. \tag{7}$$

Equality (7) is shown below as a series of lemmas, which then completes the proof of Theorem 1.1.

Lemma 2.1. *We have*

$$D_r^{(1)}(x) = -\sum_{j=1}^{r-\ell+1} \sum_{p=\ell-1}^{r-j} s(p, \ell-1)(j-1)! p E_{j,p} + \sum_{p=\ell-1}^{r} s(p, \ell-1) E_{0,p},$$

where $E_{j,p} = \binom{r-j}{p}(-1)^{j-1} y^{r-j-p} e^{(r-j-p)x}$.

Proof. By the definitions, we have

$$\begin{aligned} D_r^{(1)}(x) &= \left(\frac{d}{dx} - r\right) M_r^{(1)}(x) - M_{r+1}^{(1)}(x) \\ &= \left(\frac{d}{dx} - r\right) \sum_{j=1}^{r-\ell+1} \sum_{p=\ell-1}^{r-j} s(p, \ell-1)(j-1)! E_{j,p} + \sum_{j=1}^{r-\ell+2} \sum_{p=\ell-1}^{r-j+1} s(p, \ell-1)(j-1)! E_{j-1,p} \\ &= -\sum_{j=1}^{r-\ell+1} \sum_{p=\ell-1}^{r-j} s(p, \ell-1)(j-1)!(j+p) E_{j,p} + \sum_{j=1}^{r-\ell+1} \sum_{p=\ell-1}^{r-j} s(p, \ell-1) j! E_{j,p} \\ &\quad + \sum_{p=\ell-1}^{r} s(p, \ell-1) E_{0,p} \\ &= -\sum_{j=1}^{r-\ell+1} \sum_{p=\ell-1}^{r-j} s(p, \ell-1)(j-1)!(j+p-j) E_{j,p} + \sum_{p=\ell-1}^{r} s(p, \ell-1) E_{0,p}, \end{aligned}$$

as desired, upon separating the terms in the second sum of the second equality above for which $j = 1$ from the others (wherein j is then replaced by $j+1$). $\square$

Lemma 2.2. *We have*

$$D_r^{(1)}(x)+D_r^{(2)}(x)=-\sum_{j=1}^{r-\ell+1}\sum_{p=\ell-1}^{r-j}s(p,\ell-1)(j-1)!pE_{j,p}.$$

Proof. Let $H_p=\binom{r}{p}y^{r-p}e^{y(e^x-1)+(r-p)x}$ and $I_k=I_k\left(e^{y(1-e^x)}\right)$ for $p\geq 0$ and $k\geq 1$, with $H_p=0$ if $p<0$. Upon treating separately the terms for which $k=1$, we get

$$\begin{aligned}\left(\frac{d}{dx}-r\right)M_r^{(2)}(x)&=\sum_{p=\ell-1}^{r}s(p,\ell-1)\left(-E_{0,p}+(ye^x+r-p)H_pI_1\right)-r\sum_{p=\ell-1}^{r}s(p,\ell-1)H_pI_1\\&+\sum_{k=2}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)H_p\left(I_{k-1}+(ye^x+r-p)I_k\right)-r\sum_{k=2}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)H_pI_k\\&=-\sum_{p=\ell-1}^{r}s(p,\ell-1)E_{0,p}+\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)(ye^x-p)H_pI_k\\&+\sum_{k=2}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)H_pI_{k-1}.\end{aligned}$$

Further, we have

$$\begin{aligned}M_{r+1}^{(2)}(x)&=\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r+1}s(p,\ell-k)\binom{r+1}{p}y^{r-p+1}e^{y(e^x-1)+(r-p+1)x}I_k\\&=\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r+1}s(p,\ell-k)\left(\binom{r}{p}+\binom{r}{p-1}\right)y^{r-p+1}e^{y(e^x-1)+(r-p+1)x}I_k\\&=\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)\binom{r}{p}y^{r-p+1}e^{y(e^x-1)+(r-p+1)x}I_k\\&\quad+\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r+1}s(p,\ell-k)\binom{r}{p-1}y^{r-p+1}e^{y(e^x-1)+(r-p+1)x}I_k\\&=ye^x\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)H_pI_k+\sum_{k=1}^{\ell}\sum_{p=\ell-k-1}^{r}s(p+1,\ell-k)H_pI_k\\&=ye^x\sum_{k=1}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)H_pI_k+\sum_{k=1}^{\ell}\sum_{p=\ell-k-1}^{r}(s(p,\ell-k-1)-ps(p,\ell-k))H_pI_k.\end{aligned}$$

Combining this with the formula found above for $\left(\frac{d}{dx}-r\right)M_r^{(2)}(x)$ and simplifying, we get

$$\begin{aligned}D_r^{(2)}(x)&=\left(\frac{d}{dx}-r\right)M_r^{(2)}(x)-M_{r+1}^{(2)}(x)\\&=-\sum_{p=\ell-1}^{r}s(p,\ell-1)E_{0,p}+\sum_{k=2}^{\ell}\sum_{p=\ell-k}^{r}s(p,\ell-k)H_pI_{k-1}\\&\quad-\sum_{k=1}^{\ell-1}\sum_{p=\ell-k-1}^{r}s(p,\ell-k-1)H_pI_k\\&=-\sum_{p=\ell-1}^{r}s(p,\ell-1)E_{0,p},\end{aligned}$$

upon replacing k with $k-1$ in the third sum. Combining this result with Lemma 2.1 completes the proof. $\square$

Lemma 2.3. *Equality* (7) *holds for all* $r\geq\ell-1$.

Proof. We are left with computing $D_r^{(3)}(x)$. First note that

$$D_r^{(3)}(x)=\left(\frac{d}{dx}-r\right)M_r^{(3)}(x)-M_{r+1}^{(3)}(x)$$

$$
\begin{aligned}
&= -\sum_{j=2}^{r-\ell+1}\sum_{i=1}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!(j+p)\binom{j-1}{i}p^{\overline{i}}E_{j,p}\\
&\quad+\sum_{j=2}^{r-\ell+2}\sum_{i=1}^{j-1}\sum_{p=\ell-1}^{r-j+1} s(p,\ell-1)(j-i-1)!\binom{j-1}{i}p^{\overline{i}}E_{j-1,p}\\
&= -\sum_{j=2}^{r-\ell+1}\sum_{i=1}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!(j+p)\binom{j-1}{i}p^{\overline{i}}E_{j,p}\\
&\quad+\sum_{j=1}^{r-\ell+1}\sum_{i=1}^{j}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i)!\binom{j}{i}p^{\overline{i}}E_{j,p}\\
&= -\sum_{j=2}^{r-\ell+1}\sum_{i=1}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!\left((i+1)\binom{j}{i+1}+p\binom{j-1}{i}\right)p^{\overline{i}}E_{j,p}\\
&\quad+\sum_{j=2}^{r-\ell+1}\sum_{i=1}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!(i+1)\binom{j}{i+1}p^{\overline{i}}E_{j,p}\\
&\quad+\sum_{j=1}^{r-\ell+1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)p^{\overline{j}}E_{j,p}\\
&= -\sum_{j=2}^{r-\ell+1}\sum_{i=1}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!p\binom{j-1}{i}p^{\overline{i}}E_{j,p}\\
&\quad+\sum_{j=1}^{r-\ell+1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)p^{\overline{j}}E_{j,p},
\end{aligned}
$$

upon noting $(j+p)\binom{j-1}{i}=(i+1)\binom{j}{i+1}+p\binom{j-1}{i}$, $(j-i)!\binom{j}{i}=(j-i-1)!(i+1)\binom{j}{i+1}$ if $i<j$ and separating the terms for which $i=j$ in the second sum of the third expression for $D_r^{(3)}(x)$.

Combining the result of Lemma 2.2 with the first sum in the last expression above for $D_r^{(3)}(x)$ then gives

$$\sum_{i=1}^{3} D_r^{(i)}(x) = -A+B,$$

where

$$A=\sum_{j=1}^{r-\ell+1}\sum_{i=0}^{j-1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)(j-i-1)!p\binom{j-1}{i}p^{\overline{i}}E_{j,p}$$

and

$$B=\sum_{j=1}^{r-\ell+1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)p^{\overline{j}}E_{j,p}.$$

Interchanging the two inner sums in the expression for A, and noting $(j-i-1)!=1^{\overline{j-i-1}}$, implies

$$
\begin{aligned}
A&=\sum_{j=1}^{r-\ell+1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)pE_{j,p}\sum_{i=0}^{j-1}\binom{j-1}{i}p^{\overline{i}}1^{\overline{j-i-1}}\\
&=\sum_{j=1}^{r-\ell+1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)pE_{j,p}\cdot(p+1)^{\overline{j-1}}\\
&=\sum_{j=1}^{r-\ell+1}\sum_{p=\ell-1}^{r-j} s(p,\ell-1)p^{\overline{j}}E_{j,p}=B,
\end{aligned}
$$

which completes the proof, where we have used the identity $(u+v)^{\overline{m}}=\sum_{i=0}^{m}\binom{m}{i}u^{\overline{i}}v^{\overline{m-i}}$ (see [5, p. 245]) and the fact $p(p+1)^{\overline{j-1}}=p^{\overline{j}}$. ☐

3. Combinatorial proof of a special case

We provide a combinatorial proof of the $\ell = 1$ case of (3), which may be written as

$$\sum_{j=0}^{r}\sum_{i=0}^{n+j-1} s(r,j)\binom{n+j}{i}B_i(y)B_{n+j-i-1}(-y)$$
$$=\sum_{i=0}^{n-1}\binom{n}{i}y^rB_i^{(r)}(y)B_{n-i-1}(-y)-\sum_{j=0}^{r-1}(-1)^{r-j}y^j(r-j-1)!j^n, \qquad n,r\geq 0. \tag{8}$$

Note that (8) reduces to (2) when $y = 1$.

Combinatorial proof of (8)**.**

We may assume $r \geq 1$, as (8) clearly holds if $r = 0$. Let $\mathcal{A} = \mathcal{A}_{n,r}$ denote the set of configurations obtained as follows. First arrange the elements of $[r]$ according to a permutation having j cycles in standard form for some $j \in [r]$. We put these j cycles together with the elements of $[n]$ and refer to any one of these $n + j$ objects as an *item*. Assume that all individual elements of $[n]$ are less than all cycles of $[r]$, with cycles being ordered according to the size of their respective largest elements. We then select exactly i items and arrange them according to an arbitrary partition of size i, the blocks of which will be colored black. Next, we hold out the largest unselected item and then arrange the remaining $n + j - i - 1$ items according to an arbitrary partition, the blocks of which will be colored white. Let $\mathcal{A}$ denote the set of all possible configurations arising as i and j vary. Define the (signed) weight of $\pi \in \mathcal{A}$ by

$$\mathrm{wgt}(\pi) = (-1)^{r-j+\mu_2(\pi)}y^{\mu_1(\pi)+\mu_2(\pi)},$$

where j denotes the number of cycles and μ_1 and μ_2 are the statistics on $\mathcal{A}$ recording the number of black and white blocks, respectively. Then the left-hand side of (8) is seen to give the sum of the weights of the members of $\mathcal{A}$, upon considering all possible i and j.

We now define an involution on $\mathcal{A}$ as follows. Let $\pi \in \mathcal{A}$ and suppose some block B of either color has two or more elements of $[r]$ altogether within the cycles contained therein. Let us assume that out of all the cycles that lie within blocks of the form B, the block B' contains the cycle with the smallest element. Let k and ℓ denote the smallest and second smallest elements of $[r]$ lying within the cycles in B'. If k and ℓ lie in the same cycle, say as $(k\cdots\ell\cdots)$, then we break this cycle at ℓ to obtain the two cycles $(k\cdots)$ and $(\ell\cdots)$. Conversely, if k and ℓ belong to different cycles, then we fuse these cycles into a single cycle. Note that this operation is an involution that reverses the sign while preserving the y-weight (as no blocks of either kind are created or destroyed). Hence, we may assume that all cycles lying within the blocks of π are of length one, with at most one cycle per block. Let $\mathcal{A}'$ denote the subset of $\mathcal{A}$ consisting of those π for which this is indeed the case (i.e., the set of survivors of the involution defined above). Note however that if the item held out prior to forming the partition with white blocks is a cycle, then there is no restriction on the length of this cycle.

We now determine the sum of the weights of all members of $\mathcal{A}'$. First suppose that the item held out is a member of $[n]$. Then all cycles must belong to black blocks, by the ordering of the items, with no two cycles belonging to the same block. Let i denote the number of members of $[n]$ occurring in black blocks, where $0 \leq i \leq n-1$. Then there are $\binom{n}{i}y^rB_i^{(r)}(y)$ possibilities regarding the selection and placement of the items in black blocks, where the y^r factor accounts for the r extra blocks (containing the r 1-cycles) which would not otherwise be accounted for by $B_i^{(r)}(y)$. The remaining members of $[n]$ (minus the element that is held out) are accounted for by $B_{n-i-1}(-y)$ as they occur in white blocks. Considering all possible i then gives the first sum on the right side of (8).

Now suppose that the item held out within $\pi \in \mathcal{A}'$ is a cycle C. If C does not consist solely of elements of $[j]$ for some $j \in [r]$, then there exists a 1-cycle (t) occurring in a block of π of either color where t is less than the maximum element contained in C. We may assume that t is the smallest element in $[r]$ for which there exists such a 1-cycle. If (t) lies in block D of π, then (t) is the only cycle belonging to D and we may change the color of D, by the ordering of items. Thus, there is no contribution towards the sum of weights coming from π where C does not comprise $[j]$ for some j. On the other hand, if C does comprise $[j]$, then there are $(j-1)!$ possibilities for C with all cycles occurring in black blocks. Further, we may assume that there are no black blocks which fail to contain a 1-cycle or white blocks (which must consist solely of elements of $[n]$ in this case), for otherwise the color may be changed of the block that contains the smallest element of $[n]$ among such blocks reversing the sign. Thus, the weight is given by $(-1)^{j-1}y^{r-j}$ for each possible configuration. The number of these configurations is given by $(j-1)!(r-j)^n$ since there are $(r-j)^n$ ways in which to place the elements of $[n]$ in black blocks (as each element must go in a block containing a 1-cycle (ℓ) for some $\ell \in [j+1,r]$). Thus, the contribution towards the weight is given by $(-1)^{j-1}y^{r-j}(j-1)!(r-j)^n$. Summing over all $j \in [r]$, and replacing j with $r-j$, then gives the second sum on the right side of (8) and completes the proof. $\square$

We close with a few remarks concerning the last proof. First note that the $\ell = 0$ case of Theorem 1.1 also holds and implies

$$\sum_{j=0}^{r}\sum_{i=0}^{n+j} s(r,j)\binom{n+j}{i}B_i(y)B_{n+j-i}(-y) = \begin{cases} 0, & \textit{if } n \geq 1; \\ \delta_{r,0}, & \textit{if } n = 0. \end{cases}$$

This may be realized combinatorially by changing the color of the block containing 1 if $n \geq 1$ or of the block containing the cycle that starts with 1 if $n = 0$ and $r \geq 1$, where no item is held out in forming the configurations as described above. Further, the involution of fusing and breaking cycles as described in the second paragraph of the preceding proof can be used to provide a combinatorial explanation of formula (1) above. Note that here the cycles of some permutation of $[r]$ are arranged in an arbitrary partition together with the elements of $[n]$ and the sign is defined as $(-1)^{r-j}$, where j denotes the number of cycles. The survivors of the involution would then all have positive sign and be synonymous with the r-partitions of $[n+r]$ as the 1-cycles (ℓ) for $\ell \in [r]$ must belong to distinct blocks. Finally, it is possible to generalize the foregoing combinatorial proof of (8) to provide such a proof for Theorem 1.1, the details of which we do not pursue here for the sake of brevity.

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